Some interactions / model theory and set theory

Andrés Villaveces - *Universidad Nacional de Colombia - Bogotá*

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Contents

Around walls

A completely stupid wall...

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(A conversation in the Arctic)

Model Theory and Set Theory - walls or bridges?

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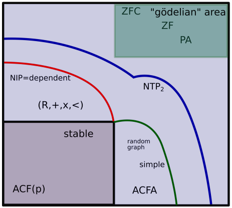

A "map" of the universe (FO).

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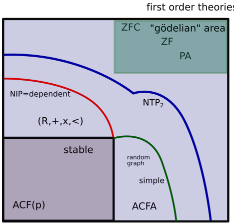

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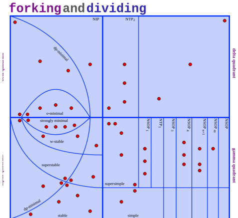
- Or else (see forkinganddividing.com / G. Conant)



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In the 1920s and 30s Gödel, Carnap, Skolem, ... studied the very well-known incompleteness of "fixed" structures, the completeness of "flexible" structures - categoricity emerged as a very special and peculiar version of completeness.

In the mid-1950s, based on many other observations Łoś conjectured that every first order theory in a countable vocabulary can only have 4 kinds of categoricity spectrum:

$$\emptyset \qquad (\aleph_0) \qquad (> \aleph_0) \qquad (Card_\infty).$$

The Shelah Conjecture (early version)

A key test problem in model theory in the past two or three decades: finding versions of the Morley Theorem and Shelah's Categoricity Transfer theorems, for wider contexts: abstract elementary classes (semantically-centered extensions of the model theory of $L_{\lambda^+,\omega}(Q)$).

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Conjecture (Shelah)

Given any cardinal λ, there exists μ_λ such that if ψ is an $L_{\omega_1,\omega}$-sentence that satisfies a "Löwenheim-Skolem" theorem down to λ and is categorical is some cardinality $\geq \mu_\lambda$, then it is categorical in all cardinalities above μ_λ.

Abstract Elementary Classes

Fix a first order vocabulary τ.
Let $\mathcal{K}$ be a class of τ-structures, $\prec = \prec_{\mathcal{K}}$ a binary relation on $\mathcal{K}$.

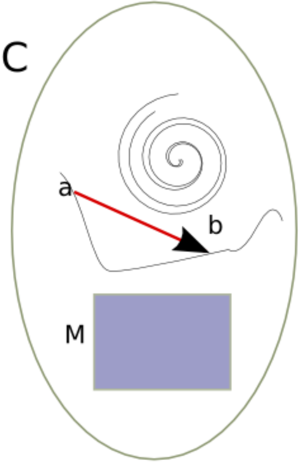
Definition

$(\mathcal{K}, \prec_{\mathcal{K}})$ is an abstract elementary class if

- $\mathcal{K}$, $\prec_{\mathcal{K}}$ are **closed under isomorphism**,
- $M, N \in \mathcal{K}, M \prec_{\mathcal{K}} N \Rightarrow M \subset N$,
- $\prec_{\mathcal{K}}$ is a partial order,
- **(Tarski-Vaught)** $M \subset N \prec_{\mathcal{K}} \bar{N}, M \prec_{\mathcal{K}} \bar{N} \Rightarrow M \prec_{\mathcal{K}} N$, y…
- **($\searrow$LS)** $\exists \kappa = LS(\mathcal{K}) \geq \aleph_0$ s.t. $\forall M \in \mathcal{K}, \forall A \subset |M|, \exists N \prec_{\mathcal{K}} M$ with $A \subset |N|$ and $\|N\| \leq |A| + LS(\mathcal{K})$,
- **(Unions of $\prec_{\mathcal{K}}$-chains)** A union of $\prec_{\mathcal{K}}$-chain in $\mathcal{K}$ belongs to $\mathcal{K}$, is a $\prec_{\mathcal{K}}$-extension of all the models in the chain and is the sup of the chain.

Galois (orbital) types

The correct notion of type in an AEC (with the amalgamation and joint embedding properties (AP, JEP), and no maximal models (NMM):



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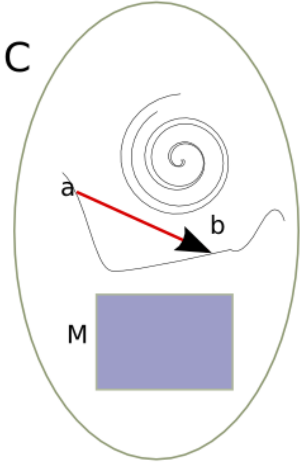

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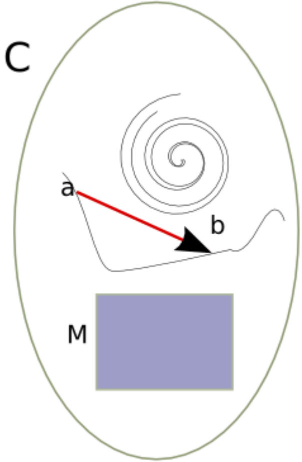

1. First build a "monster" (universal, model-homogeneous) model $\mathbb{C}$ in the class.
2. Next define $ga-tp(a/M) = ga-tp(b/M)$ if and only if there exists $f \in Aut(\mathbb{C}/M)$ s.t. $f(a) = b$.
3. Then (under AP, JEP, NMM) Galois types over M are orbits under the action of the group $Aut_M(\mathbb{C})$, the automorphisms of the monster that fix M pointwise.

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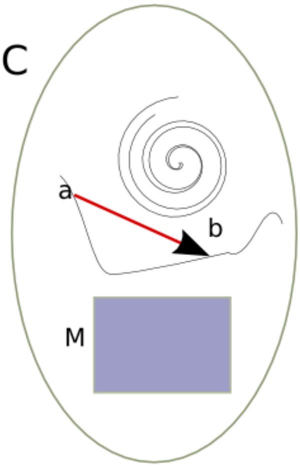

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4. (This generalizes the classical (syntactic) notion of a type.)

We now start our descent.

Grossberg-VanDieren: tameness isolated

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Their proof built on a previous proof of the "downward" transfer by Shelah but has a crucial element: isolating the notion of tameness ("buried" in Shelah's proof of the downward part - fleshing out the notion allows Grossberg/VanDieren to prove the upward categoricity).

Localizing difference

Idea: “localizing” the condition of...
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- we want: to localize this to checking that there is some $M_0 \in \mathcal{P}^*_\kappa(M)$ and $X_0 \in \mathcal{P}_\kappa(N_0)$ such that

$$\text{gatp}(X_0/M_0) \neq \text{gatp}(f(X_0)/M_0)$$

Tameness and type-shortness

Definition ((κ, λ)-tameness for μ, type shortness)

Let $\kappa < \lambda$. An aec $\mathcal{K}$ with AP and $LS(\mathcal{K}) \leq \kappa$ is

- (κ, λ)-tame for sequences of length μ if for every $M \in \mathcal{K}$ of size λ, if $p_1 \neq p_2$ are Galois types over M then there exists $M_0 \prec_{\mathcal{K}} M$ with $|M_0| \leq \kappa$ such that

$$p_1 \restriction M_0 \neq p_2 \restriction M_0$$

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$$gatp((x_{1,\alpha})_{\alpha \in I}/M) \neq gatp((x_{2,\alpha})_{\alpha \in I}/M).$$

Dual notions - stability

The two notions are clearly dual (parameters/realizations):

- In tameness, a narrow orbit (fixing large models) is controlled by the thicker orbits that approximate it (parameter locality),

These dualities are actually equivalent under stability conditions. In general, they are not.

Dual notions - stability

The two notions are clearly dual (parameters/realizations):

- In tameness, a narrow orbit (fixing large models) is controlled by the thicker orbits that approximate it (parameter locality),
- In type shortness, the orbit of a long sequence is controlled by the narrower orbits of its subsequences (realization locality)...

These dualities are actually equivalent under stability conditions. In general, they are not.

Getting Tameness from Large Cardinals

In 2013, Boney changed a bit the direction of the approach: why not look directly at the impact of large cardinals on tameness and similar notions?

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Theorem (Boney)

If κ is strongly compact and $\mathcal{K}$ is essentially below κ (i.e. $LS(\mathcal{K}) < \kappa$ or $\mathcal{K} = Mod(\psi)$ for some $L_{\kappa,\omega}$-sentence ψ) then $\mathcal{K}$ is $(< (\kappa + LS(K)^+, \lambda$-tame and $(< \kappa, \lambda)$-typeshort for all λ.

Boney and Unger proved (2015) that under strong inaccessibility of κ, the $(< \kappa, \kappa)$-tameness of all aecs implies κ's strong compactness.

Reframing slightly Boney's proof

Remember

- A cardinal κ is strongly compact iff for every $\lambda > \kappa$ there exists an elementary embedding $j : V \to M$ with critical point κ, and there exists some $Y \in M$ such that $j''\lambda \subset Y$ and $|Y|^M < j(\kappa)$.

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Definition

Let $j : V \to M$ be an elementary embedding. j has the (κ, λ)-cover property if for every X with $|X| \leq \lambda$ there exists $Y \in M$ such that $j''X \subset Y \subset j(X)$ and $|Y|^M < j(\kappa)$.

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For example, for a measurable cardinal κ, the usual embedding j has the (κ, κ)-cover property. If κ is λ-strongly compact, and U is a fine κ-complete ultrafilter on $P_\kappa(\lambda)$ then the associated j has the (κ, λ)-cover property.

THE "IMAGE" OF AN AEC UNDER $j : V \to M$

Let in general $(\mathcal{K}, \prec_{\mathcal{K}})$ be an AEC in τ.
Shelah's Presentation Theorem gives

- $\tau' \supset \tau$,
- T' a τ'-theory and
- Γ' a set of T'-types

such that

$$\mathcal{K} = PC(\tau, T', \Gamma') = \{M' \restriction \tau \mid M' \models T' \text{ and } M' \text{ omits } \Gamma'\},$$

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We define $j(\mathcal{K})$ as the class $PC^M(j(\tau), j(T'), j(\Gamma'))$.
By elementarity, $M \models j(\mathcal{K})$ is a an AEC with LS number equal to $j(LS(\mathcal{K}))$.

Attempt at getting $j(\mathcal{K}) \subset \mathcal{K}$ and $\prec_{j(\mathcal{K})} \subset \prec_{\mathcal{K}}$.

Definition

Let $\mathcal{M} \in \mathcal{K}$ (a τ-AEC). Then $j(\mathcal{M})$ is a $j(\tau)$-structure. We say that j respects $\mathcal{K}$ if the following conditions hold:

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- for every $\mathcal{M}, \mathcal{N} \in j(\mathcal{K})$, $\mathcal{M} \prec_{j(\mathcal{K})} \mathcal{N}$ implies $\mathcal{M} \upharpoonright \tau \prec_{\mathcal{K}} \mathcal{N} \upharpoonright \tau$,

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- for every $\mathcal{M} \in \mathcal{K}$, $j''\mathcal{M} \prec_{\mathcal{K}} j(\mathcal{M}) \restriction \tau$.

Examples

1. Let first $j : V \to M$ be a nontrivial elementary embedding with critical point κ and let $\mathcal{K}$ be an AEC with $LS(\mathcal{K}) < \kappa$. Then $\mathcal{K} = PC(\tau', T', \Gamma')$, with $|\tau'| + |T'| + |\Gamma'| < \kappa$; wlog we can assume $\tau', T', \Gamma' \in V_\kappa$ and therefore

$$j(\mathcal{K}) = PC^M(\tau, T', \Gamma') = (\mathcal{K} \cap M, \prec_{\mathcal{K}} \cap M),$$

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2. $\mathcal{K}$ is given as $Mod(\varphi)$ for φ in $L_{\kappa,\omega}$, with $\prec_{\mathcal{K}} = \subset^{TV}_{\mathcal{F}}$, $\mathcal{F}$ some fragment of $L_{\kappa,\omega}$. Then j respects $\mathcal{K}$.

Getting Tameness

We prove then that whenever $\mathcal{K}$ is an AEC with $LS(\mathcal{K}) < \kappa < \lambda$, and $j : V \to M$ has the (κ, λ)-cover property and respects $\mathcal{K}$ then $\mathcal{K}$ is $(< \kappa, \lambda)$-tame.

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Let $\mathcal{M} \in \mathcal{K}_\lambda$ and $p_1 = \text{gatp}(\vec{a}/\mathcal{M}, \mathcal{N}_1)$, $p_2 = \text{gatp}(\vec{b}/\mathcal{M}, \mathcal{N}_2)$ be two types such that for every $\mathcal{N} \prec_{\mathcal{K}} \mathcal{M}$ of size $< \kappa$ we have

$$p_1 \restriction \mathcal{N} = p_2 \restriction \mathcal{N}.$$

(Here, $\vec{a} = (a_i)_{i \in I}$, $\vec{b} = (b_i)_{i \in I}$.)

Getting Tameness

Let now $Y \in M$ by such that $j''|\mathcal{M}| \subset Y \subset j(|\mathcal{M}|)$ and $|Y|^M < j(\kappa)$. But in M, $LS(j(\mathcal{K})) = j(LS(\mathcal{K})) < j(\kappa)$ so there is $\mathcal{M}' \in j(\mathcal{K})$ such that $Y \subset |\mathcal{M}'|$, $\|\mathcal{M}'\| < j(\kappa)$ and $\mathcal{M}' \prec_{j(\mathcal{K})} j(\mathcal{M})$; by transitivity, $\mathcal{M}' \prec_{j(\mathcal{K})} j(\mathcal{M})$.

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By elementarity, $M \models j(p_1) \restriction \mathcal{M}' = j(p_2) \restriction \mathcal{M}'$ (in $j(\mathcal{K})$) and therefore

$$\begin{aligned} p_1' &= \mathrm{gatp}(j(\vec{a})/\mathcal{M}' \restriction \tau, j(\mathcal{N}_1) \restriction \tau) \\ &= \mathrm{gatp}(j(\vec{b})/\mathcal{M}' \restriction \tau, j(\mathcal{N}_2) \restriction \tau) = p_2' \end{aligned}$$

in $\mathcal{K}$ (again by our hypothesis on j).

Getting Tameness

Since $j''\mathcal{M} \prec_{\mathcal{K}} j(\mathcal{M})$ we get that $j''\mathcal{M} \prec_{\mathcal{K}} \mathcal{M}' \upharpoonright \tau$ (coherence axiom), so restricting we have

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Restricting "above" we get

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and therefore

$$p = q. \qquad \square$$

Back to the Reflection Property

So, we use the λ-strong compactness of κ to show first that the embedding $j : V \to M$ has the (κ, λ)-property and respects $\mathcal{K}$ and then apply the previous. One may also show that the (κ, λ)-cover of $j : V \to M$ for $\kappa > LS(\mathcal{K})$ implies

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So, we are in a good position to use the Grossberg-VanDieren theorem to conclude the consistency of the Shelah Categoricity Conjecture.

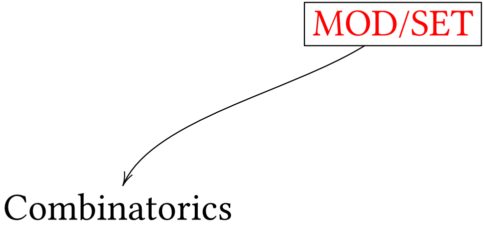
Other interactions Mod Th / Set Th

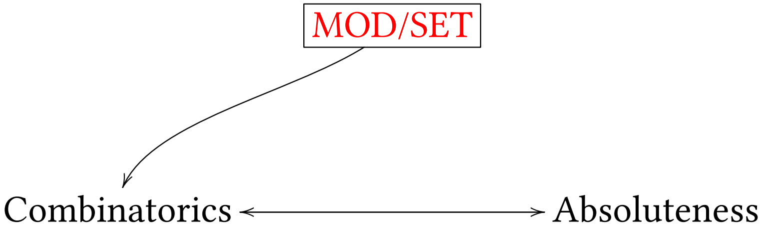
More on the two sides

Oh... I had a very strange referee report on the (proper forcing) paper. I think Moschovakis was the editor. So he thought "Saharon is a model theorist" well, he knew me - I was even a year in UCLA before, so he sent it to a model theorist. And the problem was in model theory, [of the form] "the consistency of...", and the referee report said "well, there is very little model theory". . .

Saharon Shelah, in (forthcoming) interview, 2017.

MOD/SET


MOD/SET
Combinatorics


MOD/SET
Combinatorics
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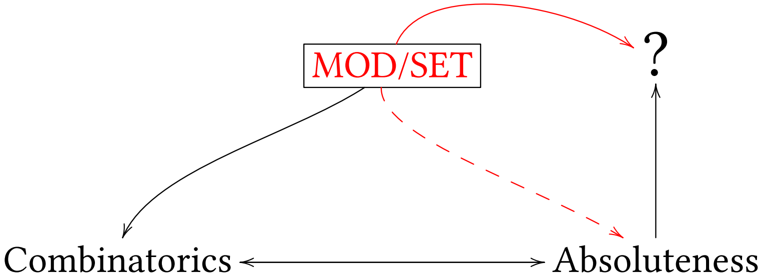
MOD/SET
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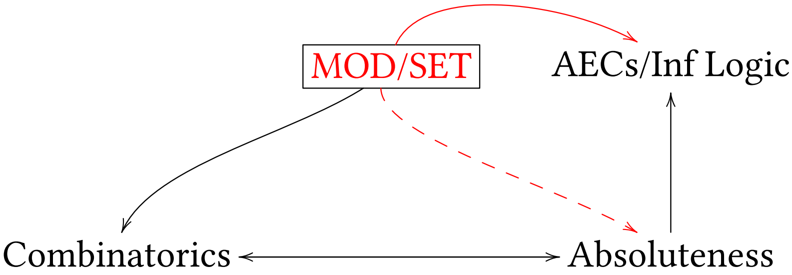
MOD/SET
AECs/Inf Logic
Combinatorics
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A dichotomic behavior

- Under Weak Diamond:

Theorem (from Sh88)

(Under $2^\kappa < 2^{\kappa^+}$). Every aec $\mathcal{K}$ with $LS(\mathcal{K}) \leq \kappa$, categorical in κ, failing AP for models of size κ has 2^{κ^+} many non-isomorphic models of cardinality κ^+.

A DICHOTOMIC BEHAVIOR

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- Example under MA:
  (MA_{ω_1}) There is a class (axiomatizable in $L_{\omega_1,\omega}(Q)$) that is $\aleph_0$-categorical, fails AP in $\aleph_0$ and is also categorical in $\aleph_1$. This can be lifted below continuum.

Forcing isomorphism/categoricity

Theorem (Asperó, V.)

The existence of a weak AEC, categorical in both $\aleph_1$ and $\aleph_2$, failing AP in $\aleph_1$, is consistent with ZFC+CH+$2^{\aleph_1} = 2^{\aleph_2}$.

The result is obtained by an ω_3-iteration over a model of GCH, where we

- Start with GCH in V.
- Build a countable support iteration of length ω_3, where
- at each stage α of the iteration you consider in $V^{\mathbb{P}_\alpha}$ two models $M_0, M_1 \in \mathcal{K}$, $|M_0| = |M_1| = \aleph_2$ (use a bookkeeping function) and
- fix $(M_i^0)_{i<\omega_2}$, $(M_i^1)_{i<\omega_2}$ resolutions of the two models with $M_i^\varepsilon = N_i \cap M_\varepsilon$ where $(N_i)_{i<\omega_2}$ is an $\in$-increasing and $\subset$-continuous of elementary substructures of some $H(\theta)$ of size $\aleph_1$ containing M_0 and M_1...

Forcing isomorphism/categoricity

- at this stage iterate with $\mathbb{Q}_\alpha$ the partial order consisting of countable partial isomorphisms p between M_0 and M_1 such that if $x \in \mathrm{dom}(p)$ and i is the minimum such that $x \in M_i^0$ then $p(x) \in M_i^1$.
- Each stage $\mathbb{Q}_\alpha$ of the iteration, and all the forcing $\mathbb{P}_{\omega_3}$ is σ-closed and $\mathbb{P}_{\omega_3}$ has the $(\aleph_2) - a.c.$ (need CH for the relevant (!) Δ-lemma).

Collapsing and its limitations

Collapsing large cardinals while keeping some of their properties has a long history of interesting results. For instance,

- Mitchell: collapsed a weakly compact to $\aleph_2$ while keeping the tree property. This was later generalized (collapsing much more) in order to get the tree property at all the $\aleph_n$'s and/or in $\aleph_{\omega+1}$ (Magidor, Cummings, Neeman, Fontanella, etc.)

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- For the "strong tree" and "supertree" properties the consistency strength seems to be around a strongly compact / supercompact respectively. (Weiss, Viale, Fontanella, Magidor).

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- ... but adapting Levy-collapse (Easton iteration) or the more sophisticated constructions mentioned cannot yield full tameness; only residual.

A reflecting / playful wall...

¡Gracias! / Thank you!