Modular Invariants, towards Real Multiplication

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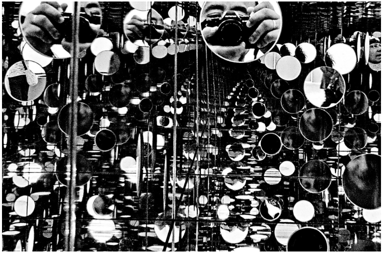

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Pensieve

Three categories for j:

$$\begin{cases} \text{DIFF} : DCF_0 \\ \text{MOD} : Th^{SF}_{L_{\omega_1,\omega}}(j) \\ \text{ARITHM} : Gal(\bar{K}/K) \end{cases}$$

Pensieve

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- The Freitag-Scanlon / Aslanyan theory - differential approach to the model theory of j, Ax-Schanuel
- The sheaf theoretic question: sheaves of Harris frameworks? Differential sheaves? Connection to universal $\check{j}$
- The Gendron extension of j is sheaf-theoretic in nature; what kind of framework may mix it? We want a Harris-style framework for quantum j, and with Zilber, we have partial results in that direction

A DREAM?

$$(\mathbb{C}, \circlearrowright_{SL_2(\mathbb{Q})}) \xrightarrow{\ddot{\mathbb{j}}} (\mathbb{C}, +, \cdot)$$

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$$\begin{array}{ccc}
\left(\mathcal{F}_1, \circlearrowright_{SL_2(\mathbb{Q})}\right) & \xrightarrow{\varphi_{\circlearrowright}} & \left(\mathcal{F}_2, \circlearrowright_{SL_2(\mathbb{Q})}\right) \\
\downarrow \ddot{\mathbb{j}}_1 & & \downarrow \ddot{\mathbb{j}}_2 \\
\left(\mathcal{F}_1, +, \cdot\right) & \xrightarrow{\varphi_F} & \left(\mathcal{F}_2, +, \cdot\right)
\end{array}$$

Classical j invariant

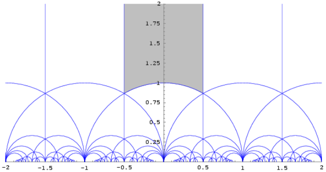
Klein defines the function (we call) "classical j"

$$j : \mathbb{H} \to \mathbb{C}$$

(where $\mathbb{H}$ is the complex upper half-plane)
through the explicit rational formula

$$j(\tau) = 12^3 \cdot \frac{g_2(\tau)^3}{g_2(\tau)^3 - 27g_3(\tau)^2}$$

with g_2 and g_3 certain rational functions (Eisenstein functions).



The fundamental domain of the modular group acting on $\mathbb{H}$

Classical j

Recall a definition of classical j:
if $\mu \in \mathbb{H}$, $\Lambda(\mu) = \mathbb{Z} + \mathbb{Z}\mu$ is the μ-lattice, and the classical torus associated to μ is $\mathbb{T}(\mu) := \mathbb{C}/\Lambda(\mu)$. (This is also a Riemann surface.)
Now, $\mathbb{T}(\mu)$ is equivalent to the elliptic curve $\mathbb{E}(\mu)$ given by
$Y^2 = X^3 - g_2(\mu)X - g_3(\mu)$.
Here, let $G_k(\mu) := \sum_{0 \neq \gamma \in \Lambda(\mu)} \gamma^{-2k} \qquad k \geq 2$ (the so-called Eisenstein series), then $g_2(\mu) = 60 \cdot G_2(\mu) g_3(\mu) = 140 \cdot G_3(\mu)$.

Basic facts about j

The following are equivalent:

1. There exists $s = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$ such that $s(\tau) = \tau'$,

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3. $j(\tau) = j(\tau')$

Classical Geometry / Arithmetic and j

Classical j is therefore an invariant of tori (=elliptic curves) with the following additional advantages:

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- (Schneider, 1937): if τ is a quadratic irrationality then $j(\tau)$ is algebraic of degree $h_{f,K}$.
- if $e^{2\pi i\tau}$ is algebraic then $j(\tau), \frac{j'(\tau)}{\pi}, \frac{j''(\tau)}{\pi^2}$ are mutually transcendental (weak Schanuel-like situation).

j-functions & Model Theory (toward categoricity)

Adam Harris provides the following theory of classical j invariants:

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- A convoluted proof of categoricity of this version of j
- Generalization of this analysis to higher dimensions (Shimura varieties).
- Analogies to pseudoexponentiation ("Zilber field") are strong, but the structure of j seems to have a much higher degree of complexity even than *exp*.

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where $\langle F, +, \cdot, 0, 1\rangle$ is an algebraically closed field of characteristic 0, $\langle H; \{g_i\}_{i\in\mathbb{N}}\rangle$ is a set together with countably many unary function symbols (the group action),

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Really, j is a ***"cover"*** *from the* <u>*action*</u> *into the field.*

The $L_{\omega_1,\omega}$-axiom - Crucial point: Standard fibers of the cover j

Let then

$$Th_{\omega_1,\omega}(j) := Th(\mathbb{C}_j) \cup \forall x \forall y (j(x) = j(y) \to \bigvee_{i<\omega} x = \gamma_i(y))$$

for $\mathbb{C}_j$ the "standard model" $(\mathbb{H}, \langle \mathbb{C}, +, \cdot, 0, 1 \rangle, j : \mathbb{H} \to \mathbb{C})$.
This captures all the first order theory of j (not the analyticity!) plus the fact that fibers are "standard" ("fibers are orbits")

Categoricity of classical j

Theorem (Harris, assuming a form of the Mumford-Tate Conj.)
The theory $T^{\infty}(j) := Th_{\omega_1,\omega}(j) + \mathrm{trdeg}(F) \geq \aleph_0$ is categorical in all infinite cardinalities.

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The theory $T^{\infty}(j) := Th_{\omega_1,\omega}(j) + \mathrm{trdeg}(F) \geq \aleph_0$ is categorical in all infinite cardinalities. I.e., given two models of same cardinality

$$M_1, M_2 \models T^{\infty}(j),$$

with $M_i = (\mathcal{H}_i, F_i, j_i : H_i \to F_i)$, $\mathcal{H}_i = (H_i, \{g^i_j\}_{j\in\mathbb{N}})$, $\mathcal{F}_i = (F_i, +_i, \cdot_i, 0, 1)$, there are isomorphisms φ_H, φ_F such that

$$\begin{array}{ccc} \mathcal{H}_1 & \xrightarrow{\varphi_H} & \mathcal{H}_2 \\ \downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle j_2} \\ \mathcal{F}_1 & \xrightarrow{\varphi_F} & \mathcal{F}_2 \end{array}$$

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Model Theory:

- Quasiminimal abstract elementary classes. These must be categorical (the model theory of these harks back to results of Shelah from the late 1980s - excellent classes, then combined with quasiminimal classes and much more dramatically simplified - in some cases - by Bays, Hart, Hyttinen, Kesälä and Kirby). [Linguistic closure, homogeneity, uniqueness of generic, CC, alg. control.]

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- On the way to the previous, reduction of types of elliptic curves to the torsion information, readable by limits of N-covers on the field structure. A quite strong form of QE.
- A theorem by Keisler on the number of types of categorical sentences of $\mathcal{L}_{\omega_1,\omega}$... (this will give a surprising twist).

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Arithmetic Geometry:
An instance of the adelic Mumford-Tate conjecture for products of elliptic curves to show this. The strategy to build an isomorphism between two models M and M' consists (as expected) in

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- realizing the field type of a finite subset of a Hecke orbit over any parameter set (algebraicity of modular curves),...
- then show that the information in the type is contained in a finite subset ("Mumford-Tate" open image theorem used here) ... every point $\tau \in \mathbb{H}$ corresponds to an elliptic curve E — the type of τ is determined by algebraic relations between torsion points of E.

Keisler's Theorem, and its consequence in arithmetic geometry

Theorem (Keisler)

If an $\mathcal{L}_{\omega_1,\omega}$-sentence ψ is $\aleph_1$-categorical then the set of complete m-types realizable in models of ψ is at most countable.

This theorem in infinitary logic is now very classical (from the late 1960s). It has a surprising recent application (due to Harris) in the proof of the equivalence between the categoricity of the $\mathcal{L}_{\omega_1,\omega}$-theory $Th^{\infty}(j)$ and a statement about a group homomorphism having finite index:

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Theorem

If $Th^\infty_{SF}(\mathbf{p})$ is categorical, then the image of the homomorphism

$$Aut(\mathbb{C}/L) \to \overline{\Gamma}^m$$

Ideas/Translations/Questions to the geometers

1. Original context: Galois representation on the Tate module of an abelian variety A (limit of torsion points). Conjecturally, the image of such a Galois representation, which is an ℓ-adic Lie group for a given prime number ℓ, is determined by the corresponding Mumford–Tate group G (knowledge of G determines the Lie algebra of the Galois image).
2. Unfolding categoricity through the geometry seems to be the main question at this point - one that the Zilber school has pushed quite far but is still in its infancy.
3. Connection to properties of extendability of local sections to global sections (in sheaf cohomology)

Same picture, much more general

Generalizing a bit the previous (but the picture is the same):

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- S a modular curve: $\mathbb{H}/\Gamma$ where Γ is a "congruence subgroup" of $GL_2(\mathbb{Q})$,
- X^+ a set with an action of $G^{ad}(\mathbb{Q})^+$,
- $p : X^+ \to S(\mathbb{C})$ satisfies
  - (SF) Standard fibers,
  - (SP) Special points,
  - (M) Modularity.

Same picture, much more general / The Bogotá seminar

If any other map $q : X^{+} \to S(\mathbb{C})$ also satisfies SF, SP and M, then there exist a $G^{ad}(\mathbb{Q})^{+}$-equivariant bijection φ and $\sigma \in Aut(\mathbb{C})$ fixing the field of definition of S such that

$$\begin{array}{ccc} X^{+} & \xrightarrow{\varphi} & X^{+} \\ \downarrow{p} & & \downarrow{q} \\ S(\mathbb{C}) & \xrightarrow{\sigma} & S(\mathbb{C}) \end{array}$$

Differential context for j

j satisfies the following order three algebraic differential equation:

$$F(x, x', x'', x''') = Sx + R(x)(x')^2 = 0$$

where S is the Schwarzian

$$Sx = \frac{x'''}{x'} - \frac{3}{2}\left(\frac{x''}{x'}\right)^2$$

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Let

$$\mathcal{J}(K) := \{F(x, x', x'', x''') = 0\},$$

where $K = (K; +, \cdot, ', 0, 1)$ a differentiably closed field with field of constants C.

Ax-Schanuel for j and derivatives

Two elements a and b in a field are modularly independent if $\Phi_N(a, b) \neq 0$ for all $N > 0$ - Φ_N is the modular polynomial.

(for any $j_1, \cdots, j_n$ in $\mathcal{J}$, the transcendence degree of $C(\bar{j}, \bar{j}', \bar{j}'')$ over C is equal to the number of different Hecke orbits of $j_1, \cdots, j_n$.

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Theorem ('15 Pila-Tsimermann (Ax-Schanuel for j with derivatives))

Let $j_1, \cdots, j_n$ be pairwise modularly independent elements of $\mathcal{J}$. Then, $j_1, \cdots, j_n, j_1', \cdots, j_n', j_1'', \cdots, j_n''$ are algebraically independent over C.

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Freitag-Scanlon / Aslanyan

In 2014, Freitag and Scanlon proved that $\mathcal{J}$ is strongly minimal and has trivial geometry. Their proof used Nishioka and a criterion of superstability due to Shelah.
In 2016, Aslanyan simplified the proof by relying on the Pila-Tsimermann result.

Freitag-Scanlon / Aslanyan (sketch of proof)

Want: any definable subset of $\mathcal{J}$ is finite or cofinite. Let $U \subset \mathcal{J}$ be definable. By stable embedding, U is definable with parameters from $\mathcal{J}$ - let $A = \{a_1, \cdots, a_n\} \subset \mathcal{J}$ be such that $U = \varphi(F, a_1, \cdots, a_n)$.

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So, $\mathrm{tp}(j/A)$ is determined by

$$F(j, j', j'', j''') = 0$$

together with

$$\{P(j, j', j'') \neq 0 \mid P(X, Y, Z) \in \mathbb{Q}(\bar{a}, \bar{a}', \bar{a}'')[X, Y, Z]\}.$$

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This gives the strong minimality.
The triviality: using the same argument, if $A \subset \mathcal{J}$ (finite) and $j \in \mathcal{J} \cup \mathrm{acl}(A)$ then there is $a \in A$ such that $j \in \mathrm{acl}(a)$.
The failure of $\aleph_0$-categoricity: if $j \in \mathcal{J}$ then for each $w \in K$, if $\Phi_N(j, w) = 0$ then $w \in \mathcal{J}$. $\mathcal{J}$ therefore realizes uncountably many types over j.

Now,

to quantum versions of j

Threading finer on the definition of classical j:

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(This is also a Riemann surface.)

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(This is also a Riemann surface.)
Now, $\mathbb{T}(\mu)$ is equivalent to the elliptic curve $\mathbb{E}(\mu)$ given by $Y^2 = X^3 - g_2(\mu)X - g_3(\mu)$. Here, let

$$G_k(\mu) := \sum_{0 \neq \gamma \in \Lambda(\mu)} \gamma^{-2k} \qquad k \geq 2$$

(the so-called Eisenstein series), then

$$g_2(\mu) = 60 \cdot G_2(\mu)$$

$$g_3(\mu) = 140 \cdot G_3(\mu).$$

Toward quantum tori: from $\mathbb{C}$ to $\mathbb{R}$

Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$, and let Λ_θ be the pseudo-lattice $\langle 1, \theta \rangle$ (the subgroup of $\mathbb{R}$ given as $\Lambda_\theta := \mathbb{Z} + \mathbb{Z} \cdot \theta$). The quotient

$$\mathbb{T}(\theta) := \mathbb{R}/\Lambda_\theta$$

is the “quantum torus”, associated to the irrational number θ. It is a one-parameter subgroup of the (classical) torus $\mathbb{T}(i)$... and also a Riemann surface.

Getting hold of quantum versions of j

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- New definition domain (from $\mathbb{H}$ to $\mathbb{R} \setminus \mathbb{Q}$)

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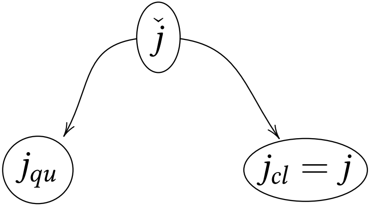
- New definition domain (from $\mathbb{H}$ to $\mathbb{R} \setminus \mathbb{Q}$)
- Topological issues resulting from the much more chaotic behavior of $\mathbb{R}$ - continuity lost in first approximations
- Rational expressions (multivalued functions now - perhaps the average of the (finite?) set of values is the robust invariant).

An example of a sheaf construction / universal j

Gendron proposes a detailed construction of a sheaf over a topological space, and a generalization of classical j called "universal j-invariant" - a specific section of a sheaf.



The specific construction of universal j

(Castaño-Bernard, Gendron)
Let ${}^*\mathbb{Z} := \mathbb{Z}^{\mathbb{N}}/\mathfrak{u}$ for some nonprincipal ultrafilter $\mathfrak{u}$ on $\mathbb{N}$. Define

$$\mathcal{H} := \{[F_i] \subset {}^*\mathbb{Z}^2 \text{ hyperfinite } \}.$$

This set is partially ordered with respect to inclusion so we may consider the Stone space

$$\mathcal{R} := \mathsf{Ult}(\mathcal{H}).$$

For each $\mathfrak{p} \in \mathcal{R}$ and $\mu \in \mathbb{H}$ one may define the j-invariant

$$j(\mu, \mathfrak{p})$$

as follows:

The construction

The idea: the classical j-invariant is an algebraic expression involving Eisenstein series which is a function of $\mu \in \mathbb{H}$. We can associate to $[F_i] \subset {}^*\mathbb{Z}^2$ a hyperfinite sum modelled on the formula of the classical j-invariant, denoted

$$j(\mu)_{[F_i]} \in {}^*\mathbb{C}.$$

We get a net

$$\{j(\mu)_{[F_i]}\}_{[F_i]\in\mathcal{H}} \subset {}^*\mathbb{C}.$$

Consider the sheaf ${}^\diamond\check{\mathbb{C}} \to \mathcal{R}$ for which the stalk over $\mathfrak{p}$ is

$${}^\diamond\mathbb{C}_{\mathfrak{p}} := ({}^*\mathbb{C})^{\mathcal{H}}/\mathfrak{p}.$$

Then we may define a section:

$$\check{j} : \mathbb{H} \times \mathcal{R} \longrightarrow {}^\diamond\check{\mathbb{C}}, \quad \check{j}(\mu, \mathfrak{p}) := \{j(\mu)_{[F_i]}\}_{[F_i]\in\mathcal{H}}/\mathfrak{p}.$$

Group actions - choosing an irrational angle

What really is at stake in these constructions is the invariance under various group actions.
For each $\theta \in \mathbb{R}$ there is a distinguished subset $\mathcal{R}_\theta \subset \mathcal{R}$ of ultrafilters which "see" θ:

$$\mathcal{R}_\theta = \{\mathfrak{p} | \ \mathfrak{p} \supset \mathfrak{c}_\theta\}$$

where $\mathfrak{c}_\theta$ is the cone filter generated by the cones

$$\text{cone}_\theta([F_i]) = \{[F_i]' \supset [F_i] | \ [F_i] \subset [F_i]' \subset {}^*\mathbb{Z}^2(\theta)\}.$$

In the above,

$${}^*\mathbb{Z}^2(\theta) = \{({}^*n^\perp, {}^*n) | \ {}^*n\theta - {}^*n^\perp \simeq 0\}.$$

Restricting to quantum and classical j

The quantum j-invariant is defined as the restriction:

$$\check{j}^{\mathsf{qu}}(\theta) := \check{j}|_{\mathcal{R}_\theta}(i, \cdot).$$

If we denote

$$\mathcal{R}_{\mathsf{cl}} = \{\mathfrak{p} |\ \mathfrak{p} \supset \mathfrak{c}\}$$

where $\mathfrak{c}$ is the filter generated by *all* cones over hyperfinite sets in ${}^*\mathbb{Z}^2$:

$$\mathsf{cone}([F_i]) = \{[F_i]' \supset [F_i] |\ [F_i]' \subset {}^*\mathbb{Z}^2\}.$$

Then the restriction

$$\check{j}^{\mathsf{cl}} := \check{j}|_{\mathcal{R}_{\mathsf{cl}}}$$

satisfies

$$\check{j}^{\mathsf{cl}}(\mu, \mathfrak{p}) \simeq j(\mu), \quad \forall \mu \in \mathbb{H},$$

where j is the usual j-invariant.

Duality I

Note the duality in the way of recovering the classical and quantum invariants:

- the classical invariant is recovered along a unique fiber ${}^{\diamond}\check{\mathbb{H}}_{\mathfrak{u}}$ (i.e., a leaf of the quotient of sheaves $\widehat{Mod}$),
- the quantum invariant is obtained by fixing the fiber parameter $i \in \mathbb{H}$ and letting $\mathfrak{u} \in Cone(\theta)$ vary: it therefore arises from a local section defined by i (a transversal of $\widehat{Mod}$).

Conjectures

The main goal is to check that if $\theta \in \mathbb{R} \setminus \mathbb{Q}$ is quadratic, then Hilbert's classfield H_K of $K = \mathbb{Q}(\theta)$ (K's maximal unramified extension) equals

$$K(j(\theta)).$$

This would give a solution to Hilbert's 12th problem for quadratic real extension (unramified case).
Again, the main point is to prove the analog of "complex multiplication" (keypoint: the algebraicity of $j(\mu)$, when $\mu \in \mathbb{Q}(\sqrt{D})$, for $D < 0$ square free - and the fact that $j(\mu)$ essentially generates the Hilbert classfield $H(\mu)$ of $\mathbb{Q}(\sqrt{D})$).
We conjecture (with Gendron) that for $\theta \in \mathbb{R}$ there exists a duality relation between the classical invariant $j(i\theta)$ and the quantum invariant $j(\theta)$.

Duality II

More precisely, we associate to $j(i\theta)$ and $j(\theta)$ two nets

$$\{j(i\theta)_\alpha\} \qquad \text{and} \qquad \{j(\theta)_\alpha\}$$

whose elements are algebraically interdependent. The two nets converge to a common limit. The classical net $\{j(i\theta)_\alpha\}$ lives along a fixed leaf of ${}^\diamond\widehat{Mod}$; the quantum net$\{j(\theta)_\alpha\}$ lives on a fixed transversal of ${}^\diamond\widehat{Mod}$.

- Building a quantum version of j (this is already done by Castaño-Bernard and Gendron - recent experiments for some values yield multivalued situations),

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- Building a quantum version of j (this is already done by Castaño-Bernard and Gendron - recent experiments for some values yield multivalued situations),
- Proving algebraicity of this new j function, for as many cases as possible.

Parallel to other themes

There are strong parallels with at least two other themes:

- Computing the values of series in non-standard models of arithmetics (or fields...) is also carried out (different details but parallel idea) by Åsa Hirvonen and Tapani Hyttinen in their analysis of Zilber's computations of Quantum Harmonic Oscillator and Quantum Free Particle.

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- The sheaf-theoretic analysis is useful according to Zilber for "Quantum Weyl Algebras" - limits of Zariski geometries and for $\mathbb{F}_1$-geometry (Cruz-Shaheen-Zilber).

New directions (with Zilber)

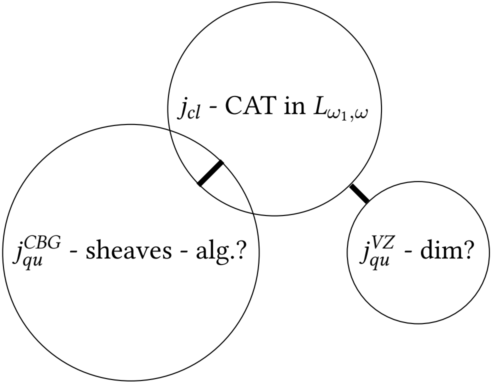
The construction of a smooth

$$(\mathbb{C}, \circlearrowright_{SL_2(\mathbb{Q})}) \xrightarrow{\mathbb{j}} (\mathbb{C}, +, \cdot)$$

by merging the Harris context with ideas inspired by Castaño-Bernard—Gendron - controlled by the predimension function arising from the Ax-Schanuel result.

Through the sheaves

The current model theoretic analysis of j looks at two possible extensions:



Two different toolkits

The two directions of generalization (to quantum tori/real multiplication on the one hand, to higher dimension varieties/Shimura on the other hand) of j maps calls different aspects of model theory (at the moment, an "amalgam" has started, but is far along the way):

- The model theory of abstract elementary classes (in particular, the theory of excellence - now "old" (1980s) but recently (2013) streamlined by Bays, Hart, Hyttinen, Kesälä, Kirby - Quasiminimal Structures and Excellence. A kind of cohomological analysis of models (and types), connected to categoricity and "smoothness". Zilber field, now j!

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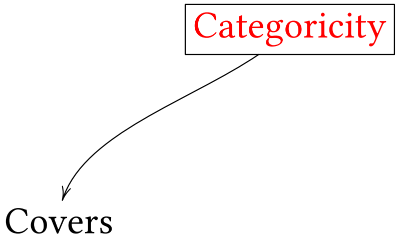
- The model theory of sheaves. A tool for (topological) "limit" structures.

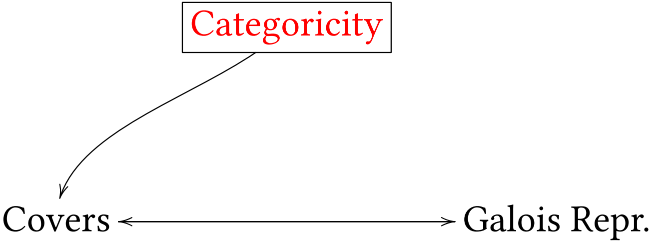
Categoricity / covers / Galois theory / ...

Categoricity

Categoricity / covers / Galois theory / ...



Categoricity / covers / Galois theory / ...



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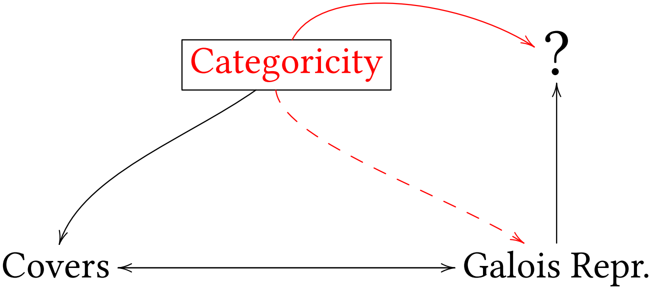

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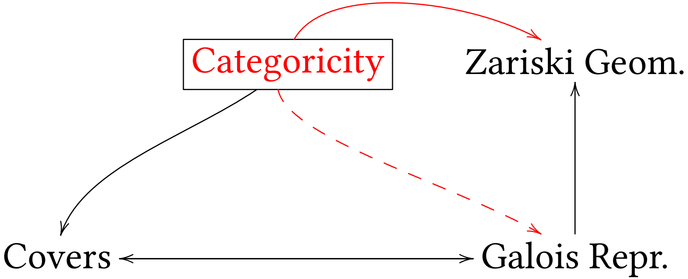
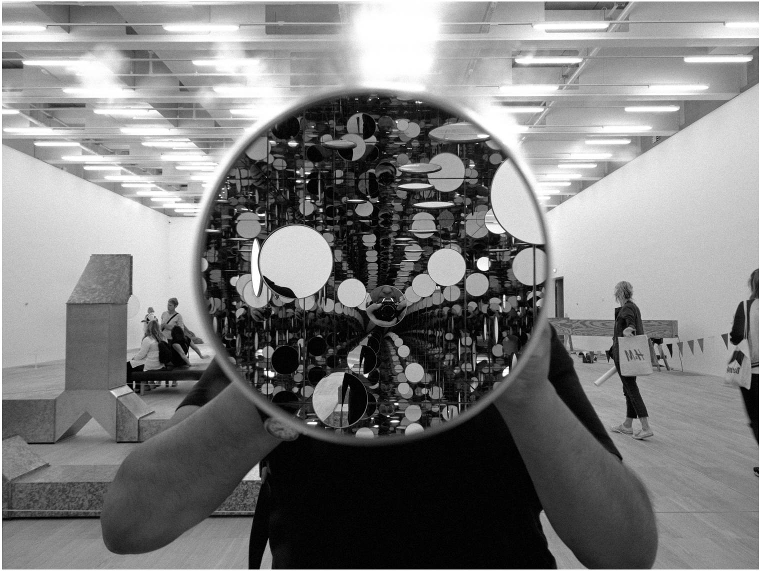


A general framework for modular functions

Summarizing interactions:

Mumford-Tate	Classical $j : \mathbb{H} \to \mathbb{C}$	Harris	Categoricity
Algebraicity Conjecture	Quantum $j : \mathbb{R} \to \mathbb{C}$	Gendron	Sheaf Models and Limits
Functional	Uniform $\mathbb{j} : (\mathbb{C}, \circlearrowright_{SL_2(\mathbb{Q})}) \to (\mathbb{C}, +, \cdot)$	V., Zilber	Sheaf Categoricity
Mumford-Tate	Shimura curves modular curves	Daw, Harris	Categoricity
?	Other uniformizing (*)	Cano, Plazas, V.	Categoricity



(merci de votre attention!)