Model Theory for the j-mapping

Andrés Villaveces
Univ. Nacional - Bogotá

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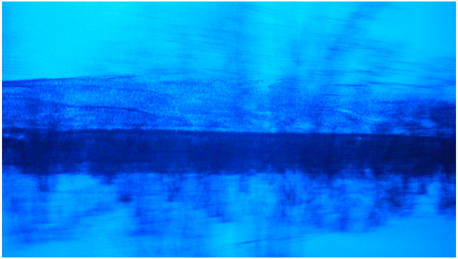

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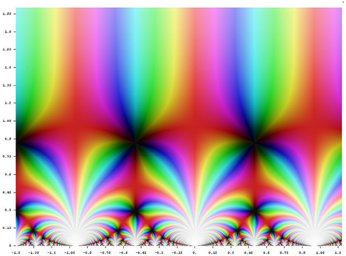
Classical j invariant



j-invariant on $\mathbb{C}$
(Wikipedia article on
j-invariant)

Klein defines the function (we call) "classical j"

$$j : \mathbb{H} \to \mathbb{C}$$

(where $\mathbb{H}$ is the complex upper half-plane)
through the explicit rational formula

$$j(\tau) = 12^3 \cdot \frac{g_2(\tau)^3}{g_2(\tau)^3 - 27 g_3(\tau)^3}$$

with g_2 and g_3 certain rational functions ("of Eisenstein").

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- $$j(\tau) = j\left(\frac{a\tau + b}{c\tau + d}\right)$$

  if $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$.

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- if $e^{2\pi i\tau}$ is algebraic then $j(\tau), \frac{j'(\tau)}{\pi}, \frac{j''(\tau)}{\pi^2}$ are mutually transcendental (Schanuel-like situation). (In January 2015, Pila and Tsimerman have announced a proof of Schanuel for j-map.)

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- (Hilbert's 12th...)

j-COVERS AND A PATH TO CATEGORICITY

Adam Harris provides a contrasting view of classical j invariants:

- An axiomatization in $L_{\omega_1,\omega}$ of j

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- A convoluted () proof of categoricity of this version of j
- Generalization of this analysis to higher dimensions (Shimura varieties).
- Analogies to pseudoexponentionation ("Zilber field") are strong, , but the structure of j seems to have a much higher degree of complexity even than *exp*.

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where $\langle F, +, \cdot, 0, 1 \rangle$ is an algebraically closed field of characteristic 0, $\langle H; \{g_i\}_{i\in\mathbb{N}} \rangle$ is a set together with countably many unary function symbols, and $j : H \to F$.

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Really, j is a **cover** from the action structure into the field $\mathbb{C}$.

The $L_{\omega_1,\omega}$-axiom - Crucial point: Standard fibers of the cover j

Let then

$$Th_{\omega_1,\omega}(j) := Th(\mathbb{C}_j) \cup \forall x \forall y (j(x) = j(y) \rightarrow \bigvee_{i<\omega} x = \gamma_i(y))$$

for $\mathbb{C}_j$ the "standard model" $(\mathbb{H}, \langle \mathbb{C}, +, \cdot, 0, 1 \rangle, j : \mathbb{H} \rightarrow \mathbb{C})$.
This captures all the first order theory of j (not the analyticity!) plus the fact that fibers are "standard" ("fibers are orbits")

Categoricity of classical j

Theorem (Harris, assuming Mumford-Tate Conj.)
The theory $Th_{\omega_1,\omega}(j) + \text{trdeg}(F) \geq \aleph_0$ is categorical in all infinite cardinalities.

CATEGORICITY OF CLASSICAL j

Theorem (Harris, assuming Mumford-Tate Conj.)
The theory $Th_{\omega_1,\omega}(j) + \mathrm{trdeg}(F) \geq \aleph_0$ is categorical in all infinite cardinalities. I.e., given two models $M_1 = (\mathcal{H}_1, F_1, j_1 : H_1 \to F_1)$ and $M_2 = (\mathcal{H}_2, F_2, j_2 : H_2 \to F_2)$ of the same infinite cardinality ($\mathcal{H}_i = (H_i, \{g^i_j\}_{j\in\mathbb{N}})$ and $\mathcal{F}_i = (F_i, +_i, \cdot_i, 0, 1)$) there are isomorphisms φ_H, φ_F such that

$$\begin{array}{ccc} \mathcal{H}_1 & \xrightarrow[\varphi_H]{} & \mathcal{H}_2 \\ \downarrow j_1 & & \downarrow j_2 \\ \mathcal{F}_1 & \xrightarrow[\varphi_F]{} & \mathcal{F}_2 \end{array}$$

In his proof, A. Harris uses an instance of the adelic Mumford-Tate conjecture for products of elliptic curves to show this. The strategy to build an isomorphism between two models M and M' consists (as expected) in

- Identifying $\mathrm{dcl}^M(\emptyset)$ with $\mathrm{dcl}^{M'}(\emptyset)$ to start the back-and-forth argument.

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- realizing the field type of a finite subset of a Hecke orbit over any parameter set (algebraicity of modular curves),...
- then show that the information in the type is contained in a finite subset ("Mumford-Tate" open image theorem used here) ... every point $\tau \in \mathbb{H}$ corresponds to an elliptic curve E – the type of τ is determined by algebraic relations between torsion points of E.

Now,

to quantum

versions of j

Threading finer on the definition of classical j:

Recall: if $\mu \in \mathbb{H}$, $\Lambda(\mu) = \mathbb{Z} + \mathbb{Z}\mu$ is the μ-lattice, and the classical torus associated to μ is

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(This is also a Riemann surface.)
Now, $\mathbb{T}(\mu)$ is equivalent to the elliptic curve $\mathbb{E}(\mu)$ given by $Y^2 = X^3 - g_2(\mu)X - g_3(\mu)$. Here, let

$$G_k(\mu) := \sum_{0 \neq \gamma \in \Lambda(\mu)} \gamma^{-2k} \qquad k \geq 2$$

(the so-called Eisenstein series), then

$$g_2(\mu) = 60 \cdot G_2(\mu)$$

$$g_3(\mu) = 140 \cdot G_3(\mu).$$

Toward quantum tori: from $\mathbb{C}$ to $\mathbb{R}$

Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$, and let Λ_θ be the pseudo-lattice $\langle 1, \theta \rangle$ (the subgroup of $\mathbb{R}$ given as $\Lambda_\theta := \mathbb{Z} + \mathbb{Z} \cdot \theta$). The quotient

$$\mathbb{T}(\theta) := \mathbb{R}/\Lambda_\theta$$

is the "quantum torus", associated to the irrational number θ. It is a one-parameter subgroup of the (classical) torus $\mathbb{T}(i)$... and also a Riemann surface.

Getting hold of quantum versions of j

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- New definition domain (from $\mathbb{H}$ to $\mathbb{R} \setminus \mathbb{Q}$)

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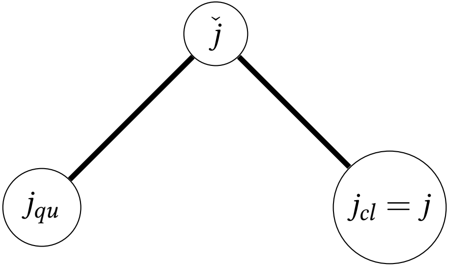
The problem:

- New definition domain (from $\mathbb{H}$ to $\mathbb{R} \setminus \mathbb{Q}$)
- Topological issues resulting from the much more chaotic behavior of $\mathbb{R}$ - continuity lost in first approximations
- Rational expressions (multivalued functions now - perhaps the average of the (finite?) set of values is <u>the</u> robust invariant).

An example of a sheaf construction / universal j

Gendron proposes a detailed construction of a sheaf over a topological space, and a generalization of classical j called "universal j-invariant" - a specific section of a sheaf.



The specific construction of universal j

(Castaño-Bernard, Gendron)
Let ${}^*\mathbb{Z} := \mathbb{Z}^{\mathbb{N}}/\mathfrak{u}$ for some nonprincipal ultrafilter $\mathfrak{u}$ on $\mathbb{N}$. Define

$$\mathcal{H} := \{[F_i] \subset {}^*\mathbb{Z}^2 \text{ hyperfinite } \}.$$

This set is partially ordered with respect to inclusion so we may consider the Stone space

$$\mathcal{R} := \mathsf{Ult}(\mathcal{H}).$$

For each $\mathfrak{p} \in \mathcal{R}$ and $\mu \in \mathbb{H}$ one may define the j-invariant

$$j(\mu, \mathfrak{p})$$

as follows:

The construction

The idea: the classical j-invariant is an algebraic expression involving Eisenstein series which is a function of $\mu \in \mathbb{H}$. We can associate to $[F_i] \subset {}^*\mathbb{Z}^2$ a hyperfinite sum modelled on the formula of the classical j-invariant, denoted

$$j(\mu)_{[F_i]} \in {}^*\mathbb{C}.$$

We get a net

$$\{j(\mu)_{[F_i]}\}_{[F_i]\in\mathcal{H}} \subset {}^*\mathbb{C}.$$

Consider the sheaf ${}^\diamond\check{\mathbb{C}} \to \mathcal{R}$ for which the stalk over $\mathfrak{p}$ is

$${}^\diamond\mathbb{C}_{\mathfrak{p}} := ({}^*\mathbb{C})^{\mathcal{H}}/\mathfrak{p}.$$

Then we may define a section:

$$\check{j} : \mathbb{H} \times \mathcal{R} \longrightarrow {}^\diamond\check{\mathbb{C}}, \quad \check{j}(\mu, \mathfrak{p}) := \{j(\mu)_{[F_i]}\}_{[F_i]\in\mathcal{H}}/\mathfrak{p}.$$

Group actions - choosing an irrational angle

What really is at stake in these constructions is the invariance under various group actions.
For each $\theta \in \mathbb{R}$ there is a distinguished subset $\mathcal{R}_\theta \subset \mathcal{R}$ of ultrafilters which "see" θ:

$$\mathcal{R}_\theta = \{\mathfrak{p} |\ \mathfrak{p} \supset \mathfrak{c}_\theta\}$$

where $\mathfrak{c}_\theta$ is the cone filter generated by the cones

$$\mathsf{cone}_\theta([F_i]) = \{[F_i]' \supset [F_i] |\ [F_i] \subset [F_i]' \subset {}^*\mathbb{Z}^2(\theta)\}.$$

In the above,

$${}^*\mathbb{Z}^2(\theta) = \{({}^*n^\perp, {}^*n) |\ {}^*n\theta - {}^*n^\perp \simeq 0\}.$$

Restricting to quantum and classical j

The quantum j-invariant is defined as the restriction:

$$\check{j}^{\mathrm{qu}}(\theta) := \check{j}|_{\mathcal{R}_\theta}(i, \cdot).$$

If we denote

$$\mathcal{R}_{\mathrm{cl}} = \{\mathfrak{p} | \ \mathfrak{p} \supset \mathfrak{c}\}$$

where $\mathfrak{c}$ is the filter generated by *all* cones over hyperfinite sets in ${}^*\mathbb{Z}^2$:

$$\mathrm{cone}([F_i]) = \{[F_i]' \supset [F_i] | \ [F_i]' \subset {}^*\mathbb{Z}^2\}.$$

Then the restriction

$$\check{j}^{\mathrm{cl}} := \check{j}|_{\mathcal{R}_{\mathrm{cl}}}$$

satisfies

$$\check{j}^{\mathrm{cl}}(\mu, \mathfrak{p}) \simeq j(\mu), \quad \forall \mu \in \mathbb{H},$$

where j is the usual j-invariant.

Duality I

Note the duality in the way of recovering the classical and quantum invariants:

- the classical invariant is recovered along a unique fiber $^{\diamond}\check{\mathbb{H}}_{\mathfrak{u}}$ (i.e., a leaf of the quotient of sheaves $\widehat{Mod}$),
- the quantum invariant is obtained by fixing the fiber parameter $i \in \mathbb{H}$ and letting $\mathfrak{u} \in Cone(\theta)$ vary: it therefore arises from a local section defined by i (a transversal of $\widehat{Mod}$).

Conjectures

The main goal is to check that if $\theta \in \mathbb{R} \setminus \mathbb{Q}$ is quadratic, then Hilbert's classfield H_K of $K = \mathbb{Q}(\theta)$ (K's maximal unramified extension) equals

$$K(j(\theta)).$$

This would give a solution to Hilbert's 12th problem for quadratic real extension (unramified case).

Again, the main point is to prove the analog of "complex multiplication" (keypoint: the algebraicity of $j(\mu)$, when $\mu \in \mathbb{Q}(\sqrt{D})$, for $D < 0$ square free - and the fact that $j(\mu)$ essentially generates the Hilbert classfield $H(\mu)$ of $\mathbb{Q}(\sqrt{D})$).

We conjecture (with Gendron) that for $\theta \in \mathbb{R}$ there exists a duality relation between the classical invariant $j(i\theta)$ and the quantum invariant $j(\theta)$.

Duality II

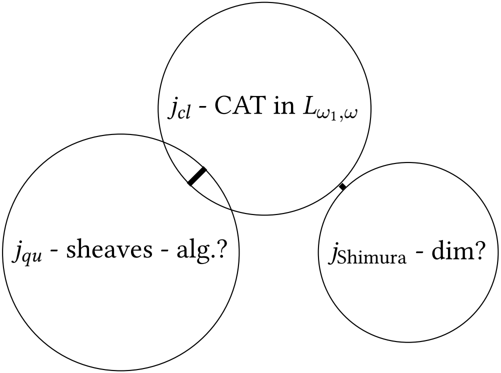
More precisely, we associate to $j(i\theta)$ and $j(\theta)$ two nets

$$\{j(i\theta)_\alpha\} \qquad \text{and} \qquad \{j(\theta)_\alpha\}$$

whose elements are algebraically interdependent. The two nets converge to a common limit. The classical net $\{j(i\theta)_\alpha\}$ lives along a fixed leaf of ${}^\diamond\widehat{Mod}$; the quantum net$\{j(\theta)_\alpha\}$ lives on a fixed transversal of ${}^\diamond\widehat{Mod}$.

Through the sheaves

The current model theoretic analysis of j looks at two possible extensions:



Two different toolkits

The two directions of generalization (to quantum tori/real multiplication on the one hand, to higher dimension varieties/Shimura on the other hand) of j maps calls different aspects of model theory (at the moment, an "amalgam" has started, but is far along the way):

- The model theory of abstract elementary classes (in particular, the theory of excellence - now "old" (1980s) but recently (2013) streamlined by Bays, Hart, Hyttinen, Kesälä, Kirby - Quasiminimal Structures and Excellence. A kind of cohomological analysis of models (and types), connected to categoricity and "smoothness". Zilber field, now j!

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- The model theory of sheaves. A tool for (topological) "limit" structures.

Model Theory for equivariant sheaves

- **Theorem:** [Padilla, V., extending Caicedo] Fix a first order vocabulary τ. Let X be a topological space, $\mathfrak{A}$ a sheaf of τ-structures over X, $\mathcal{F}$ a filter of open sets generic for $\mathfrak{A}$, and $\varphi(v_1, \cdots, v_n)$ a τ-formula. Then, given sections $\sigma_1, \cdots, \sigma_n$ of the sheaf (defined on some open set in $\mathcal{F}$), we have
  If a group G acts equivariantly on fibers (coherently) then (requiring that $\mathcal{F}$ is G-invariant)

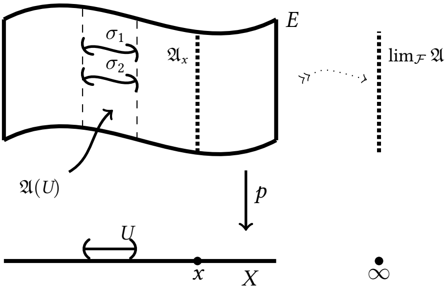
$$\mathfrak{A}^X/\mathcal{F}/G \models \varphi(\sigma_1^G/_{\sim_\mathcal{F}}, \cdots, \sigma_n^G/_{\sim_\mathcal{F}}) \quad \Leftrightarrow \quad \exists U \in \mathcal{F}, \quad \mathfrak{A} \Vdash_U \varphi(\sigma_1, \cdots, \sigma_n).$$

- Stability theory for quotients of sheaves (foliations, etc.)

A topological representation

A topological representation of our sheaf: let $E \xrightarrow{p} X$ be a local homeomorphism. We call fibers (or stalks) the preimages $p^{-1}(x)$. They are always discrete subspaces of E.

(Continuous) sections σ (the elements of the structures $\mathfrak{A}(U)$ over every open set U are partial inverses of p: $p \circ \sigma = id_U$. As usual, we identify sections σ with their images; these images form a basis for the topology of E.



Model-theoretic geometry?

Model Theoretic properties that correspond to known theorems of mathematics:

- In Harris/Zilber, Serre's Open Image Theorem corresponds to categoricity.
- More generally, Zilber has claimed that categoricity may be construed as a 21st century version of analyticity. Of course, a vague statement, but the "regularity" arising from analyticity for number fields, for number theoretical questions, is recovered

Thank you for your attention!