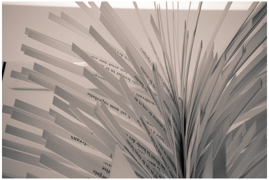

Limit Structures and Non-Locality

A discussion session

Andrés Villaveces
Universidad Nacional de Colombia - Bogotá

Logics, Decisions and Interactions
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Contents

THE WORKSHOP TRIAD

- LogicS - and limiting processes between them

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- DecisionS - (leading to emergence of logic?)

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- LogicS - and limiting processes between them
- DecisionS - (leading to emergence of logic?)
- InteractionS - the back and forth between logics, model classes, and ... logics

Ideal (limit) models in Physics - Goals

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Various questions (classical and recently posed or revisited) in Physics, Chemistry and ... point towards the need of various kinds of "ideal structures", and **tools of contrast** between "real structures" and those ideal (limit) structures.

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Implicit knowledge by the physicist of the structure of his model, not yet available to mathematicians? (Rabin, Rieffel, Zeidler)

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- Lack of a non-dependent conceptual framework!

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- Calibrating via weak/strong logics reflection phenomena,
- When combined with topology (logic on sheaves), logic may be "dynamized"

II - Model Theory

1. **Arbitrary** structures.
2. Hierarchy of types of structures (or of theories): stability theory.
3. At the “top” of the hierarchy: Hrushovski-Zilber’s Zariski Structures - generalized algebraic varieties over ACF, with relations = Zariski-closed sets.
4. One-dimensional objects of Zariski structures are exactly finite covers of algebraic curves - these correspond to “nonclassical” structures coming from non-commutativity phenomena.
5. With Model Theory on Sheaves: strong ways of controlling limit models.
6. May even go “beyond logic-dependence” and get several of the previous (Abstract Elementary Classes).

Limit and ideal models, à la Zilber

Quoting Zilber:

`The process of understanding the physical reality by working in an` ideal `model can be interpreted as follows. We assume that the ideal model` $\mathbb{M}_{\text{ideal}}$ `is being chosen from a class of "nice" structures, which allows a good theory. We suppose that the real structure` $\mathbb{M}_{\text{real}}$ `is "very similar" to` $\mathbb{M}_{\text{ideal}}$ `(...) approximated by a sequence` $\mathbb{M}_i$ `of structures and` $\mathbb{M}_{\text{real}}$ `is one of these,` $\mathbb{M}_i = \mathbb{M}_{\text{real}}$ `sufficiently close to` $\mathbb{M}_{\text{ideal}}$`. The notion of approximation must also contain both logical and topological ingredients. (...)`

This goes on...

... the reason that we wouldn't distinguish two points in the ideal model $\mathbb{M}_{\text{ideal}}$ is that the corresponding points are very close in the real world $\mathbb{M}_{\text{real}}$ so that we do not see the difference (using the tools available). In the limit of the $\mathbb{M}_i$'s this sort of difference will manifest itself as an infinitesimal. In other words, the limit passage from the sequence $\mathbb{M}_i$ to the ideal model $\mathbb{M}_{\text{ideal}}$ must happen by killing the infinitesimal differences. (...) This corresponds to taking a specialization (...) from an ultraproduct $\prod_D \mathbb{M}_i$ to $\mathbb{M}_{\text{ideal}}$.

BUT...

> ... We note that the scheme is quite delicate regarding metric issues. In principle we may have a well-defined metric (...) on the ideal structure only. Existence of a metric, especially the one that gives rise to a structure of a differentiable manifold, is one of the key reasons of why we regard some structures as "nice" or "tame". The problem of whether and when a metric on $\mathbb{M}$ can be passed to approximating structures $\mathbb{M}_i$ might be difficult, indeed we don't know how to answer this problem in some interesting cases.

Zilber's Structural Approximation [Zil2]

Given a topological structure $\mathbb{M}$ and a family of structures $\mathbb{M}_i$, $i \in I$, in the same language, $\mathbb{M}$ is approximated by $\mathbb{M}_i$ along an ultrafilter D on I if for some elementary extension $M^D \succ \prod \mathbb{M}_i/D$ of the ultraproduct there is a surjective homomorphism

$$\lim_D : \mathbb{M}^D \to \mathbb{M}.$$

Examples

These include:

1. The Gromov-Hausdorff limit of metric spaces along a non-principal ultrafilter D.
2. Structural approximation of a quantum torus at q by quantum tori at roots of unity.

Kochen-Specker's Impossibility Theorem

The common sense belief that "every physical quantity must have a value even if we do not know what it is" is challenged in Quantum Physics at the level of the formalism itself: Kochen and Specker proved in 1967 the impossibility of assigning values to all physical quantities while preserving the functional relations between them. This has a sheaf "model theoretical" flavor that was first noticed by Domenech, Freytes and De Ronde, who built a first sheaf theoretic analysis of the theorem.
Döring and Isham have constructed a sheaf "spectral presheaf" that, within the topos-theoretic realm, captures Kochen-Specker as the non-existence of global sections for those spectral presheaves, when the Hilbert space has dimension ≥ 2.

Abramsky, Brandenburger - a framework for non-locality [AbrBra]

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- The existence of a **global section** for such a sheaf ("empirical model") implies the existence of a local deterministic hidden-variable model.

Limits

Theorem (A classical Generic Model Theorem; Caicedo [Cai])

Let $\mathbb{F}$ be a generic filter for a sheaf of first order structures $\mathfrak{A}$ over X. Then

$$\begin{aligned} \mathfrak{A}[\mathbb{F}] \models \varphi(\sigma/\sim_{\mathbb{F}}) \quad &\iff \quad \{x \in X | \mathfrak{A} \Vdash_x \varphi^G(\sigma(x))\} \in \mathbb{F} \\ &\iff \quad \exists U \in \mathbb{F} \text{ such that } \mathfrak{A} \Vdash_U \varphi^G(\sigma). \end{aligned}$$

Here, φ^G is a formula equivalent classically to φ, but not necessarily in an intuitionistic framework! (The formula φ^G is sometimes called the Gödel translation of φ - in 1925, Kolmogorov had independently defined an equivalent translation.)

Addenda: after the discussion

- In joint work with Ochoa, we have found conditions on the topology of X that support a generic model theorem for continuous logic in the fibers (see [OchVil]).
- Continuous Logic may be understood in various ways; we used the "standardization" due to Ben Yaacov, Berenstein, Henson and Usvyatsov [BYBHU]

Now, to the discussion!

Questions for you all:

- Limit Structures / Limit Logics - Where else?
- Emergence of properties only after limit - Where else?
- Non-locality is one example, more connected to others than expected - Where else?
- Continuous / Discrete (back to von Neumann, ...)
- Dynamic / Static (sheaves ...)
- Syntactically Given (Logics - Infinitary, Generalized Quantifiers, etc.) / Semantically Given (Abstract Elementary Classes)
- Orbits / Maximal Consistent Sets / Types
- Transfer small languages
- Large languages controlling small languages

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