Semantics on Sheaves

Topological, Metric, etc.

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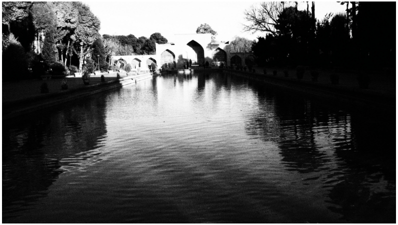

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- Zilber

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- Introduction: Why a Model Theory of Sheaves?
- Limit (generic) models - semantics
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Continuous (or even Metric!) Fibers
- Metrical Fibers
- Group Actions - Invariant Sheaves - Cohomology

Ideal (limit) models in Physics - Some Goals

- Mathematical Physics: a need of various kinds of “ideal structures”, and tools of contrast between “real structures” and those ideal (limit) structures.

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- Kochen-Specker phenomena and Bell Inequalities - related to sheaves (Abramsky) and sheaf semantics.
- "Zilber sheaves" for "Weyl algebras".

Zilber: Structural approximation

Motivated by some insatisfaction with the current state of "mathematization" of quantum field theory, Zilber speaks about the

> huge progress achieved by physicists in dealing with singularities and non-convergent sums and integrals (...Feynman path integrals) has not been matched so far ... with an adequate mathematical theory (2010)

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Implicit knowledge by the physicist of the structure of his model, not yet available to mathematicians? (Rabin, Rieffel, Zeidler)

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5. With Model Theory on Sheaves: strong ways of controlling limit models.
6. May even go "beyond logic-dependence" and get several of the previous (Abstract Elementary Classes).

Zilber's wide scale approach to structures for Physics

In a nutshell... Zariski Geometries:

$$\mathbb{M} = (M, \mathcal{C})$$

where M is a set and $\mathcal{C}$ is a collection of basic predicates. $\mathcal{C}$ is a basis of closed sets for a topology on each M^n such that

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- (Presmoothness) U irred. is presmooth if for every irred. rel. closed subsets $S_1, S_2 \subset U$ and any irreducible component S_0 of $S_1 \cap S_2$

$$\dim S_0 \geq \dim S_1 + \dim S_2 - dimU.$$

Hrushovski-Zilber's theorem

Theorem (Classification Theorem - Hrushovski-Zilber)

Any one-dimensional Zariski geometry $\mathbb{M}$ *that is "non-linear" is associated to a smooth algebraic curve* C *over an algebraically closed field* F *through a surjective map* $p : M \to C(F)$*, definable in* $\mathbb{M}$ *in such a way that the fibres are all of some finite size* N*.*

So, Zariski geometry is "almost" algebraic geometry, but the structure of the finite fibers has been studied by Zilber and found to contain "jewels" of information.
There are "not enough" definable coordinate functions $M \to F$ to encode all the structure of $\mathbb{M}$ - the usual coordinate algebra gives just $C(\mathbb{F})$.

Irma Laukkanen - Maisema - Sisätilassa

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Extended Objects / Variable Objects

Objects in the world present themselves as extended in time (or in other classical (or non-classical) “categories”):

- Physical objects, individuals, etc.

Leibniz, Peirce, Husserl, etc.

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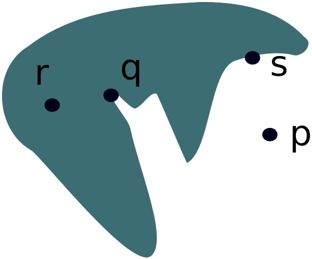
- Physical objects, individuals, etc.
- Particles, even neutrinos (for some particles, order of 10^{-20} seconds, yet still “time”)
- Concepts? Thoughts? Ideas? Visualizations? Perceptions?

Leibniz, Peirce, Husserl, etc.

Yet logic (at the limit) is "too rough"

(Really, classical logic.)



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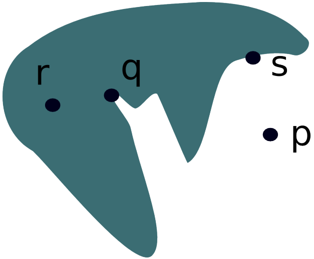

- For p and r the predicate “is in the green zone” is clear - classical logic “agrees” with perception.
- For q and s (at “limit situations”) classical logic forces one to make a decision (open, closed green zone, etc.).

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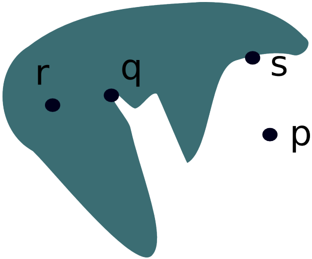

- Perception does not follow classical logic.

Physics, geometry, and "limit" phenomena

As we know since the late 1920's, Physics (wave models, quantum phenomena of "undecidability" or "uncertainty", noncommutativity of operators corresponding to formalizations of observability, etc.) has the kind of "limit phenomena" that may call for a logic of variable entities.

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Algebraic geometry of the postwar period (Leray, Cartan, Weil, and then Grothendieck reflects this same "shift of perspective": sheaves, sites, topoi.)

Instant velocity / Paradigm change

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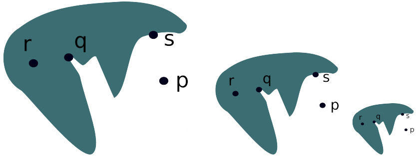
Excluded middle ~~may~~ HAS to be dropped!
The strong paradigm becomes Truth Continuity.

Truth Continuity

If an individual (an entity, a particle, etc.) has some property on some point of its domain of extension, there has to be a neighborhood of this point in this domain in which this property holds of all points.



Sheaves over topological spaces

Fix X a topological space. The pair (E, p) is a sheaf over X if and only if E is a topological space and $p : E \to X$ is a surjective local homeomorphism.

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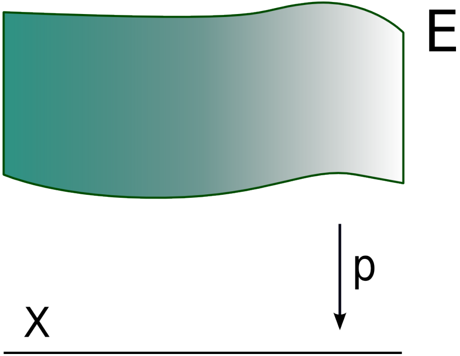

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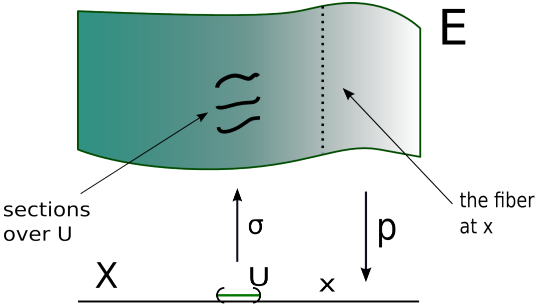
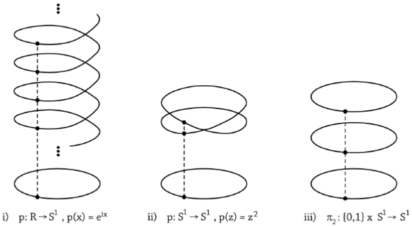
- The (images of) sections σ form a basis for the topology of E (a section is a continuous partial inverse of p defined on an open set $U \subset X$),
- If two sections σ, τ coincide at a point a then there exists an open set $U \ni a$ such that $\sigma \restriction U = \tau \restriction U$

Sections - objects



Classical and not



(from Caicedo's monograph)

Cildo Meireles - Fontes

A little history

Sheaves over topological spaces go back to **H. Weyl** (1913), in his work on Riemann surfaces.
They "reappear" strongly in **Cartan**'s seminar (1948-1952) and then catch flight with the French Algebraic Geometry School of the Postwar (**Serre**, **Leray**, etc.).
Weil: Séminaire de géométrie algébrique: study of the zeta function on finite fields.
Finally, **Grothendieck** generalizes further the frame (to sites = small categories endowed with "Grothendieck topologies"). **Deligne** then proves Weil's conjectures.

Sheaves of Structures

A sheaf of structures $\mathfrak{A}$ over X consists of:

1. A sheaf (E, p) over X,
2. On every fiber $p^{-1}(a)$ ($a \in X$), a structure

$$\mathfrak{A}_a = (E_a, (R_i^a)_i, (f_j^a)_j, (c_k^a)_k,)$$

such that $E_a = p^{-1}(a)$, and

- For every i, $R_i^{\mathfrak{A}} = \bigcup_{x \in X} R_i^{\mathfrak{A}_x}$ is open
- For every j, $f_j^{\mathfrak{A}} = \bigcup_{x \in X} f_j^{\mathfrak{A}_x}$ is continuous
- For every k, $c_k^{\mathfrak{A}} : X \to E$ such that $x \mapsto c_k^{\mathfrak{A}_x}$ is a continuous global section

Truth Continuity?

Fact

For all atomic formulas $\varphi(v)$ we have that

$$\mathfrak{A}_x \models \varphi(\sigma(x)) \textit{ iff } \exists U \ni x \forall y \in U \Big(\mathfrak{A}_y \models \varphi(\sigma(y)) \Big)$$

This also holds for positive Boolean combinations of atomic formulas.

However, this fails for negations!

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The solution to this failure is to switch to an emphasis on forcing.

Satisfaction and forcing (pointwise and local)

Three notions: satisfaction at each fiber, forcing at a point $x \in X$, forcing at a (non-empty) open set $U \subset X$:

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How do we compare them? Before diving into the definitions of the forcing notions, notice that the first one is pointwise while the second one is local. Also notice that satisfaction in $\mathfrak{A}_x$ is about values of sections at x (the $\sigma(x)$) whereas pointwise (over x) or local forcing (over U) are about the whole section σ defined on U.

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Sections are the new objects: formulas $\varphi(v_1, v_2, \cdots)$ will be "evaluated" by "replacing" v_i by a section σ_i or by its value at an element x of X, $\sigma_i(x)$.

Pointwise forcing

- For atomic φ and $t_1, \cdots, t_n$ terms,
  $\mathfrak{A} \Vdash_x (t_1 = t_2)[\vec{\sigma}] \Leftrightarrow t_1^{\mathfrak{A}_x}[\vec{\sigma}(x)] = t_2^{\mathfrak{A}_x}[\vec{\sigma}(x)]$
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- $\mathfrak{A} \Vdash_x \neg\varphi \Leftrightarrow$ for some open $U \ni x$, for every $y \in U$, $\mathfrak{A} \not\Vdash_y \varphi$.
- $\mathfrak{A} \Vdash_x (\varphi \to \psi) \Leftrightarrow$ for some open $U \ni x$, for every $y \in U$, $\mathfrak{A} \Vdash_y \varphi$ implies that $\mathfrak{A} \Vdash_y \psi$.

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- $\mathfrak{A} \Vdash_x \forall v \varphi(v, \vec{\sigma}) \Leftrightarrow$ for some $U \ni x$, for <u>every</u> $y \in U$ and <u>every</u> σ defined on y, $\mathfrak{A} \Vdash_y \varphi[\sigma, \vec{\sigma}]$.

Forcing $\neg, \to, \forall$ at x requires information "around" x. It is an exercise to check Truth Continuity for $\Vdash_x$.

Truth continuity - II

A semantics can also be defined *directly* over open sets:

$$\mathfrak{A} \Vdash_U \varphi[\sigma],$$

where U is an open set in the domain of σ.

Definition

$\mathfrak{A} \Vdash_U \varphi[\sigma]$ if and only if for every $x \in U$, $\mathfrak{A} \Vdash_x \varphi[\sigma(x)]$.

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- $\mathfrak{A} \Vdash_U (\varphi \vee \psi) \Leftrightarrow$ for some open and possibly empty V, W such that $U = V \cup W$ we have $\mathfrak{A} \Vdash_V \varphi$ and $\mathfrak{A} \Vdash_W \psi$.

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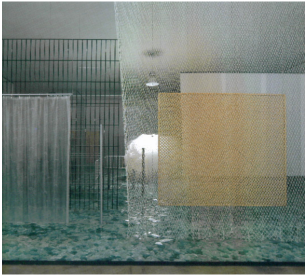
- $\mathfrak{A} \Vdash_U \forall v \varphi(v, \vec{\sigma}) \Leftrightarrow$ for every $W \subset U$ and all σ defined on W we have $\mathfrak{A} \Vdash_W \varphi(\sigma, \vec{\sigma})$.

Pointwise versus local



Sheaves	Pointwise = Local
Presheaves	They may differ

In a way, $\Vdash_U$ is more direct than $\Vdash_x$

In some steps ($\neg, \rightarrow, \forall$) of the definition of $\Vdash_x$ one needs to have access "from x" to information about forcing "around x" - this guarantees in the end the Truth Continuity paradigm.
In the definition of $\Vdash_U$, the nontrivial steps require knowledge of fibers defined over subopen sets of U. As it stands, this is non-trivial knowledge. Notice also that forcing a disjunction of two formulas "spreads" the forcing of each formula to one portion of U - the only requirement being that this may be done while still covering U.

Generic Model Theorem

The Generic Model Theorem is the version of the (Model Theoretic) Forcing Theorem for this notion. Caicedo generalized the Macintyre version to sheaves of arbitrary First Order structures. Further generalizations (adaptations) are due to Caicedo, Ochoa and V. (later!).

Generic Filters

Definition

Given $\mathfrak{A}$ a sheaf of structures over X, a generic filter $\mathbb{F}$ for $\mathfrak{A}$ is a filter of open sets of X such that

- for every $\varphi(\sigma)$ and every σ defined on $U \in \mathbb{F}$, there is some $W \in \mathbb{F}$ such that $\mathfrak{A} \Vdash_W \varphi(\sigma)$ or $\mathfrak{A} \Vdash_W \neg\varphi(\sigma)$
- for every σ defined on $U \in \mathbb{F}$, for every $\varphi(u, \sigma)$, if $\mathfrak{A} \Vdash_U \exists u \varphi(u, \sigma)$, then there exists $W \in \mathbb{F}$ and μ defined on W such that $\mathfrak{A} \Vdash_W \varphi(\mu, \sigma)$

For some topological spaces, this definition of genericity of a filter may be made more purely topological/geometrical (and less dependent on formulas and forcing). However, in the general case, this is not necessarily possible - and we must rely on this logical definition.

Existence - generic models

Fact

Generic filters exist.

Definition (Generic Models)

Given a generic filter $\mathbb{F}$ and $\mathfrak{A}(U) = \{\sigma | \mathrm{dom}(\sigma) = U\}$, let

$$\mathfrak{A}[\mathbb{F}] = \lim_{U \in \mathbb{F}} \mathfrak{A}(U) = \bigsqcup_{U \in \mathbb{F}} \mathfrak{A}(U) / \sim_{\mathbb{F}}$$

where $\sigma \sim_{\mathbb{F}} \mu$ iff there exists $W \in \mathbb{F}$ such that $\sigma \upharpoonright W = \mu \upharpoonright W$.
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where $\sigma \sim_{\mathbb{F}} \mu$ iff there exists $W \in \mathbb{F}$ such that $\sigma \upharpoonright W = \mu \upharpoonright W$. Also,

- $(\sigma_1 / \sim_{\mathbb{F}}, \ldots, \sigma_n / \sim_{\mathbb{F}}) \in R^{\mathfrak{A}[\mathbb{F}]} \Leftrightarrow \exists U \in \mathbb{F} (\sigma_1, \ldots, \sigma_n) \in R^{\mathfrak{A}(U)}$
- $f^{\mathfrak{A}[\mathbb{F}]}(\sigma_1 / \sim_{\mathbb{F}}, \ldots, \sigma_n / \sim_{\mathbb{F}}) = f^{\mathfrak{A}(U)}(\sigma_1, \ldots, \sigma_n) / \sim_{\mathbb{F}}$

Finally, the theorem...

Theorem (Generic Model Theorem)

Let $\mathbb{F}$ be a generic filter for a sheaf of topological structures $\mathfrak{A}$ over X. Then

$$\begin{aligned}\mathfrak{A}[\mathbb{F}] \models \varphi(\sigma/\sim_{\mathbb{F}}) \quad &\iff \quad \{x \in X | \mathfrak{A} \Vdash_x \varphi^G(\sigma(x))\} \in \mathbb{F} \\ &\iff \quad \exists U \in \mathbb{F} \text{ such that } \mathfrak{A} \Vdash_U \varphi^G(\sigma).\end{aligned}$$

Here, φ^G is a formula equivalent classically to φ, but not necessarily in an intuitionistic framework! (The formula φ^G is sometimes called the Gödel translation of φ - in 1925, Kolmogorov had independently defined an equivalent translation.)

Caicedo's Theorem

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More on the Generic Model Theorem

Cohen's construction of generic models for set theory is the first published result along these lines. Later, Robinson, Barwise and Keisler used generic model theorems to get Omitting Types Theorems in various logics, generalized by Caicedo. Ellerman's "ultrastalk theorem" (1976) is a GMTh for maximal filters. Miraglia also proves a similar result for Heyting-valued models.

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$$\sigma \mapsto \sigma^* = \sigma \cup \{(\infty, [\sigma]_{\sim_{\mathbb{F}}})\}.$$

The fiber "at infinity"

The model $\mathfrak{A}[\mathbb{F}]$ can be regarded as a "fiber at ∞". This may be made precise by extending X by one new point ∞, adding the generic model as the new fiber over ∞ and extending the sections by

$$\sigma \mapsto \sigma^* = \sigma \cup \{(\infty, [\sigma]_{\sim_\mathbb{F}})\}.$$

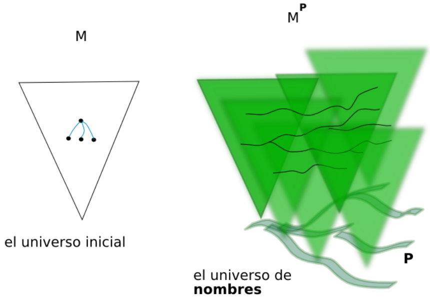
Then, the GMTh just means that in the new sheaf $\mathfrak{A}^\infty$ this fiber is classic:

$$\mathfrak{A}^\infty \Vdash_\infty \varphi(\sigma_1^*, \cdots, \sigma_n^*) \Leftrightarrow \mathfrak{A}[\mathbb{F}] \models \varphi([\sigma_1^*], \cdots, [\sigma_n^*])$$

Łoś as a first consequence

The Łoś theorem is clearly a special case of the Generic Model Theorem, corresponding to endowing X with the discrete topology. Therefore, the Model Theory of sheaves has a twisted form of compactness - of course relative to a context with no excluded middle.

The forcing theorem



The forcing theorem of Set Theory is another special case: take a partially ordered set $\mathbb{P}$, endowed with the order topology (basic open sets are downward closed sets). The Generic Model Theorem provides a model of set theory, where satisfaction is given by forcing on points. BUT in this kind of topological spaces, forcing over an open set is reducible to forcing over a point.

In general... (?)

Most topological spaces, however, do not arise from partially ordered sets. A natural question (fairly unexplored) is what other kinds of models of set theory may be obtained by forcing with such topological spaces.

Other applications of the GMTh

- Kripke models - generalized semantics
- Set-theoretic forcing
- Robinson's Joint Consistency Theorem (=Amalgamation over Models)
- Various Omitting Types Theorems (Caicedo, Brunner-Miraglia)
- Control over new kinds of limit models

Sheaves of Hilbert Spaces

Why?

Photo: Geraldo Barros

1. Hilbert Spaces are (still) a crucial tool for formalization of concepts and objects in Physics and in Chemistry
2. In Physics: really algebras of operators acting on Hilbert spaces.
3. In Chemistry: really predicates on Hilbert spaces.
4. In both, the dynamical properties of evolution of a system are relevant.

The problem of a model theory for Hilbert Spaces

So, we want to be able to put Hilbert spaces (and more structure on top of them, such as predicates for reactions, or operators for observables) on fibers.

We could in principle do that as we have seen so far, but immediately we get the problem that we may get lots of non-standard Hilbert spaces (infinitesimals, etc.).

Moreover, we want the logic to "keep track" of (say) the distance to a projection $p(v)$, the convergence of a sequence in H, isometric isomorphism, $(1+\varepsilon)$-isomorphism, etc. etc.

Finally, we need to be able to take limits of Cauchy sequences at will in our structures: metric completeness is crucial.

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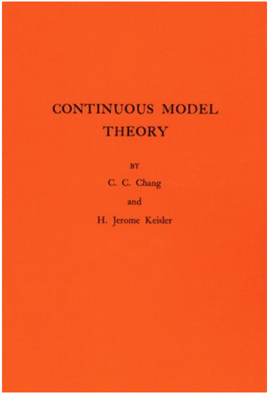
That is the rôle of Continuous Model Theory.

Continuous Model Theory - Origins



Although the origins of CMTh go back to Chang & Keisler (1966), and in some (restricted) ways to von Neumann's Continuous Geometry recent takes on Continuous Model Theory are based on formulations due to Ben Yaacov, Usvyatsov and Berenstein of Henson and Iovino's Logic for Banach Spaces.

Continuous predicates and functions

Definition

Fix (M, d) a bounded metric space. A continuous n-ary predicate is a uniformly continuous function

$$P : M^n \to [0, 1].$$

A continuous n-ary function is a uniformly continuous function

$$f : M^n \to M.$$

Metric structures

Therefore, metric structures are of the form

$$\mathcal{M} = \left(M, d, (f_i)_{i\in I}, (R_j)_{j\in \mathcal{J}}, (a_k)_{k\in K}\right)$$

Each function, relation must be endowed with a modulus of uniform continuity.

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$$\mathcal{M} = \left(M, d, (f_i)_{i \in I}, (R_j)_{j \in \mathcal{J}}, (a_k)_{k \in K}\right)$$

where the R_i and the f_j are (uniformly) continuous functions with values in $[0, 1]$, the a_k are distinguished elements of M.
Remember: M is a bounded metric space.
Each function, relation must be endowed with a modulus of uniform continuity.

Examples of FO metric structures

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- Any FO structure, endowed with the discrete metric.

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- For a probability space $(\Omega, \mathcal{B}, \mu)$, construct a metric structure $\mathcal{M}$ based on the usual measure algebra of $(\Omega, \mathcal{B}, \mu)$.
- Representations of C^*-algebras (Argoty, Berenstein, Ben Yaacov, V.).
- Valued fields.

The syntax

1. Terms: as usual.
2. Atomic formulas: $d(t_1, t_n)$ and $R(t_1, \cdots, t_n)$, if the t_i are terms. Formulas are then interpreted as functions into $[0, 1]$.
3. Connectives: continuous functions from $[0, 1]^n \to [0, 1]$. Therefore, applying connectives to formulas gives new formulas.
4. Quantifiers: $\sup_x \varphi(x)$ (universal) and $\inf_x \varphi(x)$ (existential).

Interpretation

The logical distance between $\varphi(x)$ and $\psi(x)$ is $\sup_{a\in M}|\varphi^M(a)-\psi^M(a)|$.
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Notice also that the set of connectives is too large, but it may be "densely" and uniformly generated by $0, 1, x/2, \dot{-}$: for every ε, for every connective $f(t_1, \cdots, t_n)$ there exists a connective $g(t_1, \cdots, t_n)$ generated by these four by composition such that $|f(\vec{t}) - g(\vec{t})| < \varepsilon$.

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Stability Theory

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- Categoricity for countable languages (Ben Yaacov),
- ω-stability,
- Dependent theories (Ben Yaacov),
- Not much geometric stability theory: no analog to Baldwin-Lachlan (no minimality, except some openings by Usvyatsov and Shelah in the context of $\aleph_1$-categorical Banach spaces),
- NO simplicity!!! (Berenstein, Hyttinen, V.),
- Keisler measures, NIP (Hrushovski, Pillay, etc.).

"Continuous Model Theory" beyond First Order

Several contexts, some unexplored so far.

1. Metric Abstract Elementary Classes (Hirvonen, Hyttinen - ω-stability, V. Zambrano - superstability, domination, notions of independence): an amalgam of the power of Abstract Elementary Classes with metric ideas.

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2. Continuous $L_{\omega_1\omega}$. So far, no published results as such. There are however "Lindström theorems" for Continuous First Order due to Caicedo and Iovino.
3. Sheaves of (metric) structures. Our work with Ochoa, motivated by problems originally in Chemistry. NEXT!

Sheaves of Hilbert Spaces

Why?

1. Hilbert Spaces are (still) a crucial tool for formalization of concepts and objects in Physics and in Chemistry

In the case of Chemistry, the current treatment is unsatisfactory: capturing the relevant predicates (chemical structure, chemical reaction) has depended on physics to a degree that some theoretical chemists consider excessive.

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Sheaves of Metric Structures

A sheaf of metric structures $\mathfrak{A}$ over X consists of:

1. A sheaf (E, p) over X,
2. On every fiber $p^{-1}(x)$ $(x \in X)$, a metric structure

$$(\mathfrak{A}_x, d_x) = (E_x, (R_i^x)_i, (f_j^x)_j, (c_k^x)_k, d_x, [0, 1])$$

such that $E_x = p^{-1}(x)$, (E_x, d_x) is a complete bounded metric space of diameter 1, and

- For every i, $R_i^{\mathfrak{A}} = \bigcup_{x \in X} R_i^x$ is open
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- The premetric $d^{\mathfrak{A}} := \bigcup_{x \in X} d_x : \bigcup_{x \in X} E_x^2 \to [0, 1]$ is a continuous function.

(further requirements on moduli of uniform continuity)

Truth Continuity - adapted to metric

Truth Continuity is still the guiding paradigm. Remember in the "discrete" case, negation was the first stumbling block - the first place where forcing was needed in a non-trivial way.
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Negation in continuous, metric logic, is weak: the semantics really treats $\leq$ and $\geq$ as "negations" of each other...

Truth continuity happens without the need of forcing in two basic cases:

- Formulas φ composed of max, min, $\dot{-}$ and inf: $\mathfrak{A}_x \models \varphi(x) < \varepsilon$ if and only if this happens at all y near x
- Similarly for $\varphi > \varepsilon$ when φ is built of max, min, $\dot{-}$ and sup.

Pointwise forcing

With Ochoa, we define $\mathfrak{A} \Vdash_x \varphi < \varepsilon$ and $\mathfrak{A} \Vdash_x \varphi > \varepsilon$, for $x \in X$:

- Atomic: $\mathfrak{A} \Vdash_x d(t_1, t_2) < \varepsilon \Leftrightarrow d_x(t_1^{\mathfrak{A}_x}, t_2^{\mathfrak{A}_x}) < \varepsilon$
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- $\mathfrak{A} \Vdash_x \max(\varphi, \psi) < \varepsilon \Leftrightarrow \mathfrak{A} \Vdash_x \varphi < \varepsilon$ and $\mathfrak{A} \Vdash_x \psi < \varepsilon$. Sim. for $>$.
- $\mathfrak{A} \Vdash_x \min(\varphi, \psi) \Leftrightarrow \mathfrak{A} \Vdash_x \varphi$ or $\mathfrak{A} \Vdash_x \psi$. Sim. for $>$.
- $\dots$

Pointwise forcing

With Ochoa, we define $\mathfrak{A} \Vdash_x \varphi < \varepsilon$ and $\mathfrak{A} \Vdash_x \varphi > \varepsilon$, for $x \in X$:

- Atomic: $\mathfrak{A} \Vdash_x d(t_1, t_2) < \varepsilon \Leftrightarrow d_x(t_1^{\mathfrak{A}_x}, t_2^{\mathfrak{A}_x}) < \varepsilon$
  $\mathfrak{A} \Vdash_x d(t_1, t_2) > \varepsilon \Leftrightarrow d_x(t_1^{\mathfrak{A}_x}, t_2^{\mathfrak{A}_x}) > \varepsilon$
  $\mathfrak{A} \Vdash_x R(t_1, \cdots, t_n) < \varepsilon \Leftrightarrow R^{\mathfrak{A}_x}(t_1^{\mathfrak{A}_x}, t_2^{\mathfrak{A}_x}) < \varepsilon$
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- $\mathfrak{A} \Vdash_x \min(\varphi, \psi) \Leftrightarrow \mathfrak{A} \Vdash_x \varphi$ or $\mathfrak{A} \Vdash_x \psi$. Sim. for $>$.
- $\mathfrak{A} \Vdash_x 1 \dot{-} \varphi < \varepsilon \Leftrightarrow \mathfrak{A} \Vdash_x \varphi > 1 \dot{-} \varepsilon$. Sim. for $>$.
- $\mathfrak{A} \Vdash_x \varphi \dot{-} \psi < \varepsilon$ iff and only if one of the following holds:
  - $\mathfrak{A} \Vdash_x \varphi < \psi$
  - $\mathfrak{A} \nVdash_x \varphi < \psi$ and $\mathfrak{A} \nVdash_x \varphi > \psi$
  - $\mathfrak{A} \Vdash_x \varphi > \psi$ and $\mathfrak{A} \Vdash_x \varphi < \psi + \epsilon$.
- $\mathfrak{A} \Vdash_x \varphi \dot{-} \psi > \varepsilon$ iff $\mathfrak{A} \Vdash_x \varphi > \psi + \varepsilon$
- $\cdots$

Pointwise forcing - continued

Quantifiers:

- $\mathfrak{A} \Vdash_x \inf_{s \in A_x} \varphi(s) < \varepsilon$ iff there exists a section σ such that $\mathfrak{A} \Vdash_x \varphi(\sigma) < \varepsilon$.

Pointwise forcing - continued

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- $\mathfrak{A} \Vdash_x \inf_s \varphi(s) > \varepsilon$ iff there exists an open set $U \ni x$ and a real number $\delta_x > 0$ such that for every $y \in U$ and every section σ defined on y, $\mathfrak{A} \Vdash_y \varphi(\sigma) > \varepsilon + \delta_x$

Pointwise forcing - continued

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- $\mathfrak{A} \Vdash_x \sup_s \varphi(s) < \varepsilon$ iff there exists an open set $U \ni x$ and a real number $\delta_x > 0$ such that for every $y \in U$ and every section σ defined on y, $\mathfrak{A} \Vdash_y \varphi(\sigma) < \varepsilon - \delta_x$
- $\mathfrak{A} \Vdash_x \inf_{s \in A_x} \varphi(s) > \varepsilon$ iff there exists a section σ such that $\mathfrak{A} \Vdash_x \varphi(\sigma) > \varepsilon$.

Around truth continuity

- We have $\mathfrak{A} \Vdash_x \inf_s (1 \dot{-} \varphi) > 1 \dot{-} \varepsilon$ if and only if $\mathfrak{A} \Vdash_x \sup_s \varphi < \varepsilon$.

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- We can also define $\mathfrak{A} \Vdash_x \varphi(s) \leq \varepsilon$ iff $\mathfrak{A} \nVdash_x \varphi(s) > \varepsilon$ and dually for $\geq$.
- With this, for $0 < \varepsilon' < \varepsilon$, if $\mathfrak{A} \Vdash_x \varphi(s) \leq \varepsilon'$ then $\mathfrak{A} \Vdash_x \varphi(s) < \varepsilon'$

A metric on sections? (Not yet)

So far so good, but we have (for the time being) lost the metric on the sections (so, the corresponding presheaves $\mathfrak{A}(U)$ are still missing the "metric" feature - they do not live in the correct category yet).

- Sections have different domains
- Triangle inequality is tricky
- Restrict to sections with domains in a filter of open sets
- But the ultralimit (even in that case) could fail to be complete!

Rather... a pseudometric

Fix F a filter of open sets of X. For all sections σ and μ with domain in F define

$$F_{\sigma\mu} = \{U \cap \mathrm{dom}(\sigma) \cap \mathrm{dom}(\mu) | U \in F\}.$$

Then the function

$$\rho_F(\sigma, \mu) = \inf_{U \in F_{\sigma\mu}} \sup_{x \in U} d_x(\sigma(x), \mu(x))$$

is a pseudometric on the set of sections with domain in F.

Completeness of the induced metric

Theorem (Ochoa, V.)

Let $\mathfrak{A}$ be a sheaf of metric structures defined over a regular topological space X. Let F be an ultrafilter of regular open sets. Then, the metric induced by ρ_F on $\mathfrak{A}[F]$ is complete.

Local Forcing for Metric Structures

Forcing over an open set is somewhat more tricky in this case. We have the following definition.

Definition

Let $\mathfrak{A}$ be a sheaf of metric structures defined on X, $\varepsilon > 0$, U open in X, $\sigma_1, \cdots, \sigma_n$ sections defined on U. Then

- $\mathfrak{A} \Vdash_U \varphi(\sigma) < \varepsilon \iff \exists \delta < \varepsilon \forall x \in U(\mathfrak{A} \Vdash_x \varphi(\sigma) < \delta)$
- $\mathfrak{A} \Vdash_U \varphi(\sigma) > \delta \iff \exists \varepsilon > \delta \forall x \in U(\mathfrak{A} \Vdash_x \varphi(\sigma))$

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There is an involved, equivalent, inductive definition. We also have $\mathfrak{A} \Vdash_U \inf_\sigma(1 \dot{-} \varphi(\sigma) > 1 \dot{-} \varepsilon \iff \mathfrak{A} \Vdash_U \sup_U \varphi(\sigma) < \varepsilon$, and a maximal principal principle (existence of witnesses of sections).

Metric Generic Model and the Theorem

For the appropriate notion of genericity, we build the generic model as in the discrete case. The definition of genericity guarantees the completeness of $\mathfrak{A}[F]$.

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Theorem (Metric GMTh)

Let F be a generic filter on X, $\mathfrak{A}$ a sheaf of metric structures on X and $\sigma_1, \cdots, \sigma_n$ sections. Then

1. $\mathfrak{A}[F] \models \varphi([\sigma_1]/\sim_F, \cdots, [\sigma_n]/\sim_F) < \varepsilon \iff \exists U \in F$ *such that* $\mathfrak{A} \Vdash_U \varphi(\sigma_1, \cdots, \sigma_n) < \varepsilon$
2. $\mathfrak{A}[F] \models \varphi([\sigma_1]/\sim_F, \cdots, [\sigma_n]/\sim_F) > \varepsilon \iff \exists U \in F$ *such that* $\mathfrak{A} \Vdash_U \varphi(\sigma_1, \cdots, \sigma_n) > \varepsilon$

Invariant Sheaves - Cohomology

Joint work with G. Padilla:

- A version of the generic model theorem for equivariant sheaves (sheaves with group actions on fibers).
- Cohomology for equivariant sheaves.

These apply to various sheaves constructed from actions of classical groups (e.g. $SL_2(\mathbb{Z})$ acting on fibers of arithmetic sheaves connected to modular invariants).
More recently, stability theory of sheaves.

Thank you for your attention!