Interpolation and Model Theoretic Forcing: some new perspectives

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Chapter 1

Genericity in model theory (1)

Model Companions (again)

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Model Companions (again)

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2. Approximation Theorems
3. Preservation Theorems

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We will begin by taking a new look at some of these.

Model Companions as Theories of Fraïssé Limits

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T	T^{comp}
Trivial theory	T^{∞}
R graph	R a random graph
Fields	ACF
Fields with an automorphism f	ACFA (Chatzidakis-Hrushovski)
	(=ACF with a generic automorphism)
$<$ linear order	$\mathrm{Th}(\mathbb{Q}, <)$

Fraïssé Limits

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The last step may be impossible[1].

[1] Kikyo - Model Companions of Theories with an Automorphism - JSL 65, 2000

Hrushovski constructions / Hrushovski generics

You may read the so-called "Hrushovski constructions" as containing an element of building a generic object in a way analogous to Fraïssé but with a much finer control of amalgams along the class: example of building a random forest (amalgams fail if not controlled, Hrushovski uses a pre-dimension function

$$\delta : \mathcal{K} \to \mathbb{Z}$$

in order to get the generic). This genericity is important in his later applications of model theory to algebraic geometry.

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Not clear (to me) what this would say about model companions in set theory (and the role of universally Baire sets).

Zilber: Structural approximation

Jumping (for a while) a bit in time, here is Boris Zilber (motivated by (more general) insatisfaction with the current state of "mathematization" of quantum field theory in contemporary physics:

[2] On model theory, non-commutative geometry and physics. Boris Zilber. BSL, 2010.

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`huge progress achieved by physicists in dealing with singularities and non-convergent sums and integrals (...Feynman path integrals) has not been matched so far ... with an adequate mathematical theory` (Zilber[2])

[2] On model theory, non-commutative geometry and physics. Boris Zilber. BSL, 2010.

Limit and ideal models, à la Zilber

Zilber, on the limiting process:

The process of understanding the physical reality by working in an ideal model can be interpreted as follows. We assume that the ideal model $\mathbb{M}_{\text{ideal}}$ is being chosen from a class of "nice" structures, which allows a good theory. We suppose that the real structure $\mathbb{M}_{\text{real}}$ is "very similar" to $\mathbb{M}_{\text{ideal}}$ (...) approximated by a sequence $\mathbb{M}_i$ of structures and $\mathbb{M}_{\text{real}}$ is one of these, $\mathbb{M}_i = \mathbb{M}_{\text{real}}$ sufficiently close to $\mathbb{M}_{\text{ideal}}$. The notion of approximation must also contain both logical and topological ingredients. (...)

This goes on...

... the reason that we wouldn't distinguish two points in the ideal model $\mathbb{M}_{\text{ideal}}$ is that the corresponding points are very close in the real world $\mathbb{M}_{\text{real}}$ so that we do not see the difference (using the tools available). In the limit of the $\mathbb{M}_i$'s this sort of difference will manifest itself as an infinitesimal. In other words, the limit passage from the sequence $\mathbb{M}_i$ to the ideal model $\mathbb{M}_{\text{ideal}}$ must happen by killing the infinitesimal differences. (...) This corresponds to taking a specialization (...) from an ultraproduct $\prod_D \mathbb{M}_i$ to $\mathbb{M}_{\text{ideal}}$.

Zilber's examples of structural approximation include no less than the

- Gromov-Hausdorff limit of metric spaces and
- deformation of algebraic varieties.

BUT...

... We note that the scheme is quite delicate regarding metric issues. In principle we may have a well-defined metric (...) on the ideal structure only. Existence of a metric, especially the one that gives rise to a structure of a differentiable manifold, is one of the key reasons of why we regard some structures as "nice" or "tame". The problem of whether and when a metric on $\mathbb{M}$ can be passed to approximating structures $\mathbb{M}_i$ might be difficult, indeed we don't know how to answer this problem in some interesting cases.

Close to Interpolation: Approximation Theorems

Back to the 1970s, let us check two uses of model theoretic forcing à la Robinson (for inspiration...):

- Vaught[3]: an approximation theorem saying that for every Borel set B there is an invariant Borel set B$*$ defined by an $L_{\omega_1,\omega}$ sentence (a preservation theorem saying that invariant Borel sets are those definable in $L_{\omega_1,\omega}$: Lopez-Escobar, Ryll-Nardzewsky and Scott)
- Harnik[4] refines Vaught's argument and finds dual theorems Approximation / Preservation using Model Theoretic forcing.

---

[3] Vaught: A Borel invariantization. BAMS 79, 1973.

[4] Harnik: Approximation Theorems and Model Theoretic Forcing, JSL 41, 1976.

Harnik's Approximation Theorems

For example, Harnik uses MThF to capture elementary equivalence for game-formulas:

- For any L_G-sentence φ there is an L_G-sentence φ^* which is a basic combination of $L_{\omega\omega}$-sentences such that
  1. if for all countable $\mathfrak{B} \equiv \mathfrak{A}$, $\mathfrak{B} \models \varphi$ then $\mathfrak{A} \models \varphi^*$ and
  2. if $\mathfrak{A} \models \varphi^*$ then there is a countable $\mathfrak{B} \equiv \mathfrak{A}$ such that $\mathfrak{B} \models \varphi$.
- The corresponding preservation theorem: A sentence φ in $L_{\omega_1\omega}$ (or in L_G) is preserved under elementary equivalence iff φ is equivalent to an $L_{\omega_1\omega}$-sentence (or L_G-sentence) which is a basic combination of $L_{\omega\omega}$-sentences.

Conjunctive/Disjunctive Game Formulas / L_G

L_G is an extension of $L_{\omega_1\omega}$ by the two formation rules

- $\pi(z_0, ..., z_{k-1}) =$
  $\forall u_0 \exists k_0 \forall u_1 \exists k_1 \cdots \left(\bigwedge_{n<\omega} \pi^{k_0 \ldots l_n}(u_0 v_0 \cdots v_n, z_0, \cdots, z_{k-1})\right)$
- $\sigma(z_0, ..., z_{k-1}) =$
  $\exists u_0 \forall k_0 \exists u_1 \forall k_1 \cdots \left(\bigvee_{n<\omega} \sigma^{k_0 \ldots l_n}(u_0 v_0 \cdots v_n, z_0, \cdots, z_{k-1})\right)$

Basic rules:

$$\forall k_0 \exists l_0 \forall k_1 \exists l_1 \ldots \left(\bigwedge_{n<\omega} \pi^{k_0 \ldots l_n}\right)$$

$$\exists k_0 \forall l_0 \exists k_1 \forall l_1 \ldots \left(\bigvee_{n<\omega} \sigma^{k_0 \ldots l_n}\right)$$

The role of Interpolation (elsewhere)

More recently[5], the logic L^1_κ has been built (via games, no syntax) to capture the mix

Interpolation + Weak Compactness

in the interval

$$L_{\kappa\omega} < L^1_\kappa < L_{\kappa\kappa}$$

and is also linked to a correct logic for (so called) AECs (current work with Shelah).

[5] Shelah, Nice infinitary logics – J American Math Soc 25 (2012)

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Work in progress with Väänänen and Velickovic (finding correct syntax for L^1_κ, study different logics approximating L^1_κ, understanding the connection to the combinatorics of AECs) has a flavor of MThF.

[5]Shelah, Nice infinitary logics – J American Math Soc 25 (2012)

So far, stemming from classical to L^1_κ

- The classical connection between Robinson forcing and model companions
- The connection to Fraïssé / Hrushovski limits
- The long quest by Zilber for Structural Approximation
- Vaught / Harnik: Approximation and Preservation — and possibilities for current work in L^1_κ

Chapter 2

Generic types

Generic types in AECs

In this short section I describe some recent take on model theoretic forcing from the work of Vasey and Shelah on understanding behavior of classes of models under the assumption of being "oligomorphic" (few orbits on countable sets, $\aleph_0$-stable), and an interesting role for model theoretic forcing.

Infinitary Logic vs Reflection Systems

A playground to test Infinitary Logic - and at the same time to focus on Reflection Properties directly:
Abstract Elementary Classes

$$\mathcal{K}, \prec_{\mathcal{K}})$$

(closure under limits, Löwenheim-Skolem, coherence)

"Algebraically-minded model theory" - Really?

An early origin of Abstract Elementary Classes (complementary to the Categoricity problem) was Shelah's idea of (as expressed in his paper <u>The Lazy Model-Theoretician's Guide to Stability Theory</u> 1973)

"Algebraically-minded model theory" - Really?

An early origin of Abstract Elementary Classes (complementary to the Categoricity problem) was Shelah's idea of (as expressed in his paper The Lazy Model-Theoretician's Guide to Stability Theory 1973) speaking mainly to "those who are interested in **algebraically-minded model theory,** i.e., generic models, the class of e-closed models and universal-homogeneous models **rather than elementary classes** and saturated models. These were his words in 1975. He continues: "our main point is that though stability theory was developed for the latter context, almost everything goes through in the wider context (with suitable changes in the definitions)."

What goes through, really?

This declaration (the “almost everything goes through”) entailed more than it could seem at first sight: in many ways it is true but it took a long time to build up the right notions of stability, of types, of independence.

Smooth Reflection Classes

Replacing formulas by an abstract notion of “strong embedding” between L-structures is the first important point. In the definition of AECs we do **not** declare membership in the class by satisfying some sentence or some axiomatic system.

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The relation $\models$, basic in First Order logic, takes a back seat here, and the main relation $\leq_K$ (a generalization of the elementary submodel relation $\prec$ of first order) now leads the game.

LIMIT MODELS, ANOTHER KIND

In an aec $\mathcal{K}$, we say that M is brimmed (or limit) if M is of the form

$$M = \bigcup_{i<\delta} M_i$$

for some limit ordinal δ, where for each $i < \delta$, M_{i+1} is universal over M_i.
Of course, cofinality considerations are central. And interactions between countable models, models of size $\aleph_1$ and models of size $\aleph_2$ sa well.

The $\aleph_0$-stable, with few models

If $\mathcal{K}$ is an aec, stable in $\aleph_0$ and with only countably many countable models, model theoretic forcing can be useful[6].
Notice first:

[6] Shelah-Vasey, Abstract Elementary Classes Stable in $\aleph_0$, 2017, arxiv:1702.08281v1.

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Notice first:

- The set $\{M \in \mathcal{K} \mid M$ is countable and has domain $\subset \omega\}$ is Borel.
- If $\mathcal{K}$ has amalgamation in $\aleph_0$ then $\{(M, N) \mid M \prec_{\mathcal{K}} N$ and N has domain $\subset \omega\}$ is Σ^1_1.

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With these basic tools... the class is really $PC_{\aleph_0}$.

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The forcing

$$M \Vdash \varphi[\bar{a}]$$

- M is a countable model in the class
- $\varphi(\bar{x}) \in L_{\infty,\omega}(\tau_{\mathcal{K}})$
- $\bar{a} \in {}^{\omega}M$

the “usual” way: forcing negation at M means for every countable $N \succ_{\mathcal{K}} M$, N does not force... forcing existential means densely forcing the existence of witnesses...

Nice behavior of Generic Types

With these tools, Vasey and Shelah prove that forcing is well-behaved in the presence of amalgamation:
If M is a brimmed amalgamation base, then

$$M \Vdash \varphi[\bar{a}] \qquad \text{iff} \qquad M \models \varphi[\bar{a}]$$

And this implies that Galois types are really generic types (and many other model theoretic accounts of smooth behavior).
There is some remote connection with forcing axioms (PFA) and with interesting behavior at different ordinals, if one relaxes the strong definability conditions.

Chapter 3

Forcing on Sheaves/Topoi

A unified perspective

We have seen so far:

We now take a tour through forcing on sheaves, and illustrate how it generalizes the previous (and set-theoretic forcing), and provides a natural framework.

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- Model Theoretic Forcing and its Connection with Model Companions, Interpolation and Approximation Problems.

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A unified perspective

We have see so far:

- Model Theoretic Forcing and its Connection with Model Companions, Interpolation and Approximation Problems.
- The notion of genericity in Fraïssé/Hrushovski constructions and its connection with Model Companions (and thereby with Model Theoretic Forcing).

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- Model Theoretic Forcing and its Connection with Model Companions, Interpolation and Approximation Problems.
- The notion of genericity in Fraïssé/Hrushovski constructions and its connection with Model Companions (and thereby with Model Theoretic Forcing).
- A more contemporary take on model theoretic forcing, in "Reflection Classes", allowing to capture abstract forms of homogeneity through limit (brimmed) models.

We now take a tour through forcing on sheaves, and illustrate how it generalizes the previous (and set-theoretic forcing), and provides a natural framework.

Extended Objects / Variable Objects

Objects in the world present themselves as extended in time (or in other classical (or non-classical) "categories"):

- Physical objects, individuals, etc.

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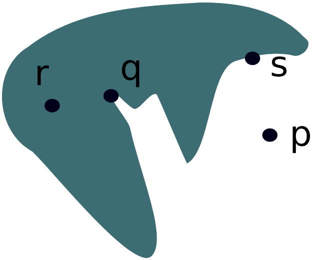
- Particles, even neutrinos (for some particles, order of 10^{-20} seconds, yet still "time")
- Concepts? Thoughts? Ideas? Visualizations? Perceptions?

Leibniz, Peirce, Husserl, etc.

Yet logic (at the limit) is "too rough"

(Really, classical logic.)



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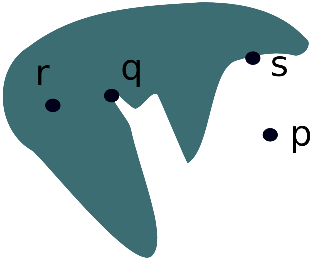

- For p and r the predicate "is in the green zone" is clear - classical logic "agrees" with perception.
- For q and s (at "limit situations") classical logic forces one to make a decision (open, closed green zone, etc.).

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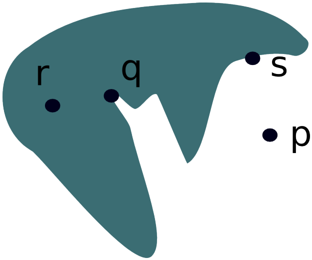

- Perception does not follow classical logic.

Physics, geometry, and "limit" phenomena

As we know since the late 1920's, Physics (wave models, quantum phenomena of "undecidability" or "uncertainty", noncommutativity of operators corresponding to formalizations of observability, etc.) has the kind of "limit phenomena" that may call for a logic of variable entities.

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Algebraic geometry of the postwar period (Leray, Cartan, Weil, and then Grothendieck reflects this same "shift of perspective": sheaves, sites, topoi.)

Instant velocity / Paradigm change

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Excluded middle may be dropped!
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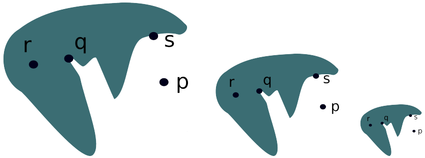
Excluded middle ~~may~~ HAS to be dropped!
The strong paradigm becomes Truth Continuity.

Truth Continuity

If an individual (an entity, a particle, etc.) has some property on some point of its domain of extension, there has to be a neighborhood of this point in this domain in which this property holds of all points.



Sheaves over topological spaces

Fix X a topological space. The pair (E, p) is a sheaf over X if and only if E is a topological space and $p : E \to X$ is a surjective local homeomorphism.

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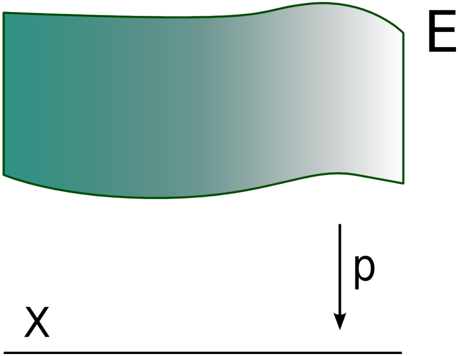

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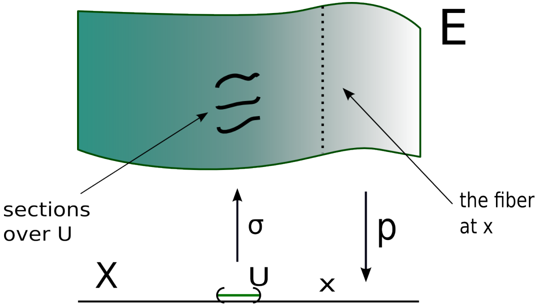
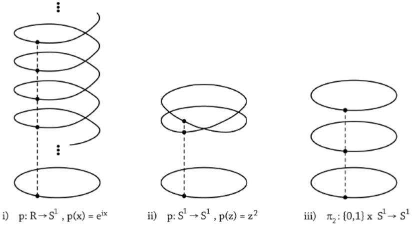
- The (images of) sections σ form a basis for the topology of E (a section is a continuous partial inverse of p defined on an open set $U \subset X$),
- If two sections σ, τ coincide at a point a then there exists an open set $U \ni a$ such that $\sigma \restriction U = \tau \restriction U$

Sections - objects



Classical and not[7]



Cildo Meireles - Fontes

[7] Caicedo: Lógica de los Haces de Estructuras - Revista Academia Colombiana de Ciencias, 1995

A little history

Sheaves over topological spaces go back to **H. Weyl** (1913), in his work on Riemann surfaces.
They "reappear" strongly in **Cartan**'s seminar (1948-1952) and then catch flight with the French Algebraic Geometry School of the Postwar (**Serre**, **Leray**, etc.).
Weil: Séminaire de géométrie algébrique: study of the zeta function on finite fields.
Finally, **Grothendieck** generalizes further the frame (to sites = small categories endowed with "Grothendieck topologies").
Deligne then proves Weil's conjectures.

Sheaves of Structures

A sheaf of structures $\mathfrak{A}$ over X consists of:

1. A sheaf (E, p) over X,
2. On every fiber $p^{-1}(a)$ ($a \in X$), a structure

$$\mathfrak{A}_a = (E_a, (R_i^a)_i, (f_j^a)_j, (c_k^a)_k,)$$

such that $E_a = p^{-1}(a)$, and

- For every i, $R_i^{\mathfrak{A}} = \bigcup_{x \in X} R_i^{\mathfrak{A}_x}$ is open
- For every j, $f_j^{\mathfrak{A}} = \bigcup_{x \in X} f_j^{\mathfrak{A}_x}$ is continuous
- For every k, $c_k^{\mathfrak{A}} : X \to E$ such that $x \mapsto c_k^{\mathfrak{A}_x}$ is a continuous global section

Truth Continuity?

Fact
For all atomic formulas $\varphi(v)$ we have that

$$\mathfrak{A}_x \models \varphi(\sigma(x)) \textit{ iff } \exists U \ni x \forall y \in U \Big(\mathfrak{A}_y \models \varphi(\sigma(y)) \Big)$$

This also holds for positive Boolean combinations of atomic formulas.
However, this fails for negations!

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This also holds for positive Boolean combinations of atomic formulas.

However, this fails for negations!

The solution to this failure is to switch to an emphasis on forcing.

Satisfaction and forcing (pointwise and local)

Three notions: satisfaction at each fiber, forcing at a point $x \in X$, forcing at a (non-empty) open set $U \subset X$:

Satisfaction and forcing (pointwise and local)

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How do we compare them? Before diving into the definitions of the forcing notions, notice that the first one is pointwise while the second one is local. Also notice that satisfaction in $\mathfrak{A}_x$ is about values of sections at x (the $\sigma(x)$) whereas pointwise (over x) or local forcing (over U) are about the whole section σ defined on U.

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Sections are the new objects: formulas $\varphi(v_1, v_2, \cdots)$ will be "evaluated" by "replacing" v_i by a section σ_i or by its value at an element x of X, $\sigma_i(x)$.

Pointwise forcing

- For atomic φ and $t_1, \cdots, t_n$ terms,
  $\mathfrak{A} \Vdash_x (t_1 = t_2)[\vec{\sigma}] \Leftrightarrow t_1^{\mathfrak{A}_x}[\vec{\sigma}(x)] = t_2^{\mathfrak{A}_x}[\vec{\sigma}(x)]$
  similarly for relation symbols.

Forcing $\neg, \rightarrow, \forall$ at x requires information "around" x. It is an exercise to check Truth Continuity for $\Vdash_x$.

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- $\mathfrak{A} \Vdash_x \exists v \varphi(v, \vec{\sigma}) \Leftrightarrow$ there exists some σ defined at x such that $\mathfrak{A} \Vdash_x \varphi[\sigma, \vec{\sigma}]$.
- $\mathfrak{A} \Vdash_x \forall v \varphi(v, \vec{\sigma}) \Leftrightarrow$ for some $U \ni x$, for <u>every</u> $y \in U$ and <u>every</u> σ defined on y, $\mathfrak{A} \Vdash_y \varphi[\sigma, \vec{\sigma}]$.

Forcing $\neg, \rightarrow, \forall$ at x requires information "around" x. It is an exercise to check Truth Continuity for $\Vdash_x$.

Truth continuity - II

A semantics can also be defined *directly* over open sets:

$$\mathfrak{A} \Vdash_U \varphi[\sigma],$$

where U is an open set in the domain of σ.

Definition
$\mathfrak{A} \Vdash_U \varphi[\sigma]$ if and only if for every $x \in U$, $\mathfrak{A} \Vdash_x \varphi[\sigma(x)]$.

Generic Filters

Definition

Given $\mathfrak{A}$ a sheaf of structures over X, a generic filter $\mathbb{F}$ for $\mathfrak{A}$ is a filter of open sets of X such that

- for every $\varphi(\sigma)$ and every σ defined on $U \in \mathbb{F}$, there is some $W \in \mathbb{F}$ such that $\mathfrak{A} \Vdash_W \varphi(\sigma)$ or $\mathfrak{A} \Vdash_W \neg\varphi(\sigma)$
- for every σ defined on $U \in \mathbb{F}$, for every $\varphi(u, \sigma)$, if $\mathfrak{A} \Vdash_U \exists u \varphi(u, \sigma)$, then there exists $W \in \mathbb{F}$ and μ defined on W such that $\mathfrak{A} \Vdash_W \varphi(\mu, \sigma)$

For some topological spaces, this definition of genericity of a filter may be made more purely topological/geometrical (and less dependent on formulas and forcing). However, in the general case, this is not necessarily possible - and we must rely on this logical definition.

Existence - generic models

Fact

Generic filters exist.

Definition (Generic Models)

Given a generic filter $\mathbb{F}$ and $\mathfrak{A}(U) = \{\sigma | \mathrm{dom}(\sigma) = U\}$, let

$$\mathfrak{A}[\mathbb{F}] = \lim_{U \in \mathbb{F}} \mathfrak{A}(U) = \bigsqcup_{U \in \mathbb{F}} \mathfrak{A}(U) / \sim_{\mathbb{F}}$$

where $\sigma \sim_{\mathbb{F}} \mu$ iff there exists $W \in \mathbb{F}$ such that $\sigma \restriction W = \mu \restriction W$.
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where $\sigma \sim_{\mathbb{F}} \mu$ iff there exists $W \in \mathbb{F}$ such that $\sigma \restriction W = \mu \restriction W$. Also,

- $(\sigma_1/\sim_{\mathbb{F}}, \ldots, \sigma_n/\sim_{\mathbb{F}}) \in R^{\mathfrak{A}[\mathbb{F}]} \Leftrightarrow \exists U \in \mathbb{F} (\sigma_1, \ldots, \sigma_n) \in R^{\mathfrak{A}(U)}$
- $f^{\mathfrak{A}[\mathbb{F}]}(\sigma_1/\sim_{\mathbb{F}}, \ldots, \sigma_n/\sim_{\mathbb{F}}) = f^{\mathfrak{A}(U)}(\sigma_1, \ldots, \sigma_n)/\sim_{\mathbb{F}}$

Limits

Theorem (A classical Generic Model Theorem)

Let $\mathbb{F}$ be a generic filter for a sheaf of topological structures $\mathfrak{A}$ over X. Then

$$\begin{aligned} \mathfrak{A}[\mathbb{F}] \models \varphi(\sigma/\sim_{\mathbb{F}}) &\iff \{x \in X | \mathfrak{A} \Vdash_x \varphi^G(\sigma(x))\} \in \mathbb{F} \\ &\iff \exists U \in \mathbb{F} \text{ such that } \mathfrak{A} \Vdash_U \varphi^G(\sigma). \end{aligned}$$

Here, φ^G is a formula equivalent classically to φ, but not necessarily in an intuitionistic framework! (The formula φ^G is sometimes called the Gödel translation of φ - in 1925, Kolmogorov had independently defined an equivalent translation.)

More on the Generic Model Theorem

Cohen's construction of generic models for set theory is the first published result along these lines. Later, Robinson, Barwise and Keisler used generic model theorems to get Omitting Types Theorems in various logics, generalized by Caicedo. Ellerman's "ultrastalk theorem" (1976) is a GMTh for maximal filters. Miraglia also proves a similar result for Heyting-valued models.

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$$\sigma \mapsto \sigma^* = \sigma \cup \{(\infty, [\sigma]_{\sim_{\mathbb{F}}})\}.$$

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Then, the GMTh just means that in the new sheaf $\mathfrak{A}^\infty$ this fiber is classic:

$$\mathfrak{A}^\infty \Vdash_\infty \varphi(\sigma_1^*, \cdots, \sigma_n^*) \Leftrightarrow \mathfrak{A}[\mathbb{F}] \models \varphi([\sigma_1^*], \cdots, [\sigma_n^*])$$

The fiber "at infinity"

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Other applications of the GMTh

- Kripke models - generalized semantics
- Set-theoretic forcing
- Robinson's Joint Consistency Theorem (=Amalgamation over Models)
- Various Omitting Types Theorems (Caicedo, Brunner-Miraglia)
- Control over new kinds of limit models

Sheaves of Hilbert Spaces

Why?

(Geraldo Barros)

1. Hilbert Spaces are (still) a crucial tool for formalization of concepts and objects in Physics and in Chemistry
2. In Physics: really algebras of operators acting on Hilbert spaces.
3. In Chemistry: really predicates on Hilbert spaces.
4. In both, the dynamical properties of evolution of a system are relevant.

The problem of a model theory for Hilbert Spaces

So, we want to be able to put Hilbert spaces (and more structure on top of them, such as predicates for reactions, or operators for observables) on fibers.
We could in principle do that as we have seen so far, but immediately we get the problem that we may get lots of non-standard Hilbert spaces (infinitesimals, etc.).
Moreover, we want the logic to “keep track” of (say) the distance to a projection $p(v)$, the convergence of a sequence in H, isometric isomorphism, $(1+\varepsilon)$-isomorphism, etc. etc.
Finally, we need to be able to take limits of Cauchy sequences at will in our structures: metric completeness is crucial.
That is the rôle of Continuous Model Theory.

Sheaves of Metric Structures

A sheaf of metric structures $\mathfrak{A}$ over X consists of:

1. A sheaf (E, p) over X,
2. On every fiber $p^{-1}(x)$ $(x \in X)$, a metric structure

$$(\mathfrak{A}_x, d_x) = (E_x, (R_i^x)_i, (f_j^x)_j, (c_k^x)_k, d_x, [0, 1])$$

such that $E_x = p^{-1}(x)$, (E_x, d_x) is a complete bounded metric space of diameter 1, and

- For every i, $R_i^{\mathfrak{A}} = \bigcup_{x \in X} R_i^x$ is open
- For every j, $f_j^{\mathfrak{A}} = \bigcup_{x \in X} f_j^x$ is continuous
- For every k, $c_k^{\mathfrak{A}} : X \to E$ such that $x \mapsto c_k^x$ is a continuous global section

(further requirements on moduli of uniform continuity)

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- For every k, $c_k^{\mathfrak{A}} : X \to E$ such that $x \mapsto c_k^x$ is a continuous global section
- The premetric $d^{\mathfrak{A}} := \bigcup_{x \in X} d_x : \bigcup_{x \in X} E_x^2 \to [0, 1]$ is a continuous function.

(further requirements on moduli of uniform continuity)

Truth Continuity - adapted to metric

Truth Continuity is still the guiding paradigm. Remember in the "discrete" case, negation was the first stumbling block - the first place where forcing was needed in a non-trivial way.
Here, in "CFO" logic, the semantics is defined on conditions of the form

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$$\varphi(x) < \varepsilon, \varphi(x) \leq \varepsilon, \cdots$$

Negation in continuous, metric logic, is weak: the semantics really treats $\leq$ and $\geq$ as "negations" of each other...

Pointwise forcing

With M. Ochoa, we define $\mathfrak{A} \Vdash_x \varphi < \varepsilon$ and $\mathfrak{A} \Vdash_x \varphi > \varepsilon$, for $x \in X$:

- Atomic: $\mathfrak{A} \Vdash_x d(t_1, t_2) < \varepsilon \Leftrightarrow d_x(t_1^{\mathfrak{A}_x}, t_2^{\mathfrak{A}_x}) < \varepsilon$
  $\mathfrak{A} \Vdash_x d(t_1, t_2) > \varepsilon \Leftrightarrow d_x(t_1^{\mathfrak{A}_x}, t_2^{\mathfrak{A}_x}) > \varepsilon$
  $\mathfrak{A} \Vdash_x R(t_1, \cdots, t_n) < \varepsilon \Leftrightarrow R^{\mathfrak{A}_x}(t_1^{\mathfrak{A}_x}, t_2^{\mathfrak{A}_x}) < \varepsilon$
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- $\mathfrak{A} \Vdash_x \max(\varphi, \psi) < \varepsilon \Leftrightarrow \mathfrak{A} \Vdash_x \varphi < \varepsilon$ and $\mathfrak{A} \Vdash_x \psi < \varepsilon$. Sim. for $>$.
- $\mathfrak{A} \Vdash_x \min(\varphi, \psi) \Leftrightarrow \mathfrak{A} \Vdash_x \varphi$ or $\mathfrak{A} \Vdash_x \psi$. Sim. for $>$.

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- $\mathfrak{A} \Vdash_x 1 \dot{-} \varphi < \varepsilon \Leftrightarrow \mathfrak{A} \Vdash_x \varphi > 1 \dot{-} \varepsilon$. Sim. for $>$.
- $\mathfrak{A} \Vdash_x \varphi \dot{-} \psi < \varepsilon$ iff and only if one of the following holds:
  - $\mathfrak{A} \Vdash_x \varphi < \psi$
  - $\mathfrak{A} \nVdash_x \varphi < \psi$ and $\mathfrak{A} \nVdash_x \varphi > \psi$
  - $\mathfrak{A} \Vdash_x \varphi > \psi$ and $\mathfrak{A} \Vdash_x \varphi < \psi + \epsilon$.
- $\mathfrak{A} \Vdash_x \varphi \dot{-} \psi > \varepsilon$ iff $\mathfrak{A} \Vdash_x \varphi > \psi + \varepsilon$
- $\cdots$

Pointwise forcing - continued

Quantifiers:

- $\mathfrak{A} \Vdash_x \inf_{s \in A_x} \varphi(s) < \varepsilon$ iff there exists a section σ such that $\mathfrak{A} \Vdash_x \varphi(\sigma) < \varepsilon$.

Pointwise forcing - continued

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- $\mathfrak{A} \Vdash_x \sup_s \varphi(s) < \varepsilon$ iff there exists an open set $U \ni x$ and a real number $\delta_x > 0$ such that for every $y \in U$ and every section σ defined on y, $\mathfrak{A} \Vdash_y \varphi(\sigma) < \varepsilon - \delta_x$
- $\mathfrak{A} \Vdash_x \inf_{s \in A_x} \varphi(s) > \varepsilon$ iff there exists a section σ such that $\mathfrak{A} \Vdash_x \varphi(\sigma) > \varepsilon$.

A metric on sections? (Not yet)

So far so good, but we have (for the time being) lost the metric on the sections (so, the corresponding presheaves $\mathfrak{A}(U)$ are still missing the "metric" feature - they do not live in the correct category yet).

- Sections have different domains
- Triangle inequality is tricky
- Restrict to sections with domains in a filter of open sets
- But the ultralimit (even in that case) could fail to be complete!

Rather... a pseudometric

Fix F a filter of open sets of X. For all sections σ and μ with domain in F define

$$F_{\sigma\mu} = \{U \cap \mathrm{dom}(\sigma) \cap \mathrm{dom}(\mu) | U \in F\}.$$

Then the function

$$\rho_F(\sigma, \mu) = \inf_{U \in F_{\sigma\mu}} \sup_{x \in U} d_x(\sigma(x), \mu(x))$$

is a pseudometric on the set of sections with domain in F.
In some cases we may get completeness of the induced metric:

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In some cases we may get completeness of the induced metric:

Lemma (Ochoa, V.)

Let $\mathfrak{A}$ be a sheaf of metric structures defined over a regular topological space X. Let F be an ultrafilter of regular open sets. Then, the metric induced by ρ_F on $\mathfrak{A}[F]$ is complete.

Other solutions include just working with pseudometrics and give up completeness, or even working with more general frameworks.

Local Forcing for Metric Structures

Forcing over an open set is somewhat more tricky in this case. We have the following definition.

Definition

Let $\mathfrak{A}$ be a sheaf of metric structures defined on X, $\varepsilon > 0$, U open in X, $\sigma_1, \cdots, \sigma_n$ sections defined on U. Then

- $\mathfrak{A} \Vdash_U \varphi(\sigma) < \varepsilon \iff \exists \delta < \varepsilon \forall x \in U (\mathfrak{A} \Vdash_x \varphi(\sigma) < \delta)$
- $\mathfrak{A} \Vdash_U \varphi(\sigma) > \delta \iff \exists \varepsilon > \delta \forall x \in U (\mathfrak{A} \Vdash_x \varphi(\sigma))$

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Let $\mathfrak{A}$ be a sheaf of metric structures defined on X, $\varepsilon > 0$, U open in X, $\sigma_1, \cdots, \sigma_n$ sections defined on U. Then

- $\mathfrak{A} \Vdash_U \varphi(\sigma) < \varepsilon \iff \exists \delta < \varepsilon \forall x \in U(\mathfrak{A} \Vdash_x \varphi(\sigma) < \delta)$
- $\mathfrak{A} \Vdash_U \varphi(\sigma) > \delta \iff \exists \varepsilon > \delta \forall x \in U(\mathfrak{A} \Vdash_x \varphi(\sigma))$

There is an involved, equivalent, inductive definition. We also have $\mathfrak{A} \Vdash_U \inf_\sigma(1 \dot{-} \varphi(\sigma)) > 1 \dot{-} \varepsilon \iff \mathfrak{A} \Vdash_U \sup_U \varphi(\sigma) < \varepsilon$, and a maximal principal principle (existence of witnesses of sections).

Metric Generic Model / Forcing Theorem

For the appropriate notion of genericity, we build the generic model as in the discrete case. The definition of genericity guarantees the completeness of $\mathfrak{A}[\mathsf{F}]$.

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Theorem (Metric GMTh)

Let F *be a generic filter on* X, $\mathfrak{A}$ *a sheaf of metric structures on* X *and* $\sigma_1, \cdots, \sigma_n$ *sections. Then*

1. $\mathfrak{A}[F] \models \varphi([\sigma_1]/\sim_F, \cdots, [\sigma_n]/\sim_F) < \varepsilon \iff \exists U \in F$ *such that* $\mathfrak{A} \Vdash_U \varphi(\sigma_1, \cdots, \sigma_n) < \varepsilon$
2. $\mathfrak{A}[F] \models \varphi([\sigma_1]/\sim_F, \cdots, [\sigma_n]/\sim_F) > \varepsilon \iff \exists U \in F$ *such that* $\mathfrak{A} \Vdash_U \varphi(\sigma_1, \cdots, \sigma_n) > \varepsilon$

A Metric Sheaf for noncommuting observables with continuous spectra

Really, a metric sheaf space for a free particle:

Definition

The triple $\mathfrak{A}_{cont} = (E, X, \pi)$ where

- $X = \mathbb{R}^+$ is the base space with the product topology.
- For $\tau \in X$ we let E_τ be a two sorted metric model where
  - $\mathcal{U}_\tau$ and $\mathcal{V}_\tau$ span the universe for each sort.
  - Every sort has is a metric space with the metric induced by the norm in $\mathcal{L}^2(\mathbb{R})$.
  - Every sort is a model in the language of a vector space, with symbols for the binary transformation $\langle , \rangle_{\mathcal{V}}$ and $\langle , \rangle_{\mathcal{U}}$, to be interpreted such that

A Metric Sheaf for noncommuting observables with continuous spectra

- $$\langle q(x_0 - x)\phi_{(\tau,t_1)}(x_0 - x), r(x_1 - x)\phi_{(\tau,t_1)}(x_1 - x)\rangle_{\mathcal{U}} = q(x_0 - x_1)r(x_0 - x_1)\phi_{(\tau,t_1+t_2)}(x_0 - x_1) \tag{1}$$
  $$\langle q(p_0 - p)\phi_{1/(\tau,t_1)}(p_0 - p), r(p_1 - p)\phi_{1/(\tau,t_1)}(p_1 - p)\rangle_{\mathcal{V}} = q(p_0 - p_1)r(p_0 - p_1)\phi_{1/(\tau,t_1+t_2)}(p_0 - p_1) \tag{2}$$
- function symbols for FT and FT^{-1} (Fourier transforms between the operators)
- The sheaf is constructed as the disjoint union of fibers: $E = \sqcup_{\tau \in X} E_\tau$
- Sections are defined such that if $\tau \in U \subset X$,

  $\sigma_{q,x_0,p_0,t}(\tau) = \left(q(x - x_0\phi_{(\tau,t)}(x, x_0)\,,\ q(p - p_0)\phi_{1/(\tau,t)}(p, p_0)\right).$
- π, the local homeomorphism, is given by $\pi(\psi) = \tau$ if $\psi \in E_\tau$.

Remarks

- The binary transformations $\langle , \rangle_{\mathcal{U}}$ and $\langle , \rangle_{\mathcal{V}}$ are not the objects usually defined as the inner product in a Hilbert space. Instead, they are our representation for the physical inner product as defined by Dirac in each sort.
- We are interested in two kinds of generic metric models:
  1. In the first kind we look at generic models that capture the limit of vanishing τ, for which we take the nonprincipal ultrafilter induced by the family of open regular sets $\{(0, 1/n) : n \in \mathbb{N}\}$. From the structure of the sheaf defined above, limit elements in the generic model coming from the $\mathcal{U}$ sort with $t = 0$ must approach Dirac's delta in position.
  2. On the other hand, the generic metric model we obtain by taking the nonprincipal ultrafilter induced by the family of open regular sets $\{(n, \infty) : n \in \mathbb{N}\}$ must contain limit elements that represent Dirac's distributions in momentum space.

Whence all this?

- Laurent Schwartz's work on distributions
- Schwartz spaces for position and momentum operators

Back to the Schwartz Space - and to the Sheaf construction

One motivation: Dirac's distribution in $\mathcal{L}^2(\mathbb{R})$:

$$\lim_{\tau\to 0} \frac{1}{\tau\sqrt{\pi}} e^{-x^2/\tau^2} = \delta(x) \tag{3}$$

(with the limit taken in the sense of distributions). This suggests that an *imperfect*[8] representation $\phi_\tau(x, x_0)$ for the physical vector state $|x_0\rangle$ in $\mathcal{L}^2(\mathbb{R})$ is

$$\phi_\tau(x, x_0) = \frac{1}{\tau\sqrt{2\pi\hbar}} e^{-(x-x_0)^2/2\hbar\tau^2}. \tag{4}$$

The family of elements $\{\phi_\tau(x, x_0)\}$ is a subset of the Schwartz space and, with the inner product in $\mathcal{L}^2(\mathbb{R})$, we find that

$$\langle \phi_\tau(x, x_0), \phi_\tau(x, x_1)\rangle = \int_{-\infty}^{\infty} dx \phi_\tau(x, x_0)\phi_\tau(x, x_1) = \phi_{\sqrt{2}\tau}(x_1, x_0). \tag{5}$$

[8] In the sense of "up to τ"

Imperfect propagator at the fiber E_τ

After many calculations we get an ugly expression for the imperfect propagator at the fiber E_τ:

$$
\begin{aligned}
\langle x_1, U(t)x_0\rangle &= \langle \phi_\tau(x, x_1), \phi_{(\tau, it/m)}(x, x_0)\rangle_{\mathcal{U}} && (6)\\
&= \phi_{(\tau, it/m)}(x_1, x_0) && (7)\\
&= \frac{1}{\sqrt{2\pi(\tau^2 + it/m)}} e^{-(x_1 - x_0)^2/2\hbar(\tau^2 + it/m)} && (8)
\end{aligned}
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Letting $\tau \to 0$, we recover the exact form for the quantum mechanical amplitude; (with any nonprincipal ultrafilter induced by the family of open regular sets $\{(0, 1/n) : n \in \mathbb{N}\}$). **Thus in the Generic model $\mathfrak{A}[\mathbb{F}]$ we recover the exact propagator as a limit element.**

Conclusions

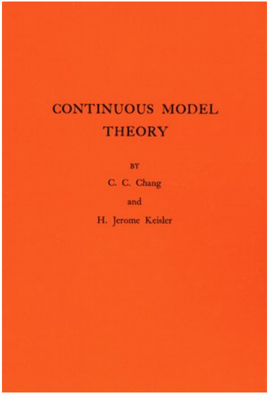
- The classical connection between Robinson forcing and model companions... and Fraïssé / Hrushovski limits
- The long quest by Zilber for Structural Approximation
- Vaught / Harnik: Approximation and Preservation — and possibilities for current work in L^1_κ
- Shelah-Vasey shed light on AECs but also possibly on forcing axioms
- Sheaf forcing seems to unify in a different way (and responds to Zilber's questions)

Of course... Muito obrigado!

Chapter 4

Continuous Model Theory

Continuous Model Theory - Origins



Although the origins of CMTh go back to Chang & Keisler (1966), and in some (restricted) ways to von Neumann's Continuous Geometryrecent takes on Continuous Model Theory are based on formulations due to Ben Yaacov, Usvyatsov and Berenstein of Henson and Iovino's Logic for Banach Spaces.

Continuous predicates and functions

Definition

Fix (M, d) a bounded metric space. A continuous n-ary predicate is a uniformly continuous function

$$P : M^n \to [0, 1].$$

A continuous n-ary function is a uniformly continuous function

$$f : M^n \to M.$$

Metric structures

Therefore, metric structures are of the form

$$\mathcal{M} = \Big(M, d, (f_i)_{i\in I}, (R_j)_{j\in J}, (a_k)_{k\in K}\Big)$$

Each function, relation must be endowed with a modulus of uniform continuity.

Metric structures

Therefore, metric structures are of the form

$$\mathcal{M} = \Big(M, d, (f_i)_{i \in I}, (R_j)_{j \in J}, (a_k)_{k \in K}\Big)$$

where the R_i and the f_j are (uniformly) continuous functions with values in $[0, 1]$, the a_k are distinguished elements of M.
Remember: M is a bounded metric space.
Each function, relation must be endowed with a modulus of uniform continuity.

Examples of FO metric structures

Example

- Any FO structure, endowed with the discrete metric.

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- Any FO structure, endowed with the discrete metric.
- Banach algebras (bounding them).
- Hilbert spaces with inner product as a binary predicate.
- For a probability space $(\Omega, \mathcal{B}, \mu)$, construct a metric structure $\mathcal{M}$ based on the usual measure algebra of $(\Omega, \mathcal{B}, \mu)$.
- Representations of C^*-algebras (Argoty, Berenstein, Ben Yaacov, V.).
- Valued fields.

The syntax

1. Terms: as usual.
2. Atomic formulas: $d(t_1, t_n)$ and $R(t_1, \cdots, t_n)$, if the t_i are terms. Formulas are then interpreted as functions into $[0, 1]$.
3. Connectives: continuous functions from $[0, 1]^n \to [0, 1]$. Therefore, applying connectives to formulas gives new formulas.
4. Quantifiers: $\sup_x \varphi(x)$ (universal) and $\inf_x \varphi(x)$ (existential).

Interpretation

The logical distance between $\varphi(x)$ and $\psi(x)$ is $\sup_{a \in M} |\varphi^M(a) - \psi^M(a)|$.

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Conditions are expressions of the form $\varphi(x)\leq\psi(y)$, $\varphi(x)\leq\psi(y)$, $\varphi(x)\geq\psi(y)$, etc.
Notice also that the set of connectives is too large, but it may be "densely" and uniformly generated by $0, 1, x/2, \dot{-}$: for every ε, for every connective $f(t_1,\cdots,t_n)$ there exists a connective $g(t_1,\cdots,t_n)$ generated by these four by composition such that $|f(\vec{t})-g(\vec{t})|<\varepsilon$.

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Stability Theory

- Stability (Ben Yaacov, Iovino, etc.),
- Categoricity for countable languages (Ben Yaacov),
- ω-stability,
- Dependent theories (Ben Yaacov),
- Not much geometric stability theory: no analog to Baldwin-Lachlan (no minimality, except some openings by Usvyatsov and Shelah in the context of $\aleph_1$-categorical Banach spaces),
- NO simplicity!!! (Berenstein, Hyttinen, V.),
- Keisler measures, NIP (Hrushovski, Pillay, etc.).

"Continuous Model Theory" beyond First Order

Several contexts, some unexplored so far.

1. Metric Abstract Elementary Classes (Hirvonen, Hyttinen - ω-stability, V. Zambrano - superstability, domination, notions of independence): an amalgam of the power of Abstract Elementary Classes with metric ideas.

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2. Continuous $L_{\omega_1\omega}$. So far, no published results as such. There are however “Lindström theorems” for Continuous First Order due to Caicedo and Iovino.
3. Sheaves of (metric) structures. Our work with Ochoa, motivated by problems originally in Chemistry. Back to main.