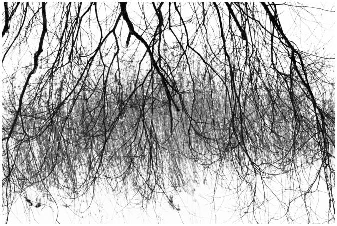


Around the Galois group of an AEC

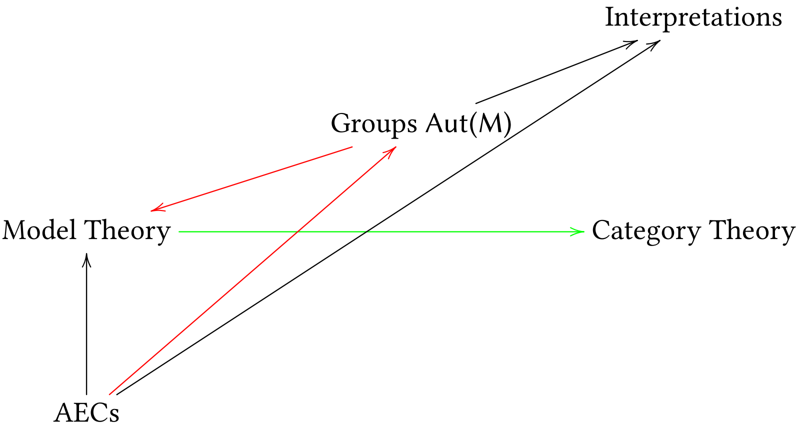
Helsinki Logic Seminar - 1.2019

Andrés Villaveces - joint work with Zaniar Ghadernezhad
Universidad Nacional de Colombia - Bogotá

January 2019

Contents

Some neighbours



Plan

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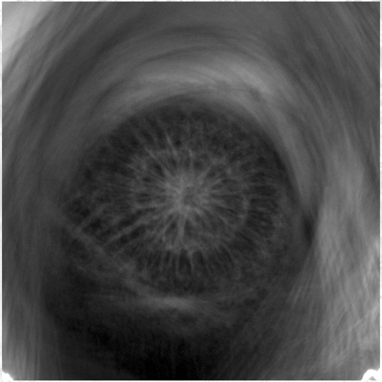

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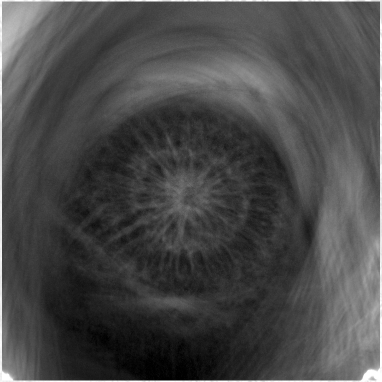

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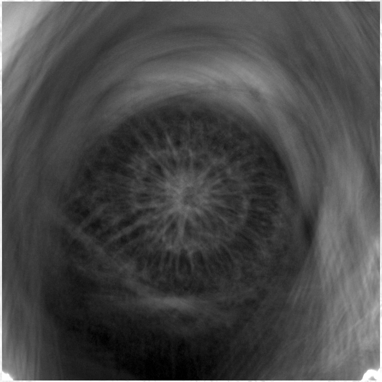

Tell me what is M!

Reconstructing models?

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- a more reasonable question: if for some (First Order) structure M we are given $\mathrm{Aut}(M)$, what can we say about $\mathrm{Th}(M)$?
- an even more reasonable question: if for some (FO) structure M we are given $\mathrm{Aut}(M)$, when can we recover all models biinterpretable with M?

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- (Koenigsmann) K and $G_{K(t)/K}$ are biinterpretable for K a perfect field with finite extensions of degree > 2 and prime to $\mathrm{char}(K)$.

Reconstructing structures - the classical blueprint (Ahlbrandt-Ziegler)

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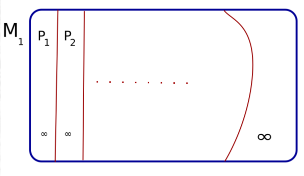
- The action $\mathrm{Aut}(M) \curvearrowright$ is (almost) $\approx$ to $\mathrm{Aut}(M) \curvearrowright M^{eq}$. So, we have recovered the action of $\mathrm{Aut}(M)$ on M^{eq} from the topology of $\mathrm{Aut}(M)$... so, if M, N are countable $\aleph_0$-categorical structures, TFAE:
  - There is a bicontinuous isomorphism from $\mathrm{Aut}(M)$ onto $\mathrm{Aut}(N)$
  - M and N are bi-interpretable.

Some Trivial Observations - Saturation is desirable!

Let M_1 be the countable saturated model of P_i $(i < \omega)$ disjoint infinite predicates and



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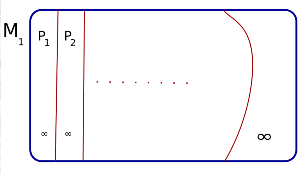
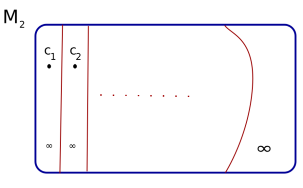


let M_2 be a countable model of an equivalence relation with infinitely many infinite classes, with exactly one constant c_i in each class but one



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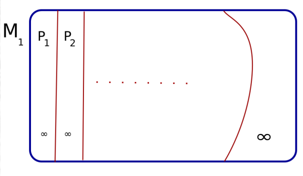

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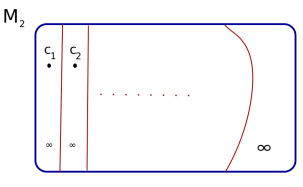

$$\text{yet } \mathrm{Aut}(M_1) \approx \mathrm{Aut}(M_2)$$

Interpretation between FO theories - Models as functors

(Makkai)

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- Morphisms correspond to definable functions: if $\mathsf{A} :: \phi(\mathsf{x})$ and $\mathsf{B} :: \psi(\mathsf{y})$, a definable morphism $\mathsf{f} : \mathsf{A} \to \mathsf{B}$ is a definable $\mathsf{f} :: \chi(\mathsf{x}, \mathsf{y})$ such that $\mathsf{T} \models \forall \mathsf{x} \forall \mathsf{y} (\chi(\mathsf{x}, \mathsf{y}) \to \varphi(\mathsf{x}) \wedge \psi(\mathsf{y}))$ and $\mathsf{T} \models \forall \mathsf{x} (\varphi(\mathsf{x}) \to \exists \mathsf{y} \chi(\mathsf{x}, \mathsf{y}))$.

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- Given any L-structure $\mathfrak{M}$ and a formula $\varphi(\mathsf{x})$, the **solution set** is $\varphi(\mathfrak{M}) = \{\mathsf{a} \in \mathsf{M}_\mathsf{x} \mid \mathfrak{M} \models \varphi(\mathsf{x})\}$.

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- Given any L-structure $\mathfrak{M}$ and a formula $\varphi(x)$, the **solution set** is $\varphi(\mathfrak{M}) = \{a \in M_x \mid \mathfrak{M} \models \varphi(x)\}$.
- With this, **we regard models of** T **as functors** from $\mathcal{T}$ to **Set**: $\mathfrak{M}(A) = \varphi(\mathfrak{M})$. Natural transformations $\equiv$ elementary maps.

Interpretation between FO theories - Models as functors

The category $\mathcal{T} = \mathrm{Def}(T)$ is Boolean (regular, and with Boolean algebras of subobjects) and extensive (co-products exist and they form an equivalence between the categories $\mathrm{Sub}(X) \times \mathrm{Sub}(Y)$ and $\mathrm{Sub}(X \sqcup Y)$).

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An **interpretation** between T_0 and T is a Boolean and extensive morphism

$$\iota : \mathcal{T}_0 \to \mathcal{T}$$

between the categories $\mathcal{T}_0$ and $\mathcal{T}$ (in the vocabularies L_0 and L). (ι preserves finite limits, induces homomorphisms of Boolean algebras in subobjects and respects images - and respects co-products)

Interpretation functor between classes of models

We lift the interpretation to classes of models:
Given $\iota : \mathcal{T}_0 \to \mathcal{T}$,

$$\iota^* : \mathsf{Mod}(\mathsf{T}) \to \mathsf{Mod}(\mathsf{T}_0)$$

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where

$$\mathfrak{M}_0 = \mathfrak{M} \circ \iota : \mathcal{T} \to \mathbf{Set}$$

and if $\sigma : \mathfrak{N} \to \mathfrak{M}$ is an elementary embedding ($\sigma = (\sigma_{\mathsf{Y}})_{\mathsf{Y} \in \mathcal{T}}$) then

$$\iota^*(\sigma) : \mathfrak{N}_0 \to \mathfrak{M}_0 : \iota^* \sigma_{\mathsf{X}} = \sigma_{\iota \mathsf{X}}$$

for each $\mathsf{X} \in \mathcal{T}_0$.

Examples - ACF, RCF

An interpretation we have known for some 200 years is the following:

$$\iota : \mathrm{Def}(\mathrm{ACF}) \to \mathrm{Def}(\mathrm{RCF})$$

$$\iota(K) = R^2 \text{ , componentwise sum}$$

$$\text{multiplication } (a, b)(\iota\cdot)(c, d) = (ac - bd, bc + ad)$$

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if $R \models \mathrm{RCF}$

$$\iota^*(R) = R[\sqrt{-1}].$$

Many other natural examples: retracts, boolean algebras in boolean rings, etc.

Stable Interpretations - a bit on Galois theory

The notion of stability is reflected in a natural way in interpretations: Remember a theory T is stable if no formula can define an infinite linear order (in tuples).
An interpretation $\iota : \mathcal{T}_0 \to \mathcal{T}$ is **stable** if for each model $\mathfrak{M}$ of T, the "expanded interpretation" $\iota^{\mathfrak{M}} : \mathcal{T}_0^{\mathfrak{M}_0} \to \mathcal{T}^{\mathfrak{M}}$ is an immersion. This means each definable in $\iota\mathsf{X}$ ($\mathsf{X} \in \mathcal{T}_0$) using parameters from $\mathcal{M}$ is the image of a definable set in X using parameters from $\mathcal{M}_0$.

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If T is a stable theory and $\iota : \mathcal{T}_0 \to \mathcal{T}$ is an interpretation, then ι is a stable interpretation and T_0 is a stable theory.
Hrushovski and Kamensky went as far as reframing a "Galois theory" of model theory for internal covers - Galois theory à la Grothendieck (Exposé IV).

The Galois group of a first order theory

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$$\mathrm{Gal}(T/A) := \mathrm{Aut}(M)/\mathrm{Autf}(M)$$

where M is a saturated model of T and

$$\mathrm{Autf}(M) = \langle \bigcup_{A \subset N \prec M} \mathrm{Aut}_N(M) \rangle$$

This is an invariant of the theory, allowing a Galois connection between definably closed submodels of M and closed subgroups of the Galois group.

Plan

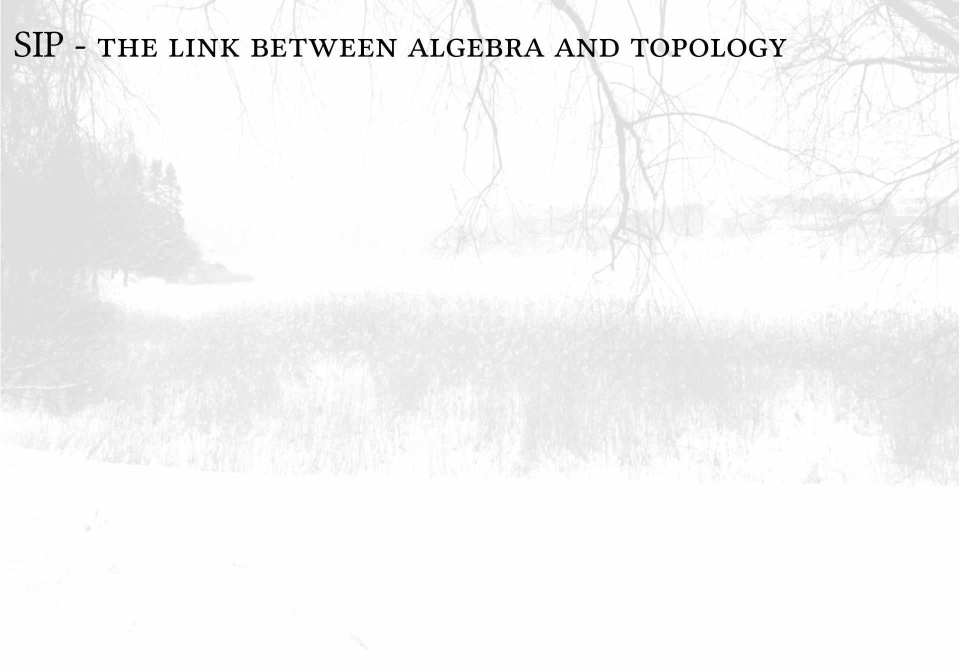
Reconstructing models / Interpretations
- The reconstruction problem
- Interpretations, category-theoretically

From Aut(M) to $\mathbf{Int}_M$: SIP
- Elementary musings and countable issues
- Smoothing SIP beyond $\aleph_0$ - Lascar-Shelah

SIP beyond first order
- Strong amalgamation classes
- SIP for homogeneous AEC
- Examples: quasiminimal classes, the Zilber field, j-invariants

Interpretations between AECs

SIP - THE LINK BETWEEN ALGEBRA AND TOPOLOGY



SIP - the link between algebra and topology

Now, to the main property of the group Aut(M) that enables us to capture its topology...

THE CLASSICAL TOPOLOGY.

Fix M for now a countable structure. The classical way of making Aut(M) into a topological space is by decreeing that basic open sets around 1_M are pointwise stabilizers of finite subsets $A_{fin} \subset M$, $\mathrm{Aut}_A(M) = \{f \in \mathrm{Aut}(M) \mid f \restriction A = 1_A\}$.
This gives Aut(M) the structure of a Polish space.

The Small Index Property (countable version)

Definition (Small Index Property - SIP)

Let M be a countable structure. M has the small index property if for any subgroup H of $\mathrm{Aut}(M)$ of index less than $2^{\aleph_0}$, there exists a finite set $A \subset M$ such that $\mathrm{Aut}_A(M) \subset H$.

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In other words, if G is "large algebraically speaking" then it is also "large topologically speaking".

Basic facts on countable SIP

SIP allows us to recover the topological structure of Aut(M) from its pure group structure:
Open neighborhoods of 1 in pointwise convergence topology = Subgroups containing pointwise stabilizers $\mathrm{Aut}_A(M)$ for some finite A.

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Open neighborhoods of 1 in pointwise convergence topology = Subgroups containing pointwise stabilizers $\mathrm{Aut}_A(M)$ for some finite A.

- SIP holds for random graph, infinite set, DLO, vector spaces over finite fields, generic relational structures, $\aleph_0$-categorical $\aleph_0$-stable structures, etc.
- It fails e.g. for $M \models ACF_0$ with ∞ transc. degree.

Galois group (of a theory)

The Galois group of a model M,

$$\mathrm{Gal}(M) := \mathrm{Aut}(M)/\mathrm{Autf}(M),$$

is invariant across saturated models of a theory[1].
Possible failures of SIP are encoded in this quotient.

[1] **Lascar, Daniel.** Automorphism Groups of Saturated Structures, ICM 2002, Vol. III - 1-3.
Lascar, Daniel. Les automorphismes d'un ensemble fortement minimal, JSL, vol. 57, n. 1. March 1992.

To the uncountable / the non-elementary

SIP for uncountable structures

We now switch focus to the uncountable, first order, case.
Fix $\lambda = \lambda^{<\lambda}$ an uncountable cardinal, and fix M a saturated model of cardinality λ.

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[2] **Sy-David Friedman, Tapani Hyttinen and Vadim Kulikov,** Generalized descriptive set theory and classification theory, Memoirs of the American Mathematical Society, 2014; Volume 230, Number 1081

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$\mathrm{Aut}(M)$ with this topology is of course no longer a Polish space. The techniques from Descriptive Set Theory that have been used for the countable case need to be replaced (Friedman, Hyttinen and Kulikov's Descriptive Set Theory for some uncountable cardinalities might become relevant to this[2]).

[2] **Sy-David Friedman, Tapani Hyttinen and Vadim Kulikov,** Generalized descriptive set theory and classification theory, Memoirs of the American Mathematical Society, 2014; Volume 230, Number 1081

Lascar-Shelah's Theorem

Theorem (Lascar-Shelah: Uncountable saturated models have the SIP)

Let M *be saturated, of cardinality* $\lambda = \lambda^{<\lambda}$ *and let* G *be a subgroup of* $\mathrm{Aut}(M)$ *such that* $[\mathrm{Aut}(M) : G] < 2^{\lambda}$*. Then there exists* $A \subset M$ *with* $|A| < \lambda$ *such that* $\mathrm{Aut}_A(M) \subset G$*.*

[3] **Daniel Lascar and Saharon Shelah,** Uncountable Saturated Structures have the Small Index Property, Bull. London Math. Soc. 25 (1993) 125-131.

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The proof[3] consists of building directly (assuming that G does not contain any open set $\mathrm{Aut}_A(M)$ around the identity) a **binary tree** of height λ of automorphisms of M in such a way that every two of them are not conjugate. This is enough but requires two crucial notions: **generic** and **existentially closed (sequences of) automorphisms**. These are obtained by assuming that G is not open.

[3] **Daniel Lascar and Saharon Shelah,** Uncountable Saturated Structures have the Small Index Property, Bull. London Math. Soc. 25 (1993) 125-131.

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Now, beyond First Order

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Although results on the reconstruction problem, so far have been stated and proved for saturated models in first order theories, the scope of the matter can go far beyond:

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- Abstract Elementary Classes with a well-behaved closure notion, and the particular case:
- Quasiminimal (qm excellent) Classes.

The setting: Strong amalgamation classes

A setting for homogeneity: let $(\mathcal{K}, \prec_{\mathcal{K}})$ be an AEC, with $\mathsf{LS}(\mathcal{K}) \leq \lambda$, $|M| = \kappa > \lambda$, $\kappa^{<\kappa} = \kappa$.
Let $\mathcal{K}^{<}(M) := \{N : N \preccurlyeq_K M, |N| < \kappa\}$ and fix $M \in \mathcal{K}$ homogeneous.

The setting: Strong amalgamation classes

A setting for homogeneity: let $(\mathcal{K}, \prec_{\mathcal{K}})$ be an AEC, with $\mathsf{LS}(\mathcal{K}) \leq \lambda$, $|M| = \kappa > \lambda$, $\kappa^{<\kappa} = \kappa$.
Let $\mathcal{K}^{<}(M) := \{N : N \preccurlyeq_K M, |N| < \kappa\}$ and fix $M \in \mathcal{K}$ homogeneous.
The topology τ^{cl}: base of open neighborhoods given by sets of the form $\mathrm{Aut}_X(M)$ where $X \in \mathcal{C}$, where
$\mathcal{C} := \left\{\mathrm{cl}^M(A) : A \subseteq M \text{ such that } |A| < \kappa\right\}$ and the "closure operator" is $\mathrm{cl}^M(A) := \bigcap_{A \subset N \prec_{\mathcal{K}} M} A$.
This class of cl^M-closed sets has enough structure for the proof of SIP.

The main result: SIP for homogeneous AEC.

With Ghadernezhad, we have proved[4]:

Theorem (SIP for $(\mathrm{Aut}(M), \mathcal{T}^{\mathrm{cl}})$ - Ghadernezhad, V.)

Let M *be a homogeneous model in an AEC* $(\mathcal{K}, \prec_{\mathcal{K}})$*, with* $|M| = \lambda = \lambda^{<\lambda} > \mathrm{LS}(\mathcal{K})$*, such that* $\mathcal{K}^{<\lambda}$ *is a strong amalgamation class. Let* $G \leq \mathrm{Aut}(M)$ *with* $[\mathrm{Aut}(M) : G] \leq \lambda$ *(this is,* G *has small index in* $\mathrm{Aut}(M)$*). Then there exists* $X \in \mathcal{C}$ *such that* $\mathrm{Aut}_X(M) \leq G$ *(i.e.,* G *is open in* $(\mathrm{Aut}(M), \mathcal{T}^{\mathrm{cl}})$*).*

[4] **Ghadernezhad, Zaniar and Villaveces, Andrés.** The Small Index Property for Homogeneous AEC's, Archive for Mathematical Logic, February 2018, Volume 57, Issue 1–2, pp 141–157.

Getting many non-conjugates

Proof (rough sketch): suppose G has small index in Aut(M) but is not open (does not contain any basic $\mathrm{Aut}_X(M)$ for $X \in \mathcal{C}$.
We have enough tools (generic sequences and strong amalgamation bases) to build a "Lascar-Shelah" tree to reach a contradiction (2^λ many branches giving automorphisms of M g_σ for $\sigma \in 2^\lambda$ such that if $\sigma \neq \tau \in 2^\lambda$ then $g_\sigma^{-1} \circ g_\tau \notin G$).
Of course, the possibility of getting these 2^λ-many automorphisms requires using the non-openness of G to get the construction going.

Lascar-Shelah tree for our situation

A λ-Lascar-Shelah tree for M and $G \leq \mathrm{Aut}(M)$ is a binary tree of height λ with, for each $s \in 2^{<\lambda}$, a model $M_s \in \mathcal{K}^{<}(M)$, $g_s \in \mathrm{Aut}(M_s)$, $h_s, k_s \in \mathrm{Aut}_{M_s}(M)$ such that

- $h_{s,0} \in G$ and $h_{s,1} \notin G$ for all $s \in \mathcal{S}$;
- $k_{s,0} = k_{s,1}$ for all $s \in \mathcal{S}$;
- for $s \in \mathcal{S}$ and all $t \in \mathcal{S}$ such that $t \leqslant s$:$h_t\,[M_s] = M_s$ (i.e. $h_t \in \mathrm{Aut}_{\{M_s\}}(M)$) and ...;
- for $s \in \mathcal{S}$ and all $t \in \mathcal{S}$ such that $t \leqslant s$: $g_s \cdot (h_t \restriction M_s) \cdot g_s^{-1} = k_t \restriction M_s$;
- for $s \in \mathcal{S}$ and $\beta <$ length$\,(s)$: $a_s \in M_s$;
- for all s, the families $\{h_t : t \leqslant s, t \in \mathcal{S}\}$ and $\{k_t : t \leqslant s, t \in \mathcal{S}\}$ are elements of $\mathcal{F}$ (i.e. they are generic).

Generic sequences and Strong amalgamation bases

The main technical tools in the construction of a LS tree are

- Guaranteeing generic sequences of automorphisms
  ($g \in \mathrm{Aut}(M)$ is generic if
  $\forall N \in \mathcal{K}^{<}(M)$ such that $g \restriction N \in \mathrm{Aut}(N)$
  $\forall N_1 \succ_{\mathcal{K}} N, N_1 \in \mathcal{K}^{<}(M)$
  $\forall h \supset g \restriction N, h \in \mathrm{Aut}(N_1)$
  $\exists \alpha \in \mathrm{Aut}_N(M)$ such that $g \supset \alpha \circ h \circ \alpha^{-1}$),
- showing they are unique up to conjugation,
- getting a generic sequence $\mathcal{F} = \left(g_i : i \in I\right)$ such that
  1. the set $\left\{i \in I : g_i \restriction M_0 = h \text{ and } g_i \notin G\right\}$ has cardinality κ for all $M_0 \in \mathcal{K}^{<}(M)$ and $h \in \mathrm{Aut}\,(M_0)$;
  2. the set $\left\{i \in I : g_i \in G\right\}$ has cardinality κ.

Another way to get generics: aut-independence

Definition
Let $A, B, C \in \mathcal{C}$. Define $A \mathop{\smile\!\!\!\!\!\!\;|}^{a}_{B} C$ if for all $f_1 \in \mathrm{Aut}(A)$ and all $f_2 \in \mathrm{Aut}(C)$ and for all $h_i \in \mathcal{O}_{f_i}$ $(i = 1, 2)$ such that $h_1 \restriction A \cap C = h_2 \restriction A \cap C$ and $h_1 \restriction B = h_2 \restriction B$ then $\mathcal{O}_g \neq \emptyset$ where $g := f_1 \cup f_2 \cup h_1 \restriction B$.

Definition
Let $A, B, C \in \mathcal{C}$. Define $A \mathop{\smile\!\!\!\!\!\!\;|}^{a-s}_{B} C$ if and only if $A' \mathop{\smile\!\!\!\!\!\!\;|}^{a}_{B} C'$ for all $A' \subseteq A$ and $B' \subseteq B$ with $A', B' \in \mathcal{C}$.

Fact
$\mathop{\smile\!\!\!\!\!\!\;|}^{a-s}$ *satisfies symmetry, monotonicity and invariance.*

Free $\mathop{\smile\!\!\!\!\!\mid}^{a-s}_{B}$ C-amalgamation.

The class $\mathcal{C}$ has the <u>free $\mathop{\smile\!\!\!\!\!\mid}^{a-s}$-amalgamation property</u> if for all $A, B, C \in \mathcal{C}$ with $A \cap B = C$ there exists $B' \in \mathcal{C}$ such that $\mathrm{ga-tp}\left(B'/C\right) = \mathrm{ga-tp}\left(B/C\right)$ (or there exists $g \in \mathrm{Aut}_C(M)$ that $g[B] = B'$) and $A \mathop{\smile\!\!\!\!\!\mid}^{a-s}_{C} B'$.

Fact

Suppose $\mathcal{C}$ has the free $\mathop{\smile\!\!\!\!\!\mid}^{a-s}$-amalgamation property. Then generic automorphisms exist.

Quasiminimal pregeometry classes

In a language L, a quasiminimal pregeometry class $\mathcal{Q}$ is a class of pairs $\langle H, \mathrm{cl}_H \rangle$ where H is an L-structure, cl_H is a pregeometry operator on H such that the following conditions hold:

1. Closed under isomorphisms,
2. For each $\langle H, \mathrm{cl}_H \rangle \in \mathcal{Q}$, the closure of any finite set is countable.
3. If $\langle H, \mathrm{cl}_H \rangle \in \mathcal{Q}$ and $X \subseteq H$, then $\langle \mathrm{cl}_H(X), \mathrm{cl}_H \restriction \mathrm{cl}_H(X) \rangle \in \mathcal{Q}$.
4. If $\langle H, \mathrm{cl}_H \rangle, \langle H', \mathrm{cl}_{H'} \rangle \in \mathcal{Q}$, $X \subseteq H$, $y \in H$ and $f : H \to H'$ is a partial embedding defined on $X \cup \{y\}$, then $y \in \mathrm{cl}_H(X)$ if and only if $f(y) \in \mathrm{cl}_{H'}(f(X))$.
5. Homogeneity over countable models.

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3. If $\langle \mathsf{H}, \mathrm{cl}_\mathsf{H} \rangle \in \mathcal{Q}$ and $\mathsf{X} \subseteq \mathsf{H}$, then $\langle \mathrm{cl}_\mathsf{H}(\mathsf{X}), \mathrm{cl}_\mathsf{H} \restriction \mathrm{cl}_\mathsf{H}(\mathsf{X}) \rangle \in \mathcal{Q}$.
4. If $\langle \mathsf{H}, \mathrm{cl}_\mathsf{H} \rangle, \langle \mathsf{H}', \mathrm{cl}_{\mathsf{H}'} \rangle \in \mathcal{Q}$, $\mathsf{X} \subseteq \mathsf{H}$, $\mathsf{y} \in \mathsf{H}$ and $\mathsf{f} : \mathsf{H} \to \mathsf{H}'$ is a partial embedding defined on $\mathsf{X} \cup \{\mathsf{y}\}$, then $\mathsf{y} \in \mathrm{cl}_\mathsf{H}(\mathsf{X})$ if and only if $\mathsf{f}(\mathsf{y}) \in \mathrm{cl}_{\mathsf{H}'}(\mathsf{f}(\mathsf{X}))$.
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These can all be generated by ONE canonical structure.

Some comments on the context

- Sébastien Vasey has proved that quasiminimal pregeometries do not require the exchange axiom of pregeometries. This makes it in principle easier to prove that classes are quasiminimal!
- Vasey has also suggested that our theorem applies to wider classes (excellent classes, and even wider: certain "non-forking frames").

Example: quasiminimal classes, "Zilber field"

- $\mathcal{Q}$ quasiminimal pregeometry class. $M \in \mathcal{Q}$ of size $\aleph_1$, $\mathcal{C} = \{\mathrm{cl}(A) \mid A \subset M, A \text{ small}\}$ then $\mathcal{C}$ has the free aut-independence amalgamation property. (Based on Haykazyan's paper on qm classes.)

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- $\mathcal{Q}$ qm pregeom. class $\rightarrow$ for every model M of $\mathcal{Q}$, $\mathrm{Aut}(M)$ has SIP,
- The "Zilber field" has SIP.
- The j-invariant has the SIP.

Plan

Toward interpretation between AECs

We already have some ingredients:

- A good and solid (category-theoretical) way of dealing with interpretation, leading to a Galois theory in the sense of Grothendieck.

So, where can we go? Interpretation is a natural way.

Toward interpretation between AECs

We <u>already</u> have some ingredients:

- A good and solid (category-theoretical) way of dealing with interpretation, leading to a Galois theory in the sense of Grothendieck.
- A criterion for reconstruction (the SIP) lifting to some AECs and their homogeneous models.

So, where can we go? Interpretation is a natural way.

Ultimate goal: reconstruction

With the ultimate goal of reconstruction in mind (what properties of an AEC are reflected by the automorphims of a large homogeneous model?) it is natural to study interpretations in various different ways.

Using specific logics to build the interpretation

Boney-Vasey have used a logic harking back to Stavi (structural logic) to capture AECs with intersections. These classes are closely related to our strong amalgamation classes with closures.
They prove that AECs with intersections correspond to classes axiomatizable by universal theories in that logic.
Other AECs can be axiomatized by other logics (work in progress with Shelah).

Interpretation of a $\mathbb{L}^{\kappa-\mathsf{struct}}$-axiomatizable classes

Given $\iota : \mathrm{Def}_{\psi_0} \to \mathrm{Def}_{\psi}$,

$$\iota^* : \mathcal{K} \to \mathcal{K}_0$$

$$\mathfrak{M} \models \mathsf{T} \mapsto \iota^*(\mathfrak{M}) = \mathfrak{M}_0$$

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where (again)

$$\mathfrak{M}_0 = \mathfrak{M} \circ \iota : \mathcal{T} \to \mathbf{Set}$$

and if $\sigma : \mathfrak{N} \to \mathfrak{M}$ is an $\mathbb{L}^{\kappa-\mathsf{struct}}$-elementary embedding ($\sigma = (\sigma_{\mathsf{Y}})_{\mathsf{Y} \in \mathrm{Def}_{\psi}}$ then

$$\iota^*(\sigma) : \mathfrak{N}_0 \to \mathfrak{M}_0 : \iota^* \sigma_{\mathsf{X}} = \sigma_{\iota \mathsf{X}}$$

for each $\mathsf{X} \in \mathrm{Def}_{\psi_0}$.

Using types to build the interpretation

A more direct approach may either use Morleyization of the vocabulary (expanding by adding all orbital types as predicates), or use Shelah's Presentation Theorem (but dealing with omitting types functorially will require additional understanding):

Theorem (Shelah)

Let $(\mathcal{K}, \leq_K)$ *be an AEC in a language* L*. Then there exist*

- *A language* $\mathsf{L}' \supset \mathsf{L}$*, with size* $\mathsf{LS}(\mathcal{K})$*,*
- *A (first order) theory* T' *in* L' *and*
- *A set of* T'*-types,* Γ'*, such that*

$$\mathcal{K} = \mathsf{PC}(\mathsf{L}, \mathsf{T}', \Gamma') := \{\mathsf{M}' \upharpoonright \mathsf{L} \mid \mathsf{M}' \models \mathsf{T}', \mathsf{M}' \textit{ omits } \Gamma'\}.$$

Moreover, if $\mathsf{M}', \mathsf{N}' \models \mathsf{T}'$*, they both omit* Γ'*,* $\mathsf{M} = \mathsf{M}' \upharpoonright \mathsf{L}$ *and* $\mathsf{N} = \mathsf{N}' \upharpoonright \mathsf{L}$*,*

$$\mathsf{M}' \subset \mathsf{N}' \Leftrightarrow \mathsf{M} \leq_K \mathsf{N}.$$

The Galois group of an AEC

This is well defined in Strong Amalgamation AECs:

$N \in \mathcal{K}, \mathcal{K}$

$$\mathrm{Gal}(\mathcal{K}/A) := \mathrm{Aut}(M)/\mathrm{Autf}(M)$$

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$$\mathrm{Gal}(\mathcal{K}/A) := \mathrm{Aut}(M)/\mathrm{Autf}(M)$$

where M is a homogeneous model in $\mathcal{K}$, $N \prec_{\mathcal{K}} M$ is small and as before

$$\mathrm{Autf}(M) = \langle \bigcup_{N \prec_{\mathcal{K}} N' \prec M} \mathrm{Aut}_{N'}(M) \rangle$$

This is an invariant of $\mathcal{K}$.
A Galois connection between definably closed submodels of M and closed subgroups of the Galois group...

Thank you for your attention!