Reflection Principles &

Abstract Elementary Classes

Andrés Villaveces
Universidad Nacional de Colombia - Bogotá

CRM - Workshop on Set-Theoretical Aspects of the Model Theory of Strong Logics, September 2016

Contents

Useful Axioms...

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- The Axiom of Choice
- The Baire Category Theorem
- Large Cardinals
- Shoenfield's Absoluteness Theorem
- The Łoś Theorem for Ultrapowers

And Even More Useful Axioms...

As if this list wasn't long enough (and didn't cover enough mathematics) I will make it even longer:

- Omitting Types (Boney's analysis of Large Cardinals, yesterday)
- Reflection Properties (really, last week's Magidor and Väänänen's minicourse)

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- Tameness!

What last week was used to study quantifiers in strong logics and earlier this week was used to study generic absoluteness is not that faraway from a property that has been used to gauge the consistency strength of Shelah's Categoricity Conjecture.

Tameness in Model Theory - long story short

Shelah's Categoricity Conjecture

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- "Semantic versions" of the model theory of $L_{\lambda^+,\omega}(Q)$.

Conjecture (Shelah - around 1980)

For every λ, there exists μ_λ such that if $\mathcal{K}$ is an AEC with $LS(\mathcal{K}) = \lambda$, categorical in some cardinality $\geq \mu_\lambda$, then $\mathcal{K}$ is categorical in all cardinalities greater than μ_λ.

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- This usually requires the development of stability theory (in some cases quite involved), so:

Proving categoricity transfer not only reveals a strong form of "semantic completeness" of the class $\mathcal{K}$ but also involves understanding deeply how models are embedded into one another and how types p are controlled by small "projections" $p \upharpoonright M$.

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- Boney (2013:) consistency of the full conjecture, under a proper class of strongly compact cardinals.
- Vasey (2016): the categoricity conjecture holds (in ZFC) for universal classes!

"Weakenings" of categoricity: superstability, NIP

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- Uniqueness of "limit" models as a form of superstability (my early joing work with Shelah, c. 1999, then more recent work with Grossberg and VanDieren on uniqueness, and with Zambrano in the continuous case),
- Other dividing lines from FO Model Theory "extend" to a (more tenuous, but more structural) "Classification Theory for AECs": NIP for example is really connected to the Genericity Pair Conjecture, a statement on the behaviour of a large groupoid of partial isomorphisms of homogeneous structures in a class - or with (again) uniqueness of pairs of models approximating a saturated model along arbitrary club sets...

Grossberg-VanDieren: tameness isolated

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Their proof built on a previous proof of the "downward" transfer by Shelah but has a crucial element: isolating the notion of tameness ("buried" in Shelah's proof of the downward part - fleshing out the notion allows Grossberg/VanDieren to prove the upward categoricity).

Localizing difference

Idea: "localizing" the condition of...
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- we want: to localize this to checking that there is some $M_0 \in \mathcal{P}^*_\kappa(M)$ and $X_0 \in \mathcal{P}_\kappa(N_0)$ such that

$$\mathrm{gatp}(X_0/M_0) \neq \mathrm{gatp}(f(X_0)/M_0)$$

Tameness and type-shortness

Definition ((κ, λ)-tameness for μ, type shortness)

Let $\kappa < \lambda$. An aec $\mathcal{K}$ with AP and $LS(\mathcal{K}) \leq \kappa$ is

- (κ, λ)-tame for sequences of length μ if for every $M \in \mathcal{K}$ of size λ, if $p_1 \neq p_2$ are Galois types over M then there exists $M_0 \prec_{\mathcal{K}} M$ with $|M_0| \leq \kappa$ such that

$$p_1 \restriction M_0 \neq p_2 \restriction M_0$$

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- (κ, λ)-typeshort over models of cardinality μ if for every $M \in \mathcal{K}$ of size μ, if $p_1 \neq p_2$ are Galois types over M and $p_i = \mathrm{gatp}(X_i/M)$ where $X_i = (x_{i,\alpha})_{\alpha<\lambda}$, there exists $I \subset \lambda$ of cardinality $\leq \kappa$ such that $p_1^I \neq p_2^I$:

$$gatp((x_{1,\alpha})_{\alpha \in I}/M) \neq gatp((x_{2,\alpha})_{\alpha \in I}/M).$$

Dual notions - stability

The two notions are clearly dual (parameters/realizations):

- In tameness, a narrow orbit (fixing large models) is controlled by the thicker orbits that approximate it (parameter locality),

These dualities are equivalences under stability conditions. In general, they are not.

Dual notions - stability

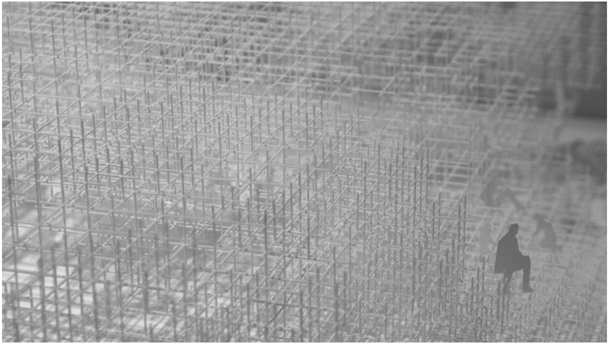
The two notions are clearly dual (parameters/realizations):

- In tameness, a narrow orbit (fixing large models) is controlled by the thicker orbits that approximate it (parameter locality),
- In type shortness, the orbit of a long sequence is controlled by the narrower orbits of its subsequences (realization locality)...

These dualities are equivalences under stability conditions. In general, they are not.

Large Cardinals & Model Theory

La double vie des grands cardinaux



Getting Tameness from Large Cardinals

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Theorem (Boney)

If κ is strongly compact and $\mathcal{K}$ is essentially below κ (i.e. $LS(\mathcal{K}) < \kappa$ or $\mathcal{K} = Mod(\psi)$ for some $L_{\kappa,\omega}$-sentence ψ) then $\mathcal{K}$ is $(< (\kappa + LS(K)^{+}, \lambda$-tame and $(< \kappa, \lambda)$-typeshort for all λ.

Boney and Unger proved (2015) that under strong inaccessibility of κ, the $(< \kappa, \kappa)$-tameness of all aecs implies κ's strong compactness.

Reframing slightly Boney's proof

For later applications, in joint work with Camilo Arosemena we have slightly reframed Boney's proof.
Remember (from yesterday, the day before, the day before...)

- A cardinal κ is strongly compact iff for every $\lambda > \kappa$ there exists an elementary embedding $j : V \to M$ with critical point κ, and there exists some $Y \in M$ such that $j''\lambda \subset Y$ and $|Y|^M < j(\kappa)$.

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Definition
Let $j : V \to M$ be an elementary embedding. j has the (κ, λ)-cover property if for every X with $|X| \leq \lambda$ there exists $Y \in M$ such that $j''X \subset Y \subset j(X)$ and $|Y|^M < j(\kappa)$.

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For example, for a measurable cardinal κ, the usual embedding j has the (κ, κ)-cover property. If κ is λ-strongly compact, and U is a fine κ-complete ultrafilter on $P_\kappa(\lambda)$ then the associated j has the (κ, λ)-cover property.

The "image" of an AEC under $j : V \to M$

Let in general $(\mathcal{K}, \prec_{\mathcal{K}})$ be an AEC in τ.
Shelah's Presentation Theorem gives

- $\tau' \supset \tau$,
- T' a τ'-theory and
- Γ' a set of T'-types

such that

$$\mathcal{K} = PC(\tau, T', \Gamma') = \{M' \restriction \tau \mid M' \models T' \text{ and } M' \text{ omits } \Gamma'\},$$

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We define $j(\mathcal{K})$ as the class $PC^M(j(\tau), j(T'), j(\Gamma'))$.
By elementarity, $M \models j(\mathcal{K})$ is a an AEC with LS number equal to $j(LS(\mathcal{K}))$.
This can be done in a canonical way, by Baldwin-Boney (2016).

Attempt at getting $j(\mathcal{K}) \subset \mathcal{K}$ and $\prec_{j(\mathcal{K})} \subset \prec_{\mathcal{K}}$.

Definition

Let $\mathcal{M} \in \mathcal{K}$ (a τ-AEC). Then $j(\mathcal{M})$ is a $j(\tau)$-structure. We say that j respects $\mathcal{K}$ if the following conditions hold:

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- for every $\mathcal{M}, \mathcal{N} \in j(\mathcal{K})$, $\mathcal{M} \prec_{j(\mathcal{K})} \mathcal{N}$ implies $\mathcal{M} \upharpoonright \tau \prec_{\mathcal{K}} \mathcal{N} \upharpoonright \tau$,
- for every $\mathcal{M} \in \mathcal{K}$, $j''\mathcal{M} \prec_{\mathcal{K}} j(\mathcal{M}) \upharpoonright \tau$.

Examples

1. Let first $j : V \to M$ be a nontrivial elementary embedding with critical point κ and let $\mathcal{K}$ be an AEC with $LS(\mathcal{K}) < \kappa$. Then $\mathcal{K} = PC(\tau', T', \Gamma')$, with $|\tau'| + |T'| + |\Gamma'| < \kappa$; wlog we can assume $\tau', T', \Gamma' \in V_\kappa$ and therefore

$$j(\mathcal{K}) = PC^M(\tau, T', \Gamma') = (\mathcal{K} \cap M, \prec_{\mathcal{K}} \cap M),$$

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   (we have to use correctness of $\models$). Clearly, j respects $\mathcal{K}$.

2. $\mathcal{K}$ is given as $Mod(\varphi)$ for φ in $L_{\kappa,\omega}$, with $\prec_{\mathcal{K}} = \subset^{TV}_{\mathcal{F}}$, $\mathcal{F}$ some fragment of $L_{\kappa,\omega}$. Then j respects $\mathcal{K}$.

Getting Tameness

We prove then that whenever $\mathcal{K}$ is an AEC with $LS(\mathcal{K}) < \kappa < \lambda$, and $j : V \to M$ has the (κ, λ)-cover property and respects $\mathcal{K}$ then $\mathcal{K}$ is $(< \kappa, \lambda)$-tame.

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Let $\mathcal{M} \in \mathcal{K}_\lambda$ and $p_1 = \mathrm{gatp}(\vec{a}/\mathcal{M}, \mathcal{N}_1)$, $p_2 = \mathrm{gatp}(\vec{b}/\mathcal{M}, \mathcal{N}_2)$ be two types such that for every $\mathcal{N} \prec_{\mathcal{K}} \mathcal{M}$ of size $< \kappa$ we have

$$p_1 \restriction \mathcal{N} = p_2 \restriction \mathcal{N}.$$

(Here, $\vec{a} = (a_i)_{i \in I}$, $\vec{b} = (b_i)_{i \in I}$.)

Getting Tameness

Let now $Y \in M$ by such that $j''|\mathcal{M}| \subset Y \subset j(|\mathcal{M}|)$ and $|Y|^M < j(\kappa)$. But in M, $LS(j(\mathcal{K})) = j(LS(\mathcal{K})) < j(\kappa)$ so there is $\mathcal{M}' \in j(\mathcal{K})$ such that $Y \subset |\mathcal{M}'|$, $\|\mathcal{M}'\| < j(\kappa)$ and $\mathcal{M}' \prec_{j(\mathcal{K})} j(\mathcal{M})$; by transitivity, $\mathcal{M}' \prec_{j(\mathcal{K})} j(\mathcal{M})$.

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By elementarity, $M \models j(p_1) \restriction \mathcal{M}' = j(p_2) \restriction \mathcal{M}'$ (in $j(\mathcal{K})$)
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$$\begin{aligned} p_1' &= \mathrm{gatp}(j(\vec{a})/\mathcal{M}' \upharpoonright \tau, j(\mathcal{N}_1) \upharpoonright \tau) \\ &= \mathrm{gatp}(j(\vec{b})/\mathcal{M}' \upharpoonright \tau, j(\mathcal{N}_2) \upharpoonright \tau) = p_2' \end{aligned}$$

in $\mathcal{K}$ (again by our hypothesis on j).

Getting Tameness

Since $j''\mathcal{M} \prec_{\mathcal{K}} j(\mathcal{M})$ we get that $j''\mathcal{M} \prec_{\mathcal{K}} \mathcal{M}' \upharpoonright \tau$ (coherence axiom), so restricting we have

$$\mathrm{gatp}(j(\vec{a})/j''\mathcal{M}, j''\mathcal{N}_1) = \mathrm{gatp}(j(\vec{b})/j''\mathcal{M}, j''\mathcal{N}_2).$$

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Restriction "above" we get

$$\mathrm{gatp}(j(\vec{a})/j''\mathcal{M}, j''\mathcal{N}_1) = \mathrm{gatp}(j(\vec{b}))/j''\mathcal{M}, j''\mathcal{N}_2),$$

and therefore

$$p = q. \qquad \square$$

Back to the Reflection Property

So, we use the λ-strong compactness of κ to show first that the embedding $j : V \to M$ has the (κ, λ)-property and respects $\mathcal{K}$ and then apply the previous. One may also show that the (κ, λ)-cover of $j : V \to M$ for $\kappa > LS(\mathcal{K})$ implies

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- $\mathcal{K}_{[\kappa,\lambda]}$ has no maximal models, and
- $\mathcal{K}_{[\kappa,\lambda]}$ has the amalgamation property (provided all models of $\mathcal{K}_\mu$ are $< \kappa$-universally closed for some $\mu \in [\kappa, \lambda]$).

Back to the Reflection Property

So, we use the λ-strong compactness of κ to show first that the embedding $j : V \to M$ has the (κ, λ)-property and respects $\mathcal{K}$ and then apply the previous. One may also show that the (κ, λ)-cover of $j : V \to M$ for $\kappa > LS(\mathcal{K})$ implies

- $\mathcal{K}_{[\kappa,\lambda]}$ has no maximal models, and
- $\mathcal{K}_{[\kappa,\lambda]}$ has the amalgamation property (provided all models of $\mathcal{K}_\mu$ are $< \kappa$-universally closed for some $\mu \in [\kappa, \lambda]$).

So, we are in a good position to use the Grossberg-VanDieren theorem to conclude the consistency of the Shelah Categoricity Conjecture.

Challenges for Set Theory?

Under a proper class of strongly compact cardinals, Boney showed that

$$\text{Every AEC } \mathcal{K} \text{ with arbitrarily large models is tame.} \tag{1}$$

(He gives weaker versions of tameness, obtained from proper classes of measurables and weakly compact cardinals.)
All this seems rather reducible to weaker large cardinals, at least for a lot of model theory!

Lower bounds

Notice that

$$\text{Every AEC } \mathcal{K} \text{ with } LS(\mathcal{K}) < \kappa \text{ is } (<\kappa, \kappa)\text{-tame} \tag{2}$$

already implies $V \neq L$: Baldwin and Shelah constructed a counterexample to $(<\kappa, \kappa)$ starting from an almost free, non-free, non-Whitehead group of cardinality κ. In L this may happen at any κ regular, not strongly compact.

On the other hand, Hart-Shelah's example of an $L_{\omega_1,\omega}$-sentence categorical in $\aleph_0, \aleph_1, \cdots, \aleph_k$ but NOT in $\aleph_{k+2}$ shows that pushing tameness FOR ALL aecs below $\aleph_\omega$ is impossible.

Collapsing and its limitations

Collapsing large cardinals while keeping some of their properties has a long history of interesting results. For instance,

- Mitchell: collapsed a weakly compact to $\aleph_2$ while keeping the tree property. This was later generalized (collapsing much more) in order to get the tree property at all the $\aleph_n$'s and/or in $\aleph_{\omega+1}$ (Magidor, Cummings, Neeman, Fontanella, etc.)

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- For the "strong tree" and "supertree" properties the consistency strength seems to be around a strongly compact / supercompact respectively. (Weiss, Viale, Fontanella, Magidor).

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- These are instances of general reflection/compactness properties. But so are tameness and type shortness.

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- The resulting classes $j(\mathcal{K})$ and (if $\mathcal{K} = PC(L, T', \Gamma')$ the classes $\mathcal{K}^{V[G]} = PC^{V[G]}(L, T', j(\Gamma'))$ exhibit interesting "residual tameness"...

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- ... but adapting Levy-collapse (Easton iteration) or the more sophisticated constructions mentioned cannot yield full tameness; only residual.

Envoi: Tameness as a geometric property?

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These are part of Matteo's list of last Monday (enhanced)

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- The Baire Category Theorem
- Large Cardinals
- Shoenfield's Absoluteness Theorem
- The Łoś Theorem for Ultrapowers
- Omitting Type Properties

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- **Local/Global glueing (of sheaves)**

How so?

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(Blackboard - time permitting! - The keyword is to use a "Grothendieck topology" - an abstraction of the notion of an "open cover" and show that the [extremely useful] glueing condition of presheaves is an instance of tameness.)

A dichotomic behavior

- Under Weak Diamond:

Theorem (from Sh88)

(Under $2^\kappa < 2^{\kappa^+}$). Every aec $\mathcal{K}$ with $LS(\mathcal{K}) \leq \kappa$, categorical in κ, failing AP for models of size κ has 2^{κ^+} many non-isomorphic models of cardinality κ^+.

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- Example under MA:
  (MA_{ω_1}) There is a class (axiomatizable in $L_{\omega_1,\omega}(Q)$) that is $\aleph_0$-categorical, fails AP in $\aleph_0$ and is also categorical in $\aleph_1$. This can be lifted below continuum.

Forcing isomorphism/categoricity

Theorem (Asperó, V.)

The existence of a weak AEC, categorical in both $\aleph_1$ and $\aleph_2$, failing AP in $\aleph_1$, is consistent with ZFC+CH+$2^{\aleph_1} = 2^{\aleph_2}$.

The result is obtained by an ω_3-iteration over a model of GCH, where we

- Start with GCH in V.
- Build a countable support iteration of length ω_3, where
- at each stage α of the iteration you consider in $V^{\mathbb{P}_\alpha}$ two models $M_0, M_1 \in \mathcal{K}$, $|M_0| = |M_1| = \aleph_2$ (use a bookkeeping function) and
- fix $(M_i^0)_{i<\omega_2}$, $(M_i^1)_{i<\omega_2}$ resolutions of the two models with $M_i^\varepsilon = N_i \cap M_\varepsilon$ where $(N_i)_{i<\omega_2}$ is an $\in$-increasing and $\subset$-continuous of elementary substructures of some $H(\theta)$ of size $\aleph_1$ containing M_0 and M_1...

Forcing isomorphism/categoricity

- at this stage iterate with $\mathbb{Q}_\alpha$ the partial order consisting of countable partial isomorphisms p between M_0 and M_1 such that if $x \in \mathrm{dom}(p)$ and i is the minimum such that $x \in M_i^0$ then $p(x) \in M_i^1$.
- Each stage $\mathbb{Q}_\alpha$ of the iteration, and all the forcing $\mathbb{P}_{\omega_3}$ is σ-closed and $\mathbb{P}_{\omega_3}$ has the $(\aleph_2) - a.c.$ (need CH for the relevant (!) Δ-lemma).

Thank you for providing so many interesting discussions!