Around the Small Index Property (on *qm* classes)

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BGU Beersheva / HUJI Jerusalem - November 2016

Contents

A "classical enigma": reconstructing from symmetry.

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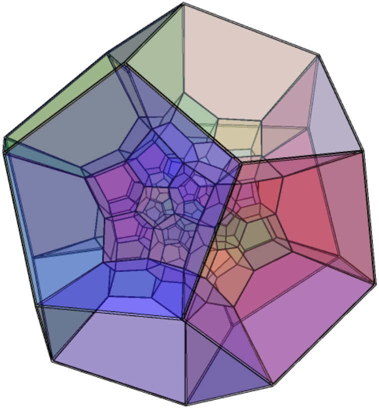

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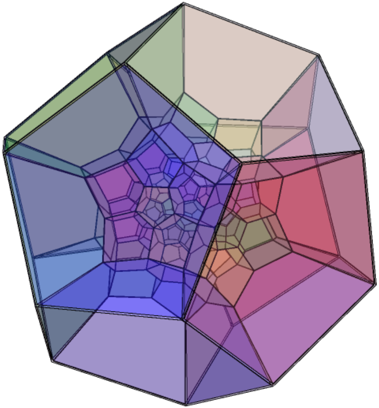

Tell me what is M!

Reconstructing models?

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- an even more reasonable question: if for some (FO) structure M we are given $Aut(M)$, when can we recover all models biinterpretable with M?
- we follow ONE line of reconstruction, different from the work of Rubin!

Where else in mathematics

The "naïve question" is quite important: What information about a model M and $Th(M)$ is contained in the group $Aut(M)$?
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These are versions of the same kind of problem - but we will not concentrate on these today. They may however be amenable to model theoretic treatment.

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- If M is $\aleph_0$-categorical, any open subgroup of $Aut(M)$ is a stabilizer $Aut_\alpha(M)$ for some imaginary α. Also $Aut(M) \curvearrowright \{H \leq Aut(M) \mid H \text{ open}\}$ (conjugation).

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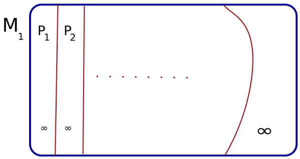
- The action $Aut(M) \curvearrowright$ is (almost) $\approx$ to $Aut(M) \curvearrowright M^{eq}$.

So, we have recovered the action of $Aut(M)$ on M^{eq} from the topology of $Aut(M)$... so, if M, N are countable $\aleph_0$-categorical structures, TFAE:

- There is a bicontinuous isomorphism from $Aut(M)$ onto $Aut(N)$
- M and N are bi-interpretable.

(Non-)Examples - Saturation needed

Let M_1 be the countable saturated model of P_i ($i < \omega$) disjoint infinite predicates and



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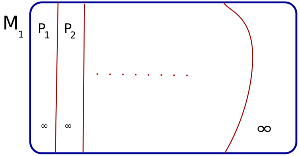
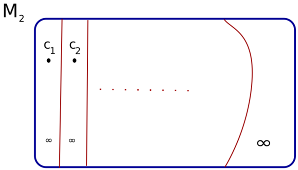


let M_2 a countable model of an equivalence relation with infinitely many infinite classes, with exactly one constant c_i in each class <u>but one</u>



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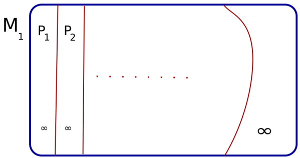

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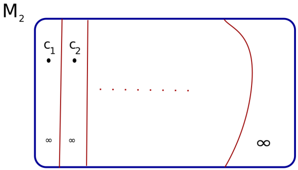

$$\text{yet } Aut(M_1) \approx Aut(M_2)$$

The Small Index Property (countable version)

Definition (Small Index Property - SIP)

Let M be a countable structure. M has the small index property if for any subgroup H of $Aut(M)$ of index less than $2^{\aleph_0}$, there exists a finite set $A \subset M$ such that $Aut_A(M) \subset H$.

Basic facts on countable SIP

SIP allows us to recover the topological structure of $Aut(M)$ from its pure group structure:
Open neighborhoods of 1 in pointwise convergence topology $=$
Subgroups containing pointwise stabilizers $Aut_A(M)$ for some finite A.

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SIP allows us to recover the topological structure of $Aut(M)$ from its pure group structure:
Open neighborhoods of 1 in pointwise convergence topology = Subgroups containing pointwise stabilizers $Aut_A(M)$ for some finite A.

- SIP holds for random graph, infinite set, DLO, vector spaces over finite fields, generic relational structures, $\aleph_0$-categorical $\aleph_0$-stable structures, etc.
- It fails e.g. for $M \models ACF_0$ with ∞ transc. degree.

To the uncountable / the non-elementary

SIP FOR UNCOUNTABLE STRUCTURES

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$Aut(M)$ with this topology is of course no longer a Polish space. The techniques from Descriptive Set Theory that have been used for the countable case need to be replaced (Friedman, Hyttinen and Kulikov have a start of Descriptive Set Theory for some uncountable cardinalities, however).

Lascar-Shelah's Theorem

Theorem (Lascar-Shelah: Uncountable saturated models have the SIP)

Let M be saturated, of cardinality $\lambda = \lambda^{<\lambda}$ and let G be a subgroup of $Aut(M)$ such that $[Aut(M) : G] < 2^{\lambda}$. Then there exists $A \subset M$ with $|A| < \lambda$ such that $Aut_A(M) \subset G$.

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The proof consists of building directly (assuming that G does not contain any open set $Aut_A(M)$ around the identity) a **binary tree** of height λ of automorphisms of M in such a way that every two of them are not conjugate. This is enough but requires two crucial notions: **generic** and **existentially closed (sequences of) automorphisms**. These are obtained by assuming that G is not open.

Beyond First Order

Although results on the reconstruction problem, so far have been stated and proved for saturated models in first order theories, the scope of the matter can go far beyond:

- Abstract Elementary Classes with well-behaved closure notions, and the particular case:

Beyond First Order

Although results on the reconstruction problem, so far have been stated and proved for saturated models in first order theories, the scope of the matter can go far beyond:

- Abstract Elementary Classes with well-behaved closure notions, and the particular case:
- Quasiminimal (qm excellent) Classes.

The setting: Strong amalgamation classes

A setting for homogeneity: let $(\mathcal{K}, \prec_{\mathcal{K}})$ be an AEC, with $LS(\mathcal{K}) \leq \lambda$, $|M| = \kappa > \lambda$, $\kappa^{<\kappa} = \kappa$.
Let $\mathcal{K}^{<}(M) := \{N : N \preccurlyeq_K M, |N| < \kappa\}$ and fix $M \in \mathcal{K}$ homogeneous.

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The topology τ^{cl}: base of open neighborhoods given by sets of the form $Aut_X(M)$ where $X \in \mathcal{C}$, where $\mathcal{C} := \left\{ \mathrm{cl}^M(A) : A \subseteq M \text{ such that } |A| < \kappa \right\}$ and the "closure operator" is $cl^M(A) := \bigcap_{A \subset N \prec_{\mathcal{K}} M} A$.
This class of cl^M-closed sets has enough structure for the proof of SIP.

The main result: SIP for homogeneous AEC.

Theorem (SIP for $(Aut(M), \mathcal{T}^{cl})$ - Ghadernezhad, V.)

Let M be a homogeneous model in an AEC $(\mathcal{K}, \prec_{\mathcal{K}})$, with $|M| = \lambda = \lambda^{<\lambda} > LS(\mathcal{K})$, such that $\mathcal{K}^{<\lambda}$ is a strong amalgamation class. Let $G \leq Aut(M)$ with $[Aut(M) : G] \leq \lambda$ (this is, G has small index in $Aut(M)$). Then there exists $X \in \mathcal{C}$ such that $Aut_X(M) \leq G$ (i.e., G is open in $(Aut(M), \mathcal{T}^{cl})$).

Getting many non-conjugates

Proof (rough sketch): suppose G has small index in $Aut(M)$ but is not open (does not contain any basic $Aut_X(M)$ for $X \in \mathcal{C}$.
We have enough tools (generic sequences and strong amalgamation bases) to build a Lascar-Shelah tree to reach a contradiction (2^λ many branches giving automorphisms of M g_σ for $\sigma \in 2^\lambda$ such that if $\sigma \neq \tau \in 2^\lambda$ then $g_\sigma^{-1} \circ g_\tau \notin G$).
Of course, the possibility of getting these 2^λ-many automorphisms requires using the non-openness of G to get the construction going.

Lascar-Shelah tree for our situation

A λ-Lascar-Shelah tree for M and $G \leq Aut(M)$ is a binary tree of height λ with, for each $s \in 2^{<\lambda}$, a model $M_s \in \mathcal{K}^{<}(M)$, $g_s \in Aut(M_s)$, $h_s, k_s \in Aut_{M_s}(M)$ such that

- $h_{s,0} \in G$ and $h_{s,1} \notin G$ for all $s \in \mathcal{S}$;
- $k_{s,0} = k_{s,1}$ for all $s \in \mathcal{S}$;
- for $s \in \mathcal{S}$ and all $t \in \mathcal{S}$ such that $t \leqslant s$:$h_t\left[M_s\right] = M_s$ (i.e. $h_t \in Aut_{\{M_s\}}\left(M\right)$) and ...;
- for $s \in \mathcal{S}$ and all $t \in \mathcal{S}$ such that $t \leqslant s$: $g_s \cdot (h_t \restriction M_s) \cdot g_s^{-1} = k_t \restriction M_s$;
- for $s \in \mathcal{S}$ and $\beta <$ length$\,(s)$: $a_s \in M_s$;
- for all s, the families $\{h_t : t \leqslant s, t \in \mathcal{S}\}$ and $\{k_t : t \leqslant s, t \in \mathcal{S}\}$ are elements of $\mathcal{F}$ (i.e. they are generic).

Generic sequences and Strong amalgamation bases

The main technical tools in the construction of a LS tree are

- Guaranteeing generic sequences of automorphisms
  ($g \in Aut(M)$ is generic if
  $\forall N \in \mathcal{K}^{<}(M)$ such that $g \upharpoonright N \in Aut(N)$
  $\forall N_1 \succ_{\mathcal{K}} N, N_1 \in \mathcal{K}^{<}(M)$
  $\forall h \supset g \upharpoonright N, h \in Aut(N_1)$
  $\exists \alpha \in Aut_N(M)$ such that $g \supset \alpha \circ h \circ \alpha^{-1}$),
- showing they are unique up to conjugation,
- getting a generic sequence $\mathcal{F} = (g_i : i \in I)$ such that
  1. the set $\{i \in I : g_i \upharpoonright M_0 = h \text{ and } g_i \notin G\}$ has cardinality κ for all $M_0 \in \mathcal{K}^{<}(M)$ and $h \in Aut(M_0)$;
  2. the set $\{i \in I : g_i \in G\}$ has cardinality κ.

Another way to get generics: aut-independence

Definition

Let $A, B, C \in \mathcal{C}$. Define $A \mathop{\smile\!\!\!\!\!\vert}_B^a C$ if for all $f_1 \in Aut(A)$ and all $f_2 \in Aut(C)$ and for all $h_i \in \mathcal{O}_{f_i}$ $(i = 1, 2)$ such that $h_1 \restriction A \cap C = h_2 \restriction A \cap C$ and $h_1 \restriction B = h_2 \restriction B$ then $\mathcal{O}_g \neq \emptyset$ where $g := f_1 \cup f_2 \cup h_1 \restriction B$.

Definition

Let $A, B, C \in \mathcal{C}$. Define $A \mathop{\smile\!\!\!\!\!\vert}_B^{a-s} C$ if and only if $A' \mathop{\smile\!\!\!\!\!\vert}_B^a C'$ for all $A' \subseteq A$ and $B' \subseteq B$ with $A', B' \in \mathcal{C}$.

Fact

$\mathop{\smile\!\!\!\!\!\vert}^{a-s}$ *satisfies symmetry, monotonicity and invariance.*

Free $\mathop{\smile\!\!\!\!\!\mid}_B^{a-s}$ C-amalgamation.

The class $\mathcal{C}$ has the free $\mathop{\smile\!\!\!\!\!\mid}^{a-s}$-amalgamation property if for all $A, B, C \in \mathcal{C}$ with $A \cap B = C$ there exists $B' \in \mathcal{C}$ such that $ga - tp\,(B'/C) = ga - tp\,(B/C)$ (or there exists $g \in Aut_C\,(M)$ that $g\,[B] = B'$) and $A \mathop{\smile\!\!\!\!\!\mid}_C^{a-s} B'$.

Fact

Suppose $\mathcal{C}$ has the free $\mathop{\smile\!\!\!\!\!\mid}^{a-s}$-amalgamation property. Then generic automorphisms exist.

Quasiminimal pregeometry classes

In a language L, a quasiminimal pregeometry class $\mathcal{Q}$ is a class of pairs $\langle H, \mathrm{cl}_H \rangle$ where H is an L-structure, cl_H is a pregeometry operator on H such that the following conditions hold:

1. Closed under isomorphisms,
2. For each $\langle H, \mathrm{cl}_H \rangle \in \mathcal{Q}$, the closure of any finite set is countable.
3. If $\langle H, \mathrm{cl}_H \rangle \in \mathcal{Q}$ and $X \subseteq H$, then $\langle \mathrm{cl}_H(X), \mathrm{cl}_H \restriction \mathrm{cl}_H(X) \rangle \in \mathcal{Q}$.
4. If $\langle H, \mathrm{cl}_H \rangle, \langle H', \mathrm{cl}_{H'} \rangle \in \mathcal{Q}$, $X \subseteq H$, $y \in H$ and $f : H \to H'$ is a partial embedding defined on $X \cup \{y\}$, then $y \in \mathrm{cl}_H(X)$ if and only if $f(y) \in \mathrm{cl}_{H'}(f(X))$.
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These can all be generated by ONE canonical structure.

Very recent updates

- In November 2016 - about a week ago, Sébastien Vasey has posted a paper on the ArXiV proving that quasiminimal pregeometries do not require the exchange axiom of pregeometries. This makes it in principle easier to prove that classes are quasiminimal!
- Vasey has also suggested that our theorem applies to wider classes (excellent classes, and even wider: certain “non-forking frames”). This is work in progress now.

Example: quasiminimal classes, "Zilber field"

- $\mathcal{Q}$ quasiminimal pregeometry class. $M \in \mathcal{Q}$ of size $\aleph_1$, $\mathcal{C} = \{cl(A) \mid A \subset M, A \text{ small}\}$ then $\mathcal{C}$ has the free aut-independence amalgamation property. (Based on Haykazyan's paper on qm classes.)

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- $\mathcal{Q}$ qm pregeom. class $\rightarrow$ for every model M of $\mathcal{Q}$, $Aut(M)$ has SIP,
- The "Zilber field" has SIP.
- The j-invariant has the SIP.

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סוף סוף ...

תודה רבה על תשומת הלב שלכם!