Model Theory, now becoming more geometric?

The case of modularity.

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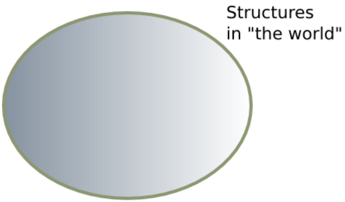
IPM - Tehran - November 2015

Indice

An ideal world



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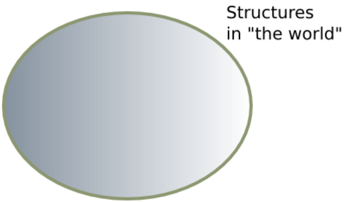

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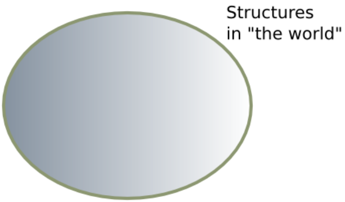

Let us just consider "from the distance" the thousands and thousands of mathematical structures that we human beings have invented/discovered along the centuries.
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Is it a homogeneous world? The response is a resounding NO, as we all know even a long time before even knowing that there actually are so many structures

Unavoidable structures?

Among those thousand of structures, the landscape is far from homogeneous - there are extremely tall "outliers" - structures that are somehow unavoidable to any mathematician of any culture:

- $\langle \mathbb{N}, +, <, \cdot, 0, 1 \rangle$ - arithmetics
- $\langle \mathbb{C}, +, \cdot, 0, 1 \rangle$ - algebraic geometry
- $\langle \mathbb{R}, +, <, \cdot, 0, 1 \rangle$ - real alg. geom.
- elliptic curves
- vector spaces (modules, etc.)
- some combinatorial graphs
- Hilbert spaces, ℓ_2, etc.
- ...

Model Theory - invariant theory

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Words like "dimension", "rank", "degree", "density character" - seem to appear attached to those structures, and control them and allow us to capture them

Model Theory: perspective and fine-grain

1. Arbitrary **structures**.
2. Hierarchy of types of structures (or their theories): Stability Theory.
3. In the "best part" of the hierarchy: generalized Zariski topology - Zariski Geometries due to Hrushovski and Zilber: algebraic varieties - "arbitrary" structures whose place in the hierarchy ends up automatically giving them strong similarity to elliptic curves.
4. Uni-dimensional objects in Zariski structures are exactly finite covers of algebraic curves - these correspond to structures built to capture non-commutative phenomena in Physics!
5. More recently, Model Theory has dealt with "limit structures" of various kinds: limiting processes of constructions. Mathematical approximation and perturbation end up being model-theoretical.
6. Beyond direct control by a logic: the hierarchy does extend (Abstract Elementary Classes)

Classical Model Theory: definability. Formulas.

Initially, Model Theory allows two basic things:

Capturing classes of models

Isolating definible sets within each class

Examples:

Classes: (models of) Peano axioms, group axioms, field axioms, algebraically closed fields, etc. (Hilbert spaces, ...).

Definable Sets: two levels: through a formula or through infinite sets of formulas that still have "solution sets" (loci)

Definable Sets

In Model Theory, a set D is definable in a structure $\mathfrak{A}$ if there exists a formula $\varphi(x)$ such that $D = \varphi(\mathfrak{A}) = \{a \in A | \mathfrak{A} \models \varphi[a]\}$.
D is then the **locus** of a formula φ, in $\mathfrak{A}$.
Classical examples of definable sets in a field include affine varieties: the elliptic curve given by

$$y^2 = x^3 + ax + bc + c$$

can be understood as the definable set D_C over $\langle \mathbb{C}, +, \cdot, a, b, c \rangle$,

$$D_C = \varphi_C(\mathbb{C}, a, b, c) = \{(x, y) \in \mathbb{C}^2 | \varphi_C(x, y, a, b, c)\},$$

where $\varphi_C(x, y, a, b, c)$ is the formula $y^2 = x^3 + ax + bc + c$.

Type-definible sets (I)

Given a and C, the type of a over C in M is the set

$$\mathrm{tp}(a/C, M) := \{\varphi(x, \bar{c}) | \bar{c} \in C, M \models \varphi[a, \bar{c}]\}.$$

1. $\mathrm{tp}(2/\mathbb{Q}, \mathbb{C}) = \{x = 1 + 1, \cdots\}$
2. $\mathrm{tp}(\sqrt{2}/\mathbb{Q}, \mathbb{C}) = \{x \cdot x = 1 + 1, \cdots\}$
3. $\mathrm{tp}(\pi/\mathbb{Q}, \mathbb{C}) = \{\neg(P(x) = 0) | P(x) \in \mathbb{Q}[x]\}$

1 (realized type): one formula, one solution.
2 is also determined by one formula (it is a principal type) but it has several solutions (finitely many): it is algebraic

3 is a non algebraic type: not determined by a single formula, infinite solutions

Type-definable sets (II)

Turning things around, we may regard a set of formulas

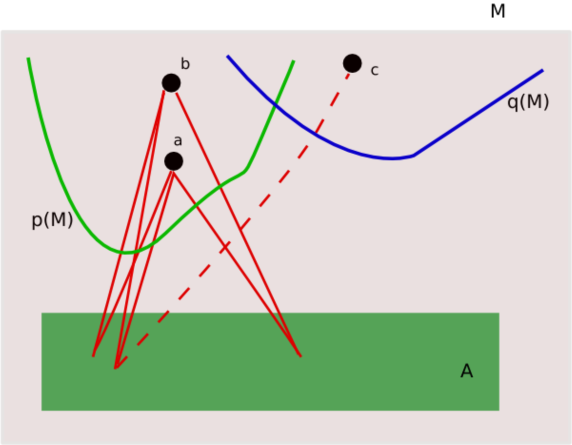
$$p = \{\varphi_i(x, \bar{b}) | i \in I\}$$

as corresponding to a definable (type-definable): look for the solutions (**locus**) in some model M yields the type-definible set

$$p(M) = \{a \in M | a \models p\}$$

Naturally, the same process can be applied to find type-definable subsets of M^2, M^3, etc. even $M^{\mathbb{N}}$...

$\mathrm{tp}(a/A, M)$, ETC.



Rôle of definable and type-definable sets

Akin to the rôle of ideals in algebraic geometry: by Hilbert's Nullstellensatz, the crucial information on varieties is captured by radical ideals.
In $(\mathbb{C}, +, \cdot, 0, 1)$, "$a$ and b have the same type over V" is captured by ideals (this is called "quantifier elimination" in logical lingo).

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More in general, types are basic blocks of information, generating, counting structures and controlling embeddings between them.

Variants

Types in model theory can be seen as

1. Sets of formulas
2. Zariski-closed sets
3. Orbits under automorphisms of monster models (strongly homogeneous, saturated)
4. More recently, measures/states or distributions.

Advantages of First Order Logic (Elementary Classes)

The classical relation

$$T \overset{\sim}{\Longleftrightarrow} Mod(T)$$

has been extremely productive in the case of First Order theories for decades (the content of a usual first course in Model Theory):

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- Theories, elementary maps, types
- Ultraproducts, compactness
- Typespaces (types topologized by definable sets)
- Unions of chains, Löwenheim-Skolem, saturated and homogeneous models.
- Quantifier elimination.
- Omitting types. Categoricity transfer.
- Stability, independence, simplicity, dependence.

General descriptions of Model theory

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More recently (2009) Hrushovski has described Model Theory as the geography of tame mathematics.

Very classical: Decidability of the complex numbers.

- Steinitz Theorem: all uncountable algebraically closed fields of characteristic zero, of same cardinality, must be isomorphic.
- The theory ACF_0 is included in $Th(\mathbb{C}, +, \cdot, 0, 1)$ – this one is obviously complete.
- The theory ACF_0 is axiomatizable (field axioms, solutions to all nonconstant polynomials, charact. zero) in a recursive way.
- The categoricity (in uncountable cardinals, Steinitz) here implies that ACF_0 is complete and therefore equal to $Th(\mathbb{C}, +, \cdot, 0, 1)$.
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Up and down phenomenon, strengthened in extreme ways later.

Classical: Ax-Kochen

For each positive integer d there is a finite set Y_d of prime numbers, such that if p is a prime not in Y_d then every homogeneous polynomial of degree d over the p-adic numbers in at least $d^2 + 1$ variables has a nontrivial zero.
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Again, (but much more sophisticated), Model Theory:

- First, Lang's theorem (same result, for the field $F_p((t))$ of formal Laurent series over finite fields) with empty Y_d.
- Then, if M and N are Henselian valued fields with equivalent valuation rings and residue fields, and their res. fields have char. 0, then $M \equiv N$.
- Then build two ultraproducts: $\prod_p F_p((t))$ and $\prod_p Q_p$ - both over all primes, the second of p-adics for each p.
- Model theory guarantees their elementary equivalence.

Hrushovski & co. - the strongest classical results

Theorem (Hrushovski)

A solution to Mordell-Lang's conjecture over fields of arbitrary characteristic.

This uses much more sophisticated ideas: analysis of "modular sets" in (generalized Zariski) contexts, analysis of unidimensional sets, ... "geometric model theory" at its deepest.

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Other developments along this line include results on differentially closed fields, Galois theory for differential-difference equations, tec. - (Pillay, Macintyre, van den Dries, …and the André-Oort Conjecture [Scanlon]).

Beyond the glory of First Order...

The results mentioned are now running deep at the heart of areas such as Galois Theory for wide contexts, Algebraic Geometry.

However, there are many reasons to try to extend the reach of Logic (and Model Theory) beyond classes of structures axiomatizable in First Order.

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3. Sometimes having more general perspective may yield simpler proofs. (L'art pour l'art)...

Examples

1 - Not captured in FO Noetherian constructions. Locally finite groups. Abelian groups under pure extensions $<_{pure}$ and many variants. End elementary extensions of models of arithmetic and set theory.

1 - Not FO in nature Banach spaces. Hilbert spaces with operators acting on them. Operator algebras of various kinds.

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3 - Generality is good (sometimes) Higher dimensional amalgams - lately, versions of cohomology for model theory, tameness phenomena, finitary classes, etc. — many of these occur more naturally when closer to the non-first order core of Model Theory…

Change of focus: from Language to ...

The many ways of shifting focus outward from First Order have actually been there for a now long time. But... isn't there the risk of losing grip?

The focus changes to

- Stronger languages (Infinitary Languages, etc.)
- Stronger quantifiers (measure quantifiers, game quantifiers, game semantics).

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- Embed convergence, continuity (measures, etc.) into the language. The analytic response - crucial in applications to Banach space theory and more recently to Operator Algebras.
- Taking seriously Grothendieck's "scheme revolution". Model Theory on sheaves and schemes and foliations. Very much in progress.

Change of Focus: from formulas to embeddings

(The high price to pay in developing the Model Theory is the absence of Compactness - a tool that shines through First Order but perhaps blocks the view of more structural phenomena behind (more "astrophysical" properties).

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$\varphi, T, \ldots$	$\prec_K$
compactness	amalgamation, tameness, …

A(N) (UN-)DECENT COMPROMISE: ABSTRACT ELEMENTARY CLASSES

Let $\mathcal{K}$ be a class of τ-structures, $\prec_{\mathcal{K}}$ a binary relation on $\mathcal{K}$.

Definition
$(\mathcal{K}, \prec_{\mathcal{K}})$ is an abstract elementary class if

- $\mathcal{K}$, $\prec_{\mathcal{K}}$ are **closed under isomorphism**,
- $M, N \in \mathcal{K}, M \prec_{\mathcal{K}} N \Rightarrow M \subset N$,
- $\prec_{\mathcal{K}}$ is a partial order,
- **(TV)** $M \subset N \prec_{\mathcal{K}} \bar{N}, M \prec_{\mathcal{K}} \bar{N} \Rightarrow M \prec_{\mathcal{K}} N$, and…

AEC, CONTINUED

- (**$\searrow$LS**) There exists a cardinal $\kappa = LS(\mathcal{K}) \geq \aleph_0$ such that for every $M \in \mathcal{K}$, for every $A \subset |M|$, there exists $N \prec_{\mathcal{K}} M$ with $A \subset |N|$ y $\|N\| \leq |A| + LS(\mathcal{K})$,
- (**Unions of $\prec_{\mathcal{K}}$-chains**) A union of a $\prec_{\mathcal{K}}$-chain in $\mathcal{K}$ belongs to $\mathcal{K}$, is a $\prec_{\mathcal{K}}$-extension of all models in the chain and is the sup of the chain.

Examples

- $(Mod(T), \prec)$, T FO,
- (Normed spaces, $\prec^{isom}$), Banach spaces form metric versions of AEC.
- $(Mod(\psi), \prec_{\mathcal{F}})$, $\psi \in \mathcal{F}$ a fragment of $L_{\omega_1\omega}$,
- $(Mod^{atom}(T), \prec)$, T FO,
- (Loc. fin. groups, $\prec$),
- $(\mathcal{K}_{exp}, \prec_{\mathcal{K}_{exp}})$, the Zilber class (2001).

Two crucial persons: Shelah and Zilber

Shelah started the Classification theory of these objects - he and other people (in Helsinki, Bogotá, Pittsburgh, Jerusalem, Oxford) have developed a quite impressive body of work in the most abstract parts of the theory but more recently have turned to applications to geometry, operator theory, etc.

Pregeometries (I)

Zilber uncovers "hidden geometry" in "strongly minimal" definable sets. A type p over C is strongly minimal iff for every formula $\varphi(x)$ $p(M) \cap \varphi(M)$ is either finite or cofinite. Minimal definability.
The following notion behaves very well on strongly minimal theories:
$a \in \mathrm{acl}(B)$ iff for some $\varphi(x, \bar{b})$ with $\bar{b} \in B$, for every $M \supset B \cup \{a\}$, $M \models \varphi[a, \bar{b}]$ a **only finitely many** a **satisfy this**.

Pregeometries (II)

acl behaves well means that

$$(p(M),\ \mathrm{cl})$$

is a **pregeometry** (or matroid).

1. $X \subset \mathrm{cl}(X) = \mathrm{cl}(\mathrm{cl}(X))$
2. $X \subset Y \to \mathrm{cl}(X) \subset \mathrm{cl}(Y)$
3. $a \in \mathrm{cl}(X) \to \exists X_0 \subset^{fin} X(a \in \mathrm{cl}(X_0))$
4. $a \in \mathrm{cl}(X \cup \{b\}) \setminus \mathrm{cl}(X) \to b \in \mathrm{cl}(X \cup \{a\})$

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There is therefore a natural notion of dimension.

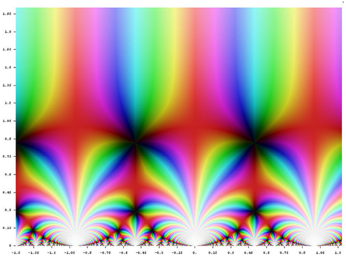
Zilber's Conjecture - Theorems

Zilber studied the dimension arising on strongly minimal sets and found that they seemed to follow a trichotomical pattern:

1. Trivial: $\dim(X \cup Y) = \dim(X) + \dim(Y)$.
2. (Loc.) modular: $\dim(X \cup Y) = \dim(X) + \dim(Y) - \dim(X \cap Y)$.
3. Alg. geom.: **as in algebraically closed fields**. The two previous fail and the model interprets an algebraically closed field!

The counterexamples to the conjecture, the attempts at recovery, the classification following this has generated among the best interactions with geometry. Part of this occurs in First Order, part of this outside.

Classical j invariant



j-invariant on $\mathbb{C}$
(Wikipedia article on j-invariant)

Klein defines the function (we call) "classical j"

$$j : \mathbb{H} \to \mathbb{C}$$

(where $\mathbb{H}$ is the complex upper half-plane)
through the explicit rational formula

$$j(\tau) = 12^3 \cdot \frac{g_2(\tau)^3}{g_2(\tau)^3 - 27 g_3(\tau)^3}$$

with g_2 and g_3 certain rational functions ("of Eisenstein".)

Basic facts about classical j

The function j is a modular invariant of elliptic curves (and classical tori).

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- j is analytic, except at ∞
- $$j(\tau) = j\left(\frac{a\tau + b}{c\tau + d}\right)$$

  if $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$.

More basic facts

The following are equivalent:

1. There exists $s \in SL_2(\mathbb{Z})$ such that $s(\tau) = \tau'$,

where $\mathbb{T}_\tau := \mathbb{C}/\Lambda$, and $\Lambda \leq \mathbb{C}$ is a lattice.

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3. $j(\tau) = j(\tau')$...

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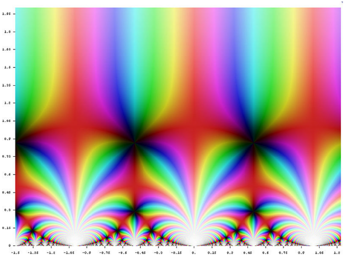
- (Schneider, 1937): if τ is a quadratic irrationality then $j(\tau)$ is algebraic of degree $h_{f,K}$.
- if $e^{2\pi i\tau}$ is algebraic then $j(\tau), \frac{j'(\tau)}{\pi}, \frac{j''(\tau)}{\pi^2}$ are mutually transcendental.

Classical j invariant



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$$j(\tau) = 12^3 \cdot \frac{g_2(\tau)^3}{g_2(\tau)^3 - 27 g_3(\tau)^3}$$

with g_2 and g_3 certain rational functions (“of Eisenstein”).

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  if $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$.

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The following are equivalent:

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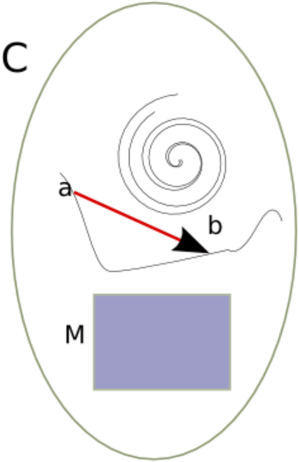
- if $e^{2\pi i\tau}$ is algebraic then $j(\tau), \frac{j'(\tau)}{\pi}, \frac{j''(\tau)}{\pi^2}$ are mutually transcendental.

Galois types

The correct notion of a type in an AEC (satisfying some stronger properties - Amalgamation, Joint Embedding, having sufficiently large models):



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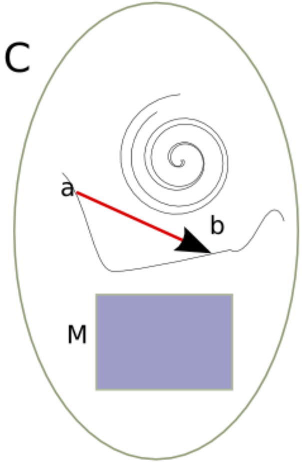

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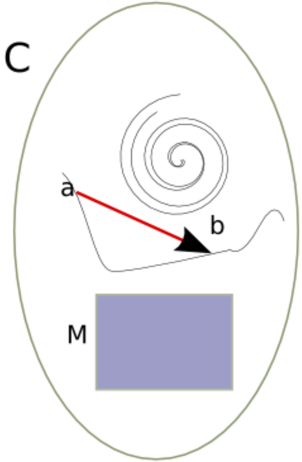

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2. We define $ga - tp(a/M) = ga - tp(b/M)$ if and only if there exists $f \in Aut(\mathbb{C}/M)$ tq $f(a) = b$.
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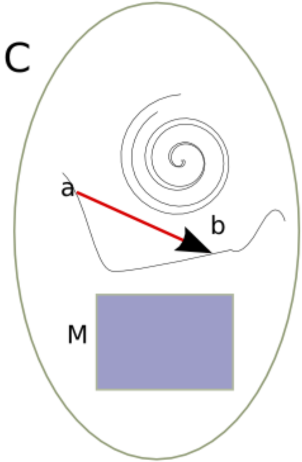

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3. Therefore, Galois types are orbits under the action of automorphisms fixing M.
4. This generalizes the classical notion of type.

Sorts for tori and j

Harris and Zilber provide a contrasting view of j invariants - directed toward categoricity and generalizations of j maps toward higher dimensions (Shimura varieties). The starting point is a view of j mappings as axiomatized in $L_{\omega_1,\omega}$:

Standard fibers

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Let L be a language for two-sorted structures of the form

$$\mathfrak{A} = \langle \langle H; \{g_i\}_{i\in\mathbb{N}} \rangle, \langle F, +, \cdot, 0, 1 \rangle, j : H \to F \rangle$$

where $\langle F, +, \cdot, 0, 1 \rangle$ is an algebraically closed field of characteristic 0, $\langle H; \{g_i\}_{i\in\mathbb{N}} \rangle$ is a set together with countably many unary function symbols, and $j : H \to F$. Let then

$$Th_{\omega_1,\omega}(j) := Th(\mathbb{C}_j) \cup \forall x \forall y (j(x) = j(y) \to \bigvee_{\gamma \in SL_2(\mathbb{Z})} x = \gamma(y)$$

for $\mathbb{C}_j$ the "standard model" $(\mathbb{H}, \langle \mathbb{C}, +, \cdot, 0, 1 \rangle, j : \mathbb{H} \to \mathbb{C})$.
(Standard fibers means "fibers are orbits")

Categoricity of classical j

The theory $Th_{\omega_1,\omega}(j) + \mathrm{trdeg}(F) \geq \aleph_0$ is categorical in all infinite cardinalities.
They use an instance of the adelic Mumford-Tate conjecture for products of elliptic curves to show this. The strategy to build an isomorphism between two models M and M' consists (as expected) in

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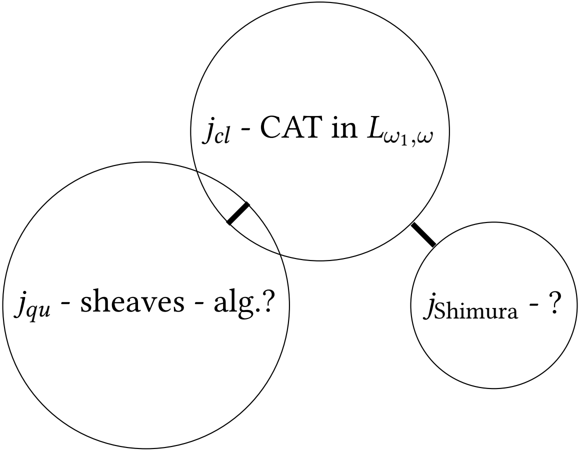
- then show that the information in the type is contained in the finite part ("Mumford-Tate" open image theorem) ... every point $\tau \in \mathbb{H}$ corresponds to an elliptic curve E – the type of τ is determined by algebraic relations between torsion points of E... determined by the Galois representation of the Tate module of E.

Two directions - Really?

The current model theoretic analysis of j looks at two possible extensions:



Two different toolkits

The two directions of generalization (to quantum tori/real multiplication on the one hand, to higher dimension varieties/Shimura on the other hand) of j maps calls different aspects of model theory (at the moment, an “amalgam” has started, but is far along the way):

- The model theory of abstract elementary classes (in particular, the theory of excellence - now “old” (1980s) but recently clarified by the “five authors”: Bays, Hart, Hyttinen, Kesälä, Kirby - Quasiminimal Structures and Excellence. A kind of cohomological analysis of models (and types), connected to categoricity and “smoothness”.

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- The model theory of sheaves (remotely based on works by Macintyre, Ellerman), developed by Caicedo and further extended by Ochoa, Padilla, V. to metric and equivariant sheaves.

Other new geometric lines

- Categorical geometry: Tannakian reconstruction, now model theoretic.
- Homotopy theory of model theory.
- Sheaves for arithmetic (modular functions, j-invariant). Transfer properties.
- Classification of group actions. Invariant model theory.