Model Theory and Non-Locality

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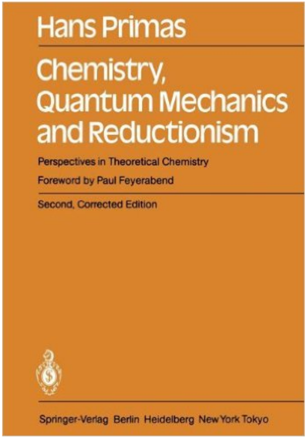
Contents

Hans Primas

In his book Chemistry, Quantum Mechanics and Reductionism (1982) Swiss chemist Hans Primas voices serious uncomfort with various fundamental questions of theoretical chemistry.



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- ... a danger to forget the original impetus of our enterprise: understanding the behavior of matter
- ... numerical qm is a most important tool for chemistry but it cannot replace thinking...

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- Do isolated quantal systems exist at all? What is a "system" (in the presence of entanglement)? (EPR, Bell's Inequalities, etc.). This is the most important open problem, according to Primas!

Primas, toward a kind of logic...

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- Is the superposition principle universally valid?
- The main stumbling block for the development of a theory of large and complex molecular systems is not computational but conceptual...
- Good theory: CONSistent, CONFirmed and INTuitable. The two mappings (from the external world into the formal framework of the theory, from the formal framework of the theory into psychic structures of the subject) of knowledge.

Primas's Questions...

- SYNTax, SEMantics, PRAGMatics - a theory must be CONSistent, CONFirmed and INTUITable
- Inner perfection (Einstein), naturalness, simplicity —
- The relationship of a theory with inner reality (pragmatics [3]) vs outer reality (semantics [2])
- METAcompleteness theorem? (intuitable   confirmed/realizable consistent)
- nowadays: predominantly operational semantic interpretation

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Original Scientific Paper

From Trees (Dendrograms and Consensus Trees) to Topology*

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chemotopology
mathematical chemistry
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cluster analysis
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DOI: 10.1007/s10910-005-9013-5

On the topological sense of chemical sets

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- Topologization/ordering of chemical structure,
- Dynamical processes in chemistry, rooted of course in quantum physics, but with an emphasis on relational emerging properties.
- This last point forces suspension of reductionism.

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Mathematical frameworks against reductionism

Mathematical Logic(s)... and more specifically

Model Theory

More recently, Theoretical Chemistry - according to J. Math. Chem. 2015

- Computational Numeric Methods,
- Big data, data mining,
- Graph Theory, dendrograms, dendimerons,
- Networks and graphs,
- Knot theory, bindings,
- Quantum Information ...

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- Computational Numeric Methods,
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- Graph Theory, dendrograms, dendimerons,
- Networks and graphs,
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- Quantum Information ...
- But Primas's questions seem to a large extent still open!

Towards a model theory of chemistry...

> *It is strange that in almost every chemical operation - all degrees of connection or separation, etc. present appear simultaneously - in different relations - and tend to remain. Relation to different tones of playing a string - the fifth, the third [interval], etc.*

Novalis - Allgemeine Brouillon - c. 1799

What is Model Theory?

Model Theory is a branch of mathematical logic, that has been described in several ways:

- Keisler (1970): model theory = logic + universal algebra

Model theory starts with the relationship between model classes (classes of structures) and their possible axiomatizations.

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- Hrushovski (2006): model theory: the geography of tame mathematics

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Too wide a question, of course - but it is worth mentioning that model theory

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- Isolates notions of genericity, of imaginary elements - it is always, in precise senses, the most general kind of "Galois theory" available to us today.
- Galois theory was born 200 years ago (almost like Chemistry...) as a theory of invariants under symmetries: instead of looking for "solutions" you look at all possible symmetries of all "possible solutions" and "filter down" by subgroups of symmetries.

Model Theory: What does it achieve?

- Model Theory isolates the extreme notion of categoricity (when is an axiomatization, a description of a phenomenon of the world, "perfect"?), and

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- Model Theory isolates the extreme notion of categoricity (when is an axiomatization, a description of a phenomenon of the world, "perfect"?), and
- Allows filtering out through hierarchies ("model theoretic stability") all possible mathematical theories and detect invariance between many of them.
- MTh is the widest theory of logical comparison (even abstract geometrical comparison) available to us

More recently...

Although Model Theory was born within mathematical logic (Gödel, Tarski, etc.) and its development has happened mostly in "dialogue" with the rest of mathematics (Shelah, Hrushovski, Zilber, Ax-Kochen, Robinson, etc.), part of Model Theory has been getting closer to topics such as

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The model theory of physics - via non-commutative geometry, modular invariants and the model theory of operator algebras (C^*-algebras, W^*-algebras, etc.)

Sheaves, infinitary logic, categoricity

- Model theory on sheaves (Macintyre, Caicedo - a "dynamization" of logic - controlled topologically to analyze simultaneously complex systems) - a system of "approximations of limit structures".
- Sheaves mixed with metric structures - Ochoa, V. [OchVil]
- Quantum Harmonic Oscillator $\to$ (and finally, Feynman integral) has resisted mathematization. Zilber, with Cruz and others has embarked in that line using sheaves.

Quantum Entanglement

One of the most fundamental phenomena of Quantum Physics, at crossroads with General Relativity - yet recently corroborated experimentally - quantum entanglement later explained by the Kochen-Specker theorem and the Bell inequalities.
This is a local/global phenomenon: the non-existence of a global section of a sheaf - Abramsky/Brandenburger.

Paths (Primas)

Back to Primas: he proposes the following questions, beyond the fixation on quantum mechanics:

- Finding an appropriate language for a theory of substances and molecules, a language accounting for typical phenomena in chemical taxonomy, quantum thermodynamics, chemical kinematics and chemical system theory.
- Contemporary quantum chemistry may not quite be false, but it is not appropriate. Explanation is missing!

Primas - EPR, Model theory?

Primas proposes various things (up to some point) with a model-theoretic flavor. Part of the difficulty comes from entanglement - previously unaccessible to model theoretic analysis. Primas proposes to study simultaneously many operator algebras ("observables" - W^*-algebras) in a common frame - what he calls the "lattice of subtheories" - and in this lattice the subtheories $T_\alpha \leq T$ each reflects a greater or smaller degree of entanglement.
Model Theory is there...

Kochen-Specker's Impossibility Theorem

The common sense belief that "every physical quantity must have a value even if we do not know what it is" is challenged in Quantum Physics at the level of the formalism itself: Kochen and Specker proved in 1967 the impossibility of assigning values to all physical quantities while preserving the functional relations between them.
This has a sheaf "model theoretical" flavor that was first noticed by Domenech, Freytes and De Ronde, who built a first sheaf theoretic analysis of the theorem.
Döring and Isham have constructed a sheaf "spectral presheaf" that, within the topos-theoretic realm, captures Kochen-Specker as the non-existence of global sections for those spectral presheaves, when the Hilbert space has dimension ≥ 2.

Abramsky, Brandenburger - a framework for non-locality.

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- The existence of a **global section** for such a sheaf ("empirical model") implies the existence of a local deterministic hidden-variable model.

"Et, comme une même ville regardée de différents côtés paraît toute autre et est comme multipliée perspectivement, il arrive de même, que par la multitude infinie des substances simples, il y a comme autant de différents univers, qui ne sont pourtant que les perspectives d'un seul selon les différents points de vue de chaque Monade."

G.W. Leibniz, Monadologie, § 57

Sheaves over topological spaces

Fix X a topological space. The pair (E, p) is a sheaf over X if and only if E is a topological space and $p : E \to X$ is a surjective local homeomorphism.

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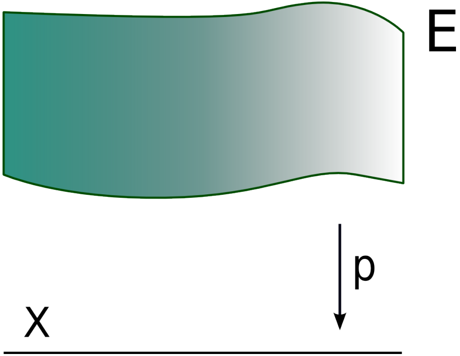

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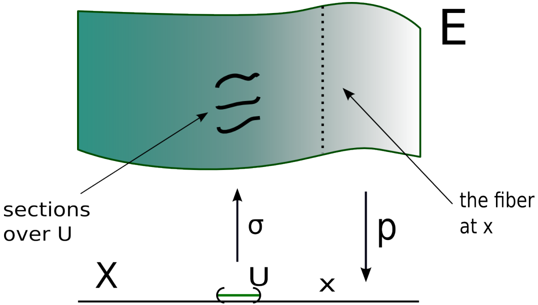
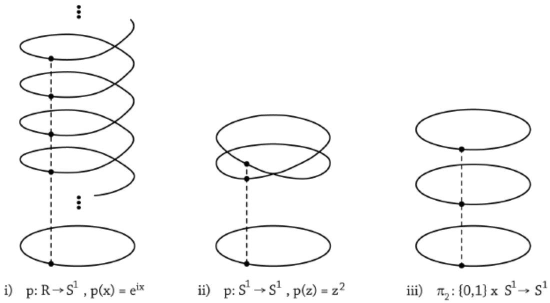
- The (images of) sections σ form a basis for the topology of E (a section is a continuous partial inverse of p defined on an open set $U \subset X$),
- If two sections σ, τ coincide at a point a then there exists an open set $U \ni a$ such that $\sigma \restriction U = \tau \restriction U$

Sections - objects



Some examples



(from Caicedo's monograph)

Sheaves of Structures

A sheaf of structures $\mathfrak{A}$ over X consists of:

1. A sheaf (E, p) over X,
2. On every fiber $p^{-1}(a)$ $(a \in X)$, a structure

$$\mathfrak{A}_a = (E_a, (R_i^a)_i, (f_j^a)_j, (c_k^a)_k,)$$

such that $E_a = p^{-1}(a)$, and

- For every i, $R_i^{\mathfrak{A}} = \bigcup_{x \in X} R_i^{\mathfrak{A}_x}$ is open
- For every j, $f_j^{\mathfrak{A}} = \bigcup_{x \in X} f_j^{\mathfrak{A}_x}$ is continuous
- For every k, $c_k^{\mathfrak{A}} : X \to E$ such that $x \mapsto c_k^{\mathfrak{A}_x}$ is a continuous global section

Truth Continuity?

Fact

For all atomic formulas $\varphi(v)$ we have that

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This also holds for positive Boolean combinations of atomic formulas.

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The solution to this failure is to switch to an emphasis on forcing.

María Clara Cortés - (Seurasaari - Talvi 2007)

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How do we compare them? Before diving into the definitions of the forcing notions, notice that the first one is pointwise while the second one is local. Also notice that satisfaction in $\mathfrak{A}_x$ is about values of sections at x (the $\sigma(x)$) whereas pointwise (over x) or local forcing (over U) are about the whole section σ defined on U.

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Sections are the new objects: formulas $\varphi(v_1, v_2, \cdots)$ will be "evaluated" by "replacing" v_i by a section σ_i or by its value at an element x of X, $\sigma_i(x)$.

Pointwise forcing

- For atomic φ and $t_1, \cdots, t_n$ terms,
  $\mathfrak{A} \Vdash_x (t_1 = t_2)[\vec{\sigma}] \Leftrightarrow t_1^{\mathfrak{A}_x}[\vec{\sigma}(x)] = t_2^{\mathfrak{A}_x}[\vec{\sigma}(x)]$
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- $\mathfrak{A} \Vdash_x \forall v \varphi(v, \vec{\sigma}) \Leftrightarrow$ for some $U \ni x$, for <u>every</u> $y \in U$ and <u>every</u> σ defined on y, $\mathfrak{A} \Vdash_y \varphi[\sigma, \vec{\sigma}]$.

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Limits

Theorem (A classical Generic Model Theorem)

Let $\mathbb{F}$ be a generic filter for a sheaf of topological structures $\mathfrak{A}$ over X. Then

$$\begin{aligned}\mathfrak{A}[\mathbb{F}] \models \varphi(\sigma/\sim_{\mathbb{F}}) &\iff \{x \in X | \mathfrak{A} \Vdash_x \varphi^G(\sigma(x))\} \in \mathbb{F} \\ &\iff \exists U \in \mathbb{F} \textit{ such that } \mathfrak{A} \Vdash_U \varphi^G(\sigma).\end{aligned}$$

Here, φ^G is a formula equivalent classically to φ, but not necessarily in an intuitionistic framework! (The formula φ^G is sometimes called the Gödel translation of φ - in 1925, Kolmogorov had independently defined an equivalent translation.)

More on the Generic Model Theorem

Cohen's construction of generic models for set theory is the first published result along these lines. Later, Robinson, Barwise and Keisler used generic model theorems to get Omitting Types Theorems in various logics, generalized by Caicedo. Ellerman's "ultrastalk theorem" (1976) is a GMTh for maximal filters. Miraglia also proves a similar result for Heyting-valued models.

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$$\sigma \mapsto \sigma^* = \sigma \cup \{(\infty, [\sigma]_{\sim_{\mathbb{F}}})\}.$$

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$$\sigma \mapsto \sigma^* = \sigma \cup \{(\infty, [\sigma]_{\sim_{\mathbb{F}}})\}.$$

Then, the GMTh just means that in the new sheaf $\mathfrak{A}^\infty$ this fiber is classic:

$$\mathfrak{A}^\infty \Vdash_\infty \varphi(\sigma_1^*, \cdots, \sigma_n^*) \Leftrightarrow \mathfrak{A}[\mathbb{F}] \models \varphi([\sigma_1^*], \cdots, [\sigma_n^*])$$

Independence, aware of metric and continuity

Definition (ε-coheir / the simplest)

Let $M \prec N$.

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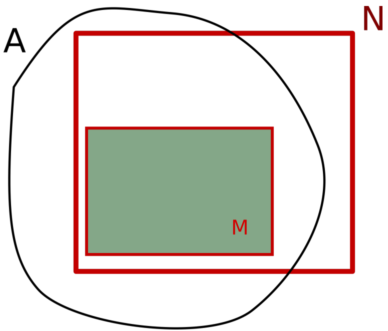

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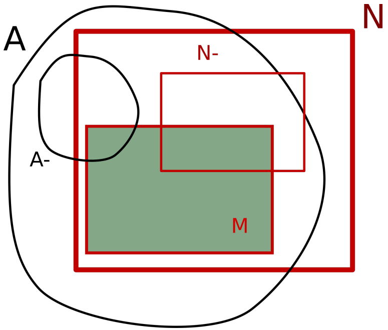

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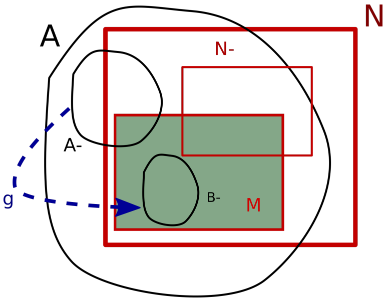
$$\exists B^- \subset M \text{ so that}$$

$$d(\mathrm{tp}(B^-/N^-), \mathrm{tp}(A^-/N^-)) < \varepsilon.$$

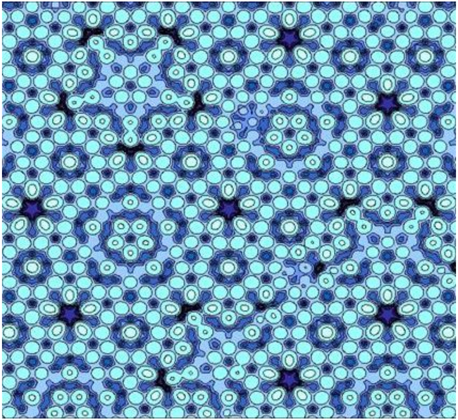


(g fixes N-)

Galois - Groups - Ambiguity Theory

Mes principales méditations depuis quelque temps étaient dirigées sur l'application à l'analyse transcendante de la théorie de l'ambiguïté. Il s'agissait de voir a priori dans une relation entre quantités ou fonctions transcendantes quels échanges on pouvait faire, quelles quantités on pouvait substituer aux quantités données sans que la relation pût cesser d'avoir lieu. Cela fait reconnaître tout de suite l'impossibilité de beaucoup d'expressions que l'on pourrait chercher. Mais je n'ai pas le temps et mes idées ne sont pas encore bien développées sur ce terrain qui est immense...
Galois

Thank you! Danke!



Potential energy surface for silver depositing on an aluminium-palladium-manganese (Al-Pd-Mn) quasicrystal surface.

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