



# Infinitary logic, large cardinals and AECs: some reflections

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Reflections on Set Theoretical Reflection  
Bagaria 60 - Montseny - Catalonia - November 2018

# CONTENTS

A Catalan Prelude

Reflection Classes (Beyond Syntax...)

Back to syntax!

Back to syntax, really?

## *Part 1*

# A Catalan Prelude

# A LULLIAN PRELUDE - LOGIC IN CATALONIA IN THE 13TH CENTURY

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In the 13th Century: Llull.

# LLULL: LOGIC IN CATALONIA IN THE 13TH CENTURY



from Atlas Català

# LLULL: LOGIC IN CATALONIA IN THE 13TH CENTURY



(detail)



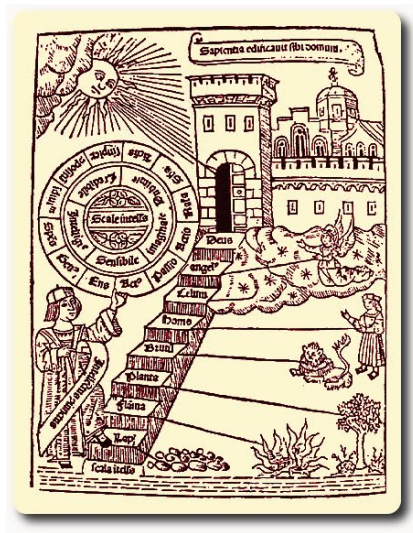
# LLULL: EPISTEMOLOGY AND ONTOLOGY OF THE UNIVERSE

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- ▶ Descriptions of the world through strong images:
- ▶ Trees (of Science), Spheres (of Predicates) and

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- ▶ Synthesis of the three cultures thriving (Sefarad, Al-Andalus, Hispania)
- ▶ Descriptions of the world through strong images:
- ▶ Trees (of Science), Spheres (of Predicates) and
- ▶ Reflection of Imago Dei through Imago Mundi,
- ▶ The universe as a system of categories reflecting one another but
- ▶ anchored in different “models” reflecting one another.  
(Superposition of planes, all of them reflecting the original, “divine” plane...)

# RAMON LLULL - A SCALE



# REFLECTIONS / ABOVE REFLECTING BELOW / SIMILITUDE

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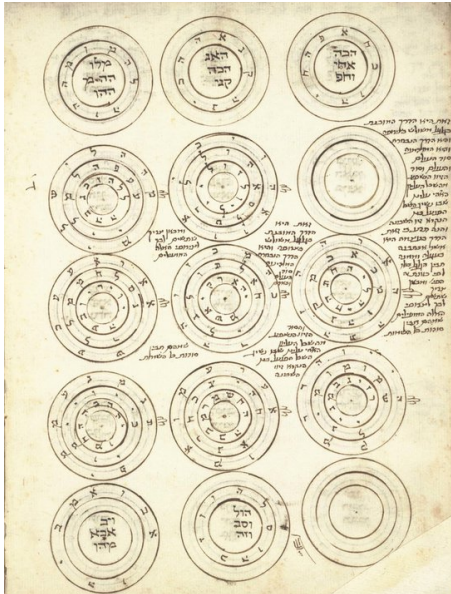
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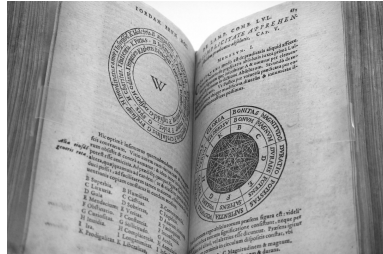
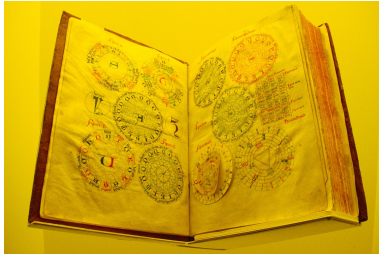
But let us listen directly to Lull:

- ▶ Entre semblança i semblança ha disposició e fi e proporció e concordança... (car) totes semblances en cors sustentades són de una comuna semblança.
- ▶ ... ciències esteses en moltes veritats, ço és saber, en lurs semblances (...); el seu encercament està en pujant o en davallant de les coses **dejús** a les **dessús** e de les **dessús** a les **dejús**, e en los efectus d'aquelles han d'elles coneixença...

# NEW LOGICS? - ABULAFIA / LLULL



# LLULL





## *Part 2*

# Reflection Classes (Beyond Syntax...)

# ORIGINS: INFINITARY LOGIC

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More generally, what is the behavior of the function

$I(\psi, \lambda) := |\{M \models \psi \mid |M| = \lambda\}| / \approx$ , for a sentence  $\psi$  of the logic  $L_{\omega_1, \omega}$ ?

# LONG STORY SHORT

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The attempts:

- ▶ (Keisler) Use “sequentially homogeneous” models. But sequential homogeneity is a consequence of categoricity...
- ▶ (Shelah) The role of models of size  $\aleph_n$  ( $n < \omega$ ) in the decomposition of large models, the role of dimension-like obstructions.
- ▶ (Shelah) Forcing-like approach to types that would eventually become “Galois types”.

# “ALGEBRAICALLY-MINDED MODEL THEORY” - REALLY?

Another early origin of Abstract Elementary Classes, complementary to the Categoricity problem, was Shelah's idea of (as expressed in his paper The Lazy Model-Theoretician's Guide to Stability Theory 1973)

# “ALGEBRAICALLY-MINDED MODEL THEORY” - REALLY?

Another early origin of Abstract Elementary Classes, complementary to the Categoricity problem, was Shelah's idea of (as expressed in his paper The Lazy Model-Theoretician's Guide to Stability Theory 1973) speaking mainly to “those who are interested in **algebraically-minded model theory**, i.e., generic models, the class of e-closed models and universal-homogeneous models **rather than elementary classes** and saturated models. These were his words in 1975. He continues: “our main point is that though stability theory was developed for the latter context, almost everything goes through in the wider context (with suitable changes in the definitions).”



# WHAT GOES THROUGH, REALLY?

This declaration (the “almost everything goes through”) entailed more than it could seem at first sight: in many ways it is true but it took a long time to build up the right notions of stability, of types, of independence.

# SMOOTH REFLECTION CLASSES

Replacing formulas by an abstract notion of “strong embedding” between  $L$ -structures is the first important point. In the definition of AECs we do **not** declare membership in the class by satisfying some sentence or some axiomatic system.

# SMOOTH REFLECTION CLASSES

Replacing formulas by an abstract notion of “strong embedding” between  $L$ -structures is the first important point. In the definition of AECs we do **not** declare membership in the class by satisfying some sentence or some axiomatic system.

The relation  $\models$ , basic in First Order logic, takes a back seat here, and the main relation  $\leq_K$  (a generalization of the elementary submodel relation  $\prec$  of first order) now leads the game.

## FORMALISM FREENESS?

All of this approach very much goes in line with other situations in mathematics where versions of “Formalism Freeness” (Kennedy) take up center stage. One of them is computability (Turing, Post, Gödel, Kleene, Church). Another one is Model Theory as a “generalized Galois theory”, as happens in AECs.

Kennedy, Magidor, Väänänen: Inner Models for Different Logics. Yet another take on formalism freeness, dual to Abstract Elementary Classes.

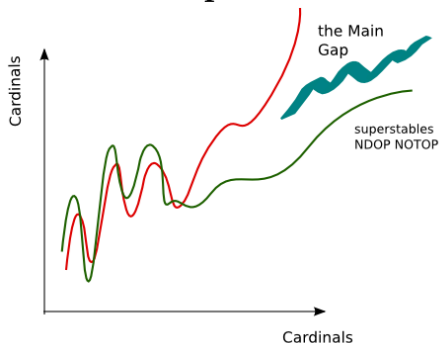
# THE DEFINITION [ABSTRACT ELEMENTARY CLASS]

Fix a language  $L$ . A class  $\mathcal{K}$  of  $L$ -structures, together with a binary relation  $\leq_K$  on  $\mathcal{K}$  is an abstract elementary class (for short, AEC) if:

1. Both  $\mathcal{K}$  and  $\leq_K$  are closed under isomorphism. This means two things: first, if  $M' \approx M \in \mathcal{K}$  then  $M' \in \mathcal{K}$ ; second, if  $M', N'$  are  $L$ -structures with  $M' \subset N'$ ,  $M' \approx M$ ,  $N' \approx N$  and  $M \leq_K N$  then  $M' \leq_K N'$ .
2. If  $M, N \in \mathcal{K}$ ,  $M \leq_K N$  then  $M \subset N$ ,
3.  $\leq_K$  is a partial order,
4. **(Coherence)** If  $M \subset N \leq_K N'$  and  $M \leq_K N'$  then  $M \leq_K N$ ,
5. **(LS)** There is a cardinal (called “the Löwenheim-Skolem number” of the class)  $\kappa = LS(\mathcal{K}) \geq \aleph_0$  such that if  $M \in \mathcal{K}$  and  $A \subset |M|$ , then there is  $N \leq_K M$  with  $A \subset |N|$  and  $|N| \leq |A| + LS(\mathcal{K})$ ,
6. **(Unions of  $\leq_K$ -chains)** If  $(M_i)_{i < \delta}$  is a  $\leq_K$ -increasing chain of length  $\delta$  ( $\delta$  a limit ordinal), then
  - ▶  $\bigcup_{i < \delta} (M_i)_{i < \delta} \in \mathcal{K}$ ,
  - ▶ for each  $j < \delta$ ,  $M_j \leq_K \bigcup_{i < \delta} M_i$ ,
  - ▶ if for each  $i < \delta$ ,  $M_i \leq_K N \in \mathcal{K}$  then  $\bigcup_{i < \delta} M_i \leq_K N$ .

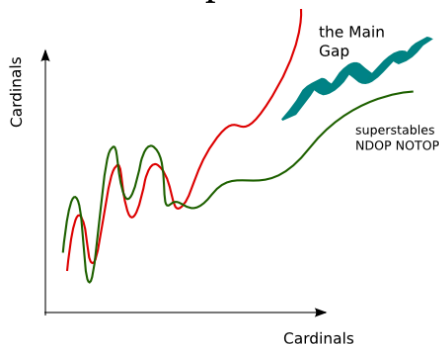
# MAIN CONJECTURE: THE MAIN GAP

## The Main Gap Theorem for FO logic



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## The Main Gap Theorem for FO logic



The “gold standard” of mathematical logic, of model theory, in various ways, and the main conjecture in AECs.

At this point, we have the following situation:

- ▶ So far, no control on possible axiomatization of the class  $\mathcal{K}$ . The emphasis is placed on its being closed under the constructions specified in the axioms. However, later (in subsection) we focus on the logical control of these classes. Remember Shelah's "algebraically-minded model theory".
- ▶ These are not necessarily amalgamation classes: there is no amalgamation axiom. However, many AECs do satisfy the amalgamation property. Furthermore, the model theory will depend on the kind of amalgamation possible in the class.



## *Part 3*

Back to syntax!

# HOW TO DEAL WITH THESE AECs?

## Theorem (Presentation Theorem, Shelah)

*Let  $(\mathcal{K}, \leq_K)$  be an AEC in a language  $L$ . Then there exist*

- ▶ *A language  $L' \supset L$ , with size  $LS(\mathcal{K})$ ,*
- ▶ *A (first order) theory  $T'$  in  $L'$  and*
- ▶ *A set of  $T'$ -types,  $\Gamma'$ , such that*

$$\mathcal{K} = PC(L, T', \Gamma') := \{M' \restriction L \mid M' \models T', M' \text{ omits } \Gamma'\}.$$

*Moreover, if  $M', N' \models T'$ , they both omit  $\Gamma'$ ,  $M = M' \restriction L$  and  $N = N' \restriction L$ ,*

$$M' \subset N' \Leftrightarrow M \leq_K N.$$

## Corollary (“Hanf” number of an AEC)

*If an AEC  $\mathcal{K}$  has a model of cardinality  $\geq \beth_{(2^{LS(\mathcal{K})})^+}$  then it has arbitrarily large models.*

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Proof: Use the Hanf number for PC classes (and the undefinability of well-ordering). □

## Theorem (Shelah)

*Let  $(\mathcal{K}, \leq_K)$  be an AEC with amalgamation and arbitrarily large models. If  $\mathcal{K}$  is categorical in  $\lambda > LS(\mathcal{K})$  then it is  $\mu$ -galois-stable for each cardinal  $\mu \in [LS(\mathcal{K}), \lambda)$ .*

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And many other results on Stability Theory for a.e.c.’s really hinge on this “syntactic embedding” of EM models coming from First Order logic! The proof hinges on Ehrenfeucht-Mostowski models (whose existence in AECs with large enough models is given by the Presentation Theorem).

# BEYOND THE PRESENTATION THEOREM

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# THE LOGIC $L^1_\kappa$

A logic called  $L^1_\kappa$  (Shelah, 2007), in between  $L_{\kappa,\omega}$  and  $L_{\kappa,\kappa}$  (for  $\kappa$  singular strong limit):

$$L_{\kappa,\omega} \subset L^1_\kappa \subset L_{\kappa,\kappa}$$

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that has **many** desirable properties:

- ▶ **Undefinability** of well-order (very weak compactness)
- ▶ **Interpolation** (“balancing” the interpolation problem between  $L_{\kappa,\omega}$  and  $L_{\kappa,\kappa}$ ):  
if  $\phi \rightarrow \psi \in L_{\kappa,\omega}$ ,  $\phi$  has vocabulary  $L_1$ ,  $\psi$  has vocabulary  $L_2$  then there is  $\theta$  in the common vocabulary  $L_1 \cap L_2$  such that  $\phi \vdash \theta \vdash \psi$ ... BUT  $\theta \in L_{\kappa,\kappa}$ .
- ▶ Downward Löwenheim-Skolem
- ▶ **Maximality** for the previous properties (“Lindström”): any logic above  $L_{\kappa,\omega}$  satisfying undefinability of well-order, occurrence below  $\kappa$  (for  $\kappa = \beth_\kappa$  strong limit) interpolation and LS must be  $\leq L_\kappa^1$ .

# THE CONNECTION WITH A.E.C.'S

(Work in progress, with Shelah)

For any a.e.c.  $\mathcal{K}$  with  $\tau = \tau_{\mathcal{K}}$ ,  $\kappa = LST_{\mathcal{K}}$ ,  $\lambda = \beth_2(\kappa + |\tau|)^+$  there exists  $\psi_{\mathcal{K}} \in L_{\lambda^+, \kappa^+}(\tau)$  such that  $\mathcal{K} = Mod(\psi_{\mathcal{K}})$ .

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Moreover,

$$\psi_{\mathcal{K}} \in L^1_{\kappa^*}, \quad \prec_{\mathcal{K}} \approx \prec_{L^1_{\kappa^*}}$$

## *Part 4*

Back to syntax, really?

# REALLY, BACK TO SYNTAX???

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# REALLY, BACK TO SYNTAX???

There are many issues related to  $L_{\kappa}^1$ :

- ▶ No actual definition of the syntax
- ▶ No “Consistency Properties” attached to the logic
- ▶ Only partial understanding of its expressive power

# ON THE “DELAYED” EHRENFUCHT-FRAÏSSÉ GAME

The syntax is really defined in terms of an Ehrenfeucht-Fraïssé “partial equivalence” game  $\mathfrak{D}_{\Gamma, \theta, \alpha}(M_1, M_2)$ :

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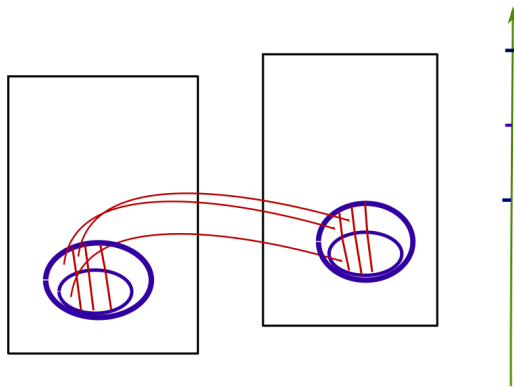
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- ▶ Player II breaks the sequence into  $\omega$  parts and chooses a sequence in  $M_2$ ,
- ▶ Player I acts following the challenge from the breakup, on the FIRST piece and plays another sequence,
- ▶ Player II acts following the challenge from the breakup, on the SECOND piece and plays another sequence,

# INFINITE DEBTS, FINITE TIME

This game has been described in terms of “Infinite debts, finite time to pay them off”.



# INFINITE DEBTS, FINITE TIME

This game has been described in terms of “Infinite debts, finite time to pay them off”. The point:

Playing the game, I “opens up” space for possible answers - possible functions - and “simulates” the role of the expansion by predicates from the Presentation Theorem.

Mysteries:

- ▶ a strong syntax for  $L_{\kappa}^1$ ,
- ▶ info from a.e.c.’s?

## EXAMPLES OF SENTENCES IN $L_{\lambda}^1$

- Let  $\text{cf } \lambda > \omega$ ,  $\phi_n(x_1, \dots, x_n) \in L_{\lambda}^1$  for all  $n < \omega$ . Then

$$\exists (x_{\alpha})_{\alpha < \lambda} \bigwedge_{\alpha_1, \dots, \alpha_n < \lambda} \phi_n(x_{\alpha_1}, \dots, x_{\alpha_n})$$

is in  $L_{\lambda}^1$ .

- So, the sentence  $\exists (x_{\alpha})_{\alpha < \omega_1} \bigwedge_{\alpha < \beta < \omega_1} (x_{\alpha} < x_{\beta})$  is in  $L_{\omega_2}^1$ .
- (Väänänen) Under GCH, there is a sentence of  $L_{\aleph_{\omega_2}}^1$  which has a model of cardinality  $\aleph_{\alpha}$  for  $\alpha < \aleph_2$  if and only if  $\text{cf } (\aleph_{\alpha}) = \omega_1$ .



# A “LINDSTRÖM THEOREM” FOR $L^1_\kappa$

Using (Strong) Undefinability of Well Ordering

Theorem (Shelah’s “Lindström theorem” for  $L^1_\kappa$ )

*Suppose  $\kappa = \beth_\kappa$ . Then  $L^1_\kappa$  is the unique logic  $\mathcal{L}$  such that:*

- ▶  $\mathcal{L} \subset H(\kappa)$
- ▶  $\mathcal{L}$  has the  $< \kappa$ -occurrence property
- ▶  $L_{\kappa\omega} \leq \mathcal{L}$
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- ▶  $\mathcal{L}$  has the  $< \kappa$ -occurrence property
- ▶  $L_{\kappa\omega} \leq \mathcal{L}$
- ▶  $\mathcal{L}$  has strong *undefinability of well-order*

A corollary of this result is Interpolation: if  $\kappa = \beth_\kappa$ ,  $\phi_1$ , a  $\tau_1$ -sentence and  $\phi_2$ , a  $\tau_2$ -sentence (both of  $L_\kappa^1$ ) are such that  $\models \phi_1 \rightarrow \phi_2$  then there is a  $(\tau_1 \cap \tau_2)$ -sentence  $\psi$  of  $L_\kappa^1$  such that  $\models \phi_1 \rightarrow \psi \rightarrow \phi_2$ .

# THE LOGICS $L^1_{\kappa\lambda+}$ .

(Joint work with Väänänen)

Suppose  $\text{cf}(\lambda) > \omega$  and  $2^\lambda < \kappa$ .

Given  $\Phi$  a function ( $I \mapsto \phi_I$ ) from  $\mathcal{P}(\lambda)$  to formulas such that  $\phi_I$  has free variables among  $x_\alpha$ ,  $\alpha \in I$ , let

$$\forall \vec{x} \bigvee_f \bigwedge_n \phi_{f^{-1}(n)}(\vec{x}')$$

and

$$\exists \vec{x} \bigwedge_f \bigvee_n \phi_{f^{-1}(n)}(\vec{x}')$$

with  $\vec{x}$  a sequence of length  $\lambda$ ,  $f : \lambda \rightarrow \omega$ ,  $n \in \text{ran}(f)$ ,  $\vec{x}'$  the subsequence of  $\vec{x}$  corresponding to indices in  $f^{-1}(n)$ .

# A DESCENDING CHAIN OF LENGTH $\lambda$

The sentence

$$\exists \vec{x} \bigwedge_f \bigvee_n \bigwedge \{x_\alpha < x_\beta \mid f(\alpha) = f(\beta) = n, \beta < \alpha\}$$

says the linear order  $<$  has a descending chain of length  $\lambda$

# A MODIFIED DELAYED EF GAME, $DG_\delta(M, N)$

Similar to the original delayed EF game except the partitioning by player II is finer:

First round of the game:

$I$	$II$
$\delta_0 < \delta$ $\{a_\alpha^0 : \alpha < \lambda\}$	
	$f_0 : \lambda \rightarrow \omega$
$n_0$	
	$\{b_\alpha^0 : f_0(\alpha) = n_0\}$

(1)

Position after first round:

$$(\delta_0, \{(a_\alpha^0, b_\alpha^0) : f_0(\alpha) = n_0\}).$$

If  $\delta_0 = 0$ , the game ends and II wins if

$$h_0 = \{(a_\alpha^0, b_\alpha^0) : f_0(\alpha) = n_0\}$$

is a partial isomorphism. Otherwise I wins.

# SECOND ROUND OF $DG_\delta(M, N)$ :

$I$	$II$
$\delta_1 < \delta_0$ $\{a_\alpha^1 : \alpha < \lambda\}$	
	$f_1 : \lambda \rightarrow \omega$
$n_1$	
	$\{b_\alpha^1 : f_1(\alpha) = n_1\}$

(2)

Again, if  $\delta_1 = 0$ , the game ends and II wins if  $h_0 \cup h_1$ , where

$$h_1 = \{(a_\alpha^1, b_\alpha^1) : f_1(\alpha) = n_1\},$$

is a partial isomorphism. Otherwise I wins. The position now is  $(\delta_1, h_0 \cup h_1)$ .

The game continues until  $\delta_i = 0$ .

# SYNTAX

## Proposition

*The relation “ $\text{II}$  has a winning strategy in  $DG^\delta(M, N)$ ” is transitive.*

The syntax may be reconstructed from formulas of the form  $\psi_{N, \vec{a}}^\delta$  that capture the notion

$$M \equiv_{L^1_{\infty, \lambda^+}}^\delta N$$

(originally defined as  $\text{II}$  having a winning strategy for the game  $DG_\delta(M, N)$ ).

## IN CONCLUSION...

- Smooth Reflection Classes (or Abstract Elementary Classes: emphasis on pure reflection phenomena as opposed to definability)



## IN CONCLUSION...

- ▶ Smooth Reflection Classes (or Abstract Elementary Classes: emphasis on pure reflection phenomena as opposed to definability)
- ▶ Yet logic makes a (cruel but sweet) return, first through the Presentation Theorem (at the price of having to use the  $\Sigma_1^1$  operation).

## IN CONCLUSION...

- ▶ Smooth Reflection Classes (or Abstract Elementary Classes: emphasis on pure reflection phenomena as opposed to definability)
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- ▶ A variant of that logic now has syntax; another variant captures (without the  $\Sigma_1^1$  operation) the phenomenon of Smooth Reflection

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