LANGUAGE, LOGIC AND NONELEMENTARY CLASSES: EXTERNAL/INTERNAL INTERACTIONS.

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Abstract. In this minicourse, I will explore three independent issues around the interaction between language and nonelementary classes: emergence and control (The Presentation Theorem for AECs), invariance and groups (the SIP property for homogeneous classes) and interpolation (comparing different AECs, lifting classical comparisons from Abstract Model Theory). The minicourse will be roughly self-contained (for people with some training in logic). It will combine classical parts with more recent constructions. The main focus is on the interaction between language and classes of structures (through logic), as highlighted by three different theorems. The course is not strongly cumulative: the three topics have independent motivations.

The level will be kept basic and accessible, with occasional forays into more technical matters. The aim of the minicourse, in addition to presenting a perspective into "linguistic" aspects of AECs, will be to open room for "philosophical" aspects of AECs typically left aside in courses. The second and third topic have never been presented in the form of a minicourse, so in this sense this will be a "first run".

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Part 1. **The Presentation Theorem**

[From the Abstract: This is a classical theorem due to Shelah, basic in AECs. I will give basic definitions, a statement of the theorem, a sketch of proof and various connections to other topics (I will describe its huge impact in the development of the Model Theory of AECs).]

1. Abstract Elementary Classes

The move from control of a class of structures by the logic (satisfaction of formulas—*tò lógos*) to "strong embeddings" between models (*tò harmótton*—encasing[1]) is strongly in the direction of *formalism freeness*.

	Concept	Model Theory	Emphasis
Elementary	tò lógos	$M \models \varphi$	Linguistic control
Nonelementary	tò harmótton	$M \prec_{\mathcal{K}} N$	Embedding

Earlier classes included Fraïssé classes, Jónsson classes. Definition of AEC and related properties (AP, Galois type, JEP, tameness). What model theory could be developed with no compactness? With apparently so few tools to grasp meaningful properties?

Definition 1. [Abstract Elementary Class] Fix a language L. A class $\mathcal{K}$ of L-structures, together with a binary relation $\prec_{\mathcal{K}}$ on $\mathcal{K}$ is an *abstract elementary class* (for short, AEC) if:

(1) Both $\mathcal{K}$ and $\prec_{\mathcal{K}}$ are closed under isomorphism. This means two things: first, if $M' \approx M \in \mathcal{K}$ then $M' \in \mathcal{K}$; second, if M', N' are L-structures with $M' \subset N'$, $M' \approx M$, $N' \approx N$ and $M \prec_{\mathcal{K}} N$ then $M' \prec_{\mathcal{K}} N'$.
(2) If $M, N \in \mathcal{K}$, $M \prec_{\mathcal{K}} N$ then $M \subset N$,
(3) $\prec_{\mathcal{K}}$ is a partial order,
(4) (**Coherence**) If $M \subset N \prec_{\mathcal{K}} N'$ and $M \prec_{\mathcal{K}} N'$ then $M \prec_{\mathcal{K}} N$,
(5) (**LS**) There is a cardinal (called "the Löwenheim-Skolem number" of the class) $\kappa = LS(\mathcal{K}) \geqslant \aleph_0$ such that if $M \in \mathcal{K}$ and $A \subset |M|$, then there is $N \prec_{\mathcal{K}} M$ with $A \subset |N|$ and $|N| \leqslant |A| + LS(\mathcal{K})$,
(6) (**Unions of $\prec_{\mathcal{K}}$-chains**) If $(M_i)_{i<\delta}$ is a $\prec_{\mathcal{K}}$-increasing chain of length δ (δ a limit ordinal), then
   - $\bigcup_{i<\delta}(M_i)_{i<\delta} \in \mathcal{K}$,
   - for each $j < \delta$, $M_j \prec_{\mathcal{K}} \bigcup_{i<\delta} M_i$,
   - if for each $i < \delta$, $M_i \prec_{\mathcal{K}} N \in \mathcal{K}$ then $\bigcup_{i<\delta} M_i \prec_{\mathcal{K}} N$ (**smoothness**)

The right notion of type - orbital/Galois type, the right notions of AP, JEP, etc.
At this point, we have the following situation:

- So far, no control on possible axiomatization of the class $\mathcal{K}$. The emphasis is placed on its being closed under the constructions specified in the axioms. However, later (in subsection 2) we focus on the logical control of these classes. Remember Shelah's "algebraically-minded model theory".
- These are not necessarily amalgamation classes: there is no amalgamation axiom. However, many AECs do satisfy the amalgamation property. Furthermore, the model theory will depend on the kind of amalgamation possible in the class.

[1]The connection between these concepts and model theory will be further explored in forthcoming work with J. Kennedy. See Patočka for a discussion on the emergence and role of *tò harmótton* in Greek Aesthetics.

2. The Presentation Theorem (Classical form - traces from outside)

Although the definition places no emphasis whatsoever on formulas or theories, there is a general *Presentation Theorem* that enables in some cases to extract properties from first order logic, or from various infinitary logics, and translate them into properties of the class $\mathcal{K}$.

Theorem 2. *Let* $(\mathcal{K}, \prec_{\mathcal{K}})$ *be an AEC in a language* L*. Then there exist*

- *A language* $L' \supset L$*, with size* $LS(\mathcal{K})$*,*
- *a (first order) theory* T' *in* L' *and*
- *a set of* T'*-types,* Γ'*, such that*

$$\mathcal{K} = PC(L, T', \Gamma') := \{M' \restriction L \mid M' \models T', M' \text{ omits } \Gamma'\}.$$

Moreover, if $M', N' \models T'$*, they both omit* Γ'*,* $M = M' \restriction L$ *and* $N = N' \restriction L$*, then*

$$M' \subset N' \Leftrightarrow M \prec_{\mathcal{K}} N.$$

Proof. Let $L' = L \cup \{F^n_i \mid i < LS(\mathcal{K}), n < \omega\}$ and let T' be the theory consisting of the following axioms:

- $\exists x(x = x)$,
- $\forall x_0 \cdots \forall x_{n-1} F^n_i(x_0, \cdots, x_{n-1}) = x_i$, for $i < n$.

Notice that this does not determine the values of F^n_i for $i \geqslant n$.

Now let $M' \models T'$ and $a \in M$ a tuple. For each subtuple b of a let

$$U_b := \{F^m_i(b) \mid i < LS(\mathcal{K})\}, \qquad \text{with } \ell(b) = m.$$

Then $U_b \subset M'$. Whether U_b is or not the universe of a submodel of $M' \restriction L$ is determined by the quantifier-free type (in the language L') of a over the empty set,

$$\mathrm{qftp}_{L'}(a/\emptyset, M').$$

Now, if U_b is indeed the universe of a submodel of $M \restriction L$ then this submodel is "generated" (entirely determined) by b and M' - call it M_b.

The isomorphism type of M_b, and its membership in $\mathcal{K}$, are also determined by $\mathrm{qftp}_{L'}(a/\emptyset, M')$, as well as whether $M_b \prec_{\mathcal{K}} M_a$:

Fact 3. *If* $\mathrm{qftp}_{L'}(a_1/\emptyset, M') = \mathrm{qftp}_{L'}(a_2/\emptyset, M')$ *then*

- $M_{b_1} \approx M_{b_2}$ *whenever* b_1 *is a subtuple of* a_1*,* b_2 *is the corresponding subtuple of* a_2 *and* U_{b_1}*,* U_{b_2} *are the universes of a submodel of* $M' \restriction L$*.*
- *In this case,* $M_{b_1} \in \mathcal{K}$ *iff* $M_{b_2} \in \mathcal{K}$*.*
- $M_{b_1} \prec_{\mathcal{K}} M_{a_1}$ *iff* $M_{b_2} \prec_{\mathcal{K}} M_{a_2}$*.*

Let now Γ' be the set of all qf types of the form $\mathrm{qftp}_{L'}(a/\emptyset, M')$ for $M' \models T'$ and a a tuple in M' that *do not* satisfy the two conditions:

- For every subtuple b of a the set U_b is the universe of a submodel M_b of $M' \restriction L$ and $M_b \in \mathcal{K}$.
- For every subtuple b of a, $M_b \prec_{\mathcal{K}} M_a$.

Fact 4. $\mathcal{K} = PC(L, T', \Gamma')$.

Proof of 4. If $M \in \mathcal{K}$, define the expansion M' of M by induction on $n < \omega$ (on the values of the functions F^n_i for $i < LS(\mathcal{K})$) in such a way that for each $a \in M$ of length n, the set $U_a = \{F^n_i(a) \mid i < LS(\mathcal{K})\}$ is a $\mathcal{K}$-submodel M_a of M, and the assignment is consistent with T'.

$\underline{n = 0}$: Let $M_\emptyset \prec_{\mathcal{K}} M$ of size $LS(\mathcal{K})$ and $U_\emptyset = \{F^0_i \mid i < LS(\mathcal{K})\}$ an enumeration of its universe.

$\underline{n \to n+1}$: let $M_a \prec_{\mathcal{K}} M$ of size $LS(\mathcal{K})$ extend the union of the U_b's for b a subtuple of a. Now enumerate U_a by $\{F^n_i(a) \mid i < LS(\mathcal{K})\}$, in such a way that $F^n_i(a) = a_i$ for $i < n+1$.

Check that $M' \models T'$ and omits Γ'.

Now, if $M \in PC(L, T', \Gamma')$ then let $M' \models T'$ omitting Γ' be such that $M' \upharpoonright L = M$. By the omission of Γ', all sets U_a, for each $a \in M$, are the universe of a submodel $M_a \subset M$, and $M_a \in \mathcal{K}$. Now the $\prec_{\mathcal{K}}$-system $(M_a \mid a \in M)$ is clearly directed since $M_a, M_b \prec_{\mathcal{K}} M_{ab}$. Then we have that $\bigcup_{a\in M} M_a \in \mathcal{K}$. Now, by the axiomatization of T', we have $a \in U_a$, hence $\bigcup_{a\in M} M_a = M$. Therefore $M \in \mathcal{K}$. □

3. Applications

3.1. Categoricity implies Stability.

Use the Shelah presentation theorem for the EM models. More in general...

Theorem 5. *Let $(\mathcal{K}, \prec_{\mathcal{K}})$ be an AEC with amalgamation and arbitrarily large models. If $\mathcal{K}$ is categorical in $\lambda > LS(\mathcal{K})$ then it is μ-galois-stable for each cardinal $\mu \in [LS(\mathcal{K}), \lambda)$.*

Proof. Let $\mathcal{K}$ be as in the statement and assume that the conclusion fails for some minimal $\mu \in [LS(\mathcal{K}), \lambda)$. Let $M \in \mathcal{K}_\mu$ be such that $|ga - S(M)| > \mu$. Now amalgamate over M all models containing realizations of types in $|ga - S(M)|$. By AP and NMM we can without loss of generality assume this amalgam is of size $\geqslant \lambda$, by Downward Löwenheim-Skolem, we can find $N_1 \in \mathcal{K}_\lambda$ with $> \mu$ realizations of types in $|ga - S(M)|$.

We restrict ourselves to the case λ regular - the proof can be extended to arbitrary λ.

Now let $\mathcal{K} = PC(L, T', \Gamma')$. We can assume wlog that L' has Skolem functions. Since $\mathcal{K}$ has arbitrarily large models, it has Ehrenfeucht-Mostowski models. Let $N_2 := EM(\lambda) \upharpoonright L$, $N_2 \in \mathcal{K}$ and $|N_2| = \lambda$.

Claim 6. N_2 realizes only μ many types of $ga - S(M')$ for each $M' \prec_{\mathcal{K}} N_2$ of size μ.

Proof of claim (sketch). Given $M' \prec_{\mathcal{K}} N$ of size μ, let $J \subset \lambda$ with $|J| = \mu$ and $M' \subset EM(J) \upharpoonright L$. Since $(\lambda, <)$ is well-ordered, it can only realize μ many $\{<\}$-types over J. Also, if a and b are sequences in λ realizing the same cut over J, there exists $(I', <')$ extending $(\lambda, <)$ such that a and b are automorphic over J in I' (e.g. $I' = {}^{\omega>}\lambda$). Then,

$$gatp(\tau(a)/EM(J) \upharpoonright L; N') = gatp(\tau(b)/EM(J) \upharpoonright L; N')$$

hence

$$gatp(\tau(a)/M'; N') = gatp(\tau(b)/M'; N')$$

(for any term τ of the Skolem language) Then N_1 and N_2 are two non-isomorphic models in $\mathcal{K}$, of size λ. This contradicts the hypothesis of categoricity in λ. □

3.2. Hanf Numbers.

4. The Presentation Theorem (Relation form)

The Shelah presentation theorem has advantages and drawbacks.

	Advantages	Drawbacks
Shelah Pres. Thm	$\lvert L'\rvert \leqslant LS(\mathcal{K})$, $\lvert\Gamma\rvert \leqslant 2^{LS(\mathcal{K})}$, $\mathcal{L}_{\left(2^{LS(\mathcal{K})}\right)^+,\omega}$	Non functorial! Not canonical!
Relational Pres. Thm	Functorial! Canonicity...	$LS(\mathcal{K})$-ary predicates, $\mathcal{L}_{(2^\kappa)^+,\kappa^+}$, $\kappa = LS(\mathcal{K})$

Definition 7. (Vasey) A functorial expansion of an AEC $\mathcal{K}$ in L is an AEC $\hat{\mathcal{K}}$ in $\hat{L} \supset L$ such that

- Each $M \in \mathcal{K}$ has a unique expansion to a $\hat{M} \in \hat{\mathcal{K}}$,
- if $f : M \approx M'$ then $f : \hat{M} \approx \widehat{M'}$,
- if $M \prec_{\mathcal{K}} M'$ then $\hat{M} \prec_{\hat{\mathcal{K}}} \widehat{M'}$.

Examples include Morley's adding predicates for each definable set, Chang adding predicates for each $L_{\omega_1,\omega}$-definable set, Shelah's T^{eq}, expanding by adding orbits in AECs.

Baldwin and Boney have the following relational variant of the presentation theorem, canonical (functorial) but with the high price of needing a much larger language and infinitary predicates.

The context is as before: fix an AEC $\mathcal{K}$ in language L and let $\kappa = LS(\mathcal{K})$. We also assume that $\mathcal{K}$ has no models of cardinality $< \kappa$. We fix a collection of compatible enumerations for models M of $\mathcal{K}_\kappa$. This means that each $M \in \mathcal{K}$ has an enumeration $\mathbf{m}^M = \langle m_i^M \mid i < \kappa \rangle$ and if $M \approx M'$ then there is some fixed isomorphism $f_{M,M'} : M \to M'$ such that $f_{M,M'}(m_i^M) = m_i^{M'}$, and such that if $M \approx M' \approx M''$ then $f_{M,M''} = f_{M',M''} \circ f_{M,M'}$.

For each isomorphism type $[M]_\approx$ and $[M \prec N]_\approx$ with $M, N \in \mathcal{K}_\kappa$ we add to the language a corresponding symbol

$$R_{[M]}(\mathbf{x}) \quad \text{and} \quad R_{[M \prec N]}(\mathbf{x}, \mathbf{y}).$$

(These are κ-ary and $\kappa \cdot 2$-ary, respectively.)

After all these choices, the presentation theory T^*.

Long axiomatization, but the essence is:

- $R_{[M]}(\mathbf{x})$ holds iff the map $\left(x_i \mapsto m_i^M\right)$ is an isomorphism
- $R_{[M \prec N]}(\mathbf{x}, \mathbf{y})$ holds iff
  - the map $\left(x_i \mapsto m_i^M\right)$ is an isomorphism,
  - the map $\left(y_i \mapsto m_i^N\right)$ is an isomorphism and
  - $x_i = y_j$ iff $m_i^M = m_j^N$.

The actual axioms are lengthy but they can be written in the logic $\mathcal{L}_{(2^\kappa)^+,\kappa^+}(L^*)$.

The statement gives more information on the way the models interact.

Theorem 8. *(Relational Presentation Theorem).*

- *If* $M^* \models T^*$ *then* $M^* \upharpoonright L \in \mathcal{K}$. *In this case, if* $M_0 \in \mathcal{K}$ *then* $M^* \models R_{[M_0]}(\mathbf{m})$ *implies that* $\mathbf{m}$ *is an enumeration of a strong substructure of* $M^* \upharpoonright L$.
- *Every* $M \in \mathcal{K}$ *has a unique expansion* M^* *that models* T^*.
- *If* $M \prec_{\mathcal{K}} N$ *then* $M^* \subset N^*$.
- *If* $M^* \subset N^*$ *both model* T^* *then* $M \prec_{\mathcal{K}} N$.
- *If* $M \prec_{\mathcal{K}} N$ *and* $M^* \models T^*$ *are such that* $M^* \upharpoonright L = M$ *then there is* $N^* \models T$ *such that* $M^* \subset N^*$ *and* $N^* \upharpoonright L = N$.

The proof is similar in spirit to the previous, but with much more delicate verification.

Part 2. The Small Index Property (SIP) for Homogeneous Classes

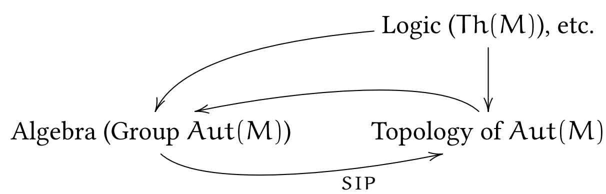
[From the Abstract: this recent work of mine and Zaniar Ghadernezhad extends results of Lascar and Shelah to some AECs. I will explain the role of the SIP in understanding the "Reconstruction Problem" (when can one reconstruct a structure M from its symmetries Aut(M)). I will explore consequences and I will describe our setting for generalizing Lascar-Shelah to some non-elementary classes.]

5. The Reconstruction Problem

A classical enigma: reconstructing from symmetry.

At a very classical extreme, there is the old enigma: There is some object M. I give you the symmetries of the object M.

Tell me what is M!



5.1. Reconstructing models?

In Model Theory (and in other parts of Mathematics!), the same naïve enigma has important variants. The main version is usually called "The Reconstruction Problem":

- if for some (First Order) structure M we are *given* the group $Aut(M)$, what can we say about M? (If no more conditions on M, in general, one cannot say much! by e.g. Ehrenfeucht-Mostowski).
- A more reasonable question: if for some (First Order) structure M we are *given* $Aut(M)$, what can we say about $Th(M)$?
- An even more reasonable question: if for some (FO) structure M we are given $Aut(M)$, when can we recover **all models biinterpretable with** M?

The question is then naïve in principle: What information about a model M and $Th(M)$ is contained in the group $Aut(M)$?

- Hodges isolated a property (Small Index Property - SIP) of a model M (or of its theory $Th(M)$) enabling us to capture the Polish topology of $Aut(M)$ from the pure group structure, in the case of saturated M. Lascar, Hodkinson, etc. use descriptive set theory to prove SIP.
- (Anabelian geometry) the *anabelian* question: recover the isomorphism class of a variety X from its étale fundamental group $\pi_1(X)$. Neukirch, Uchida, for algebraic number fields.
- (Koenigsmann) K and $G_{K(t)/K}$ are biinterpretable for K a perfect field with finite extensions of degree > 2 and prime to $char(K)$.

5.2. **The Small Index Property - Limitations.**

Definition 9. [Small Index Property - SIP] Let M be a countable structure. M has the small index property if for any subgroup H of $\mathrm{Aut}(M)$ of index less than $2^{\aleph_0}$, there exists a finite set $A \subset M$ such that $\mathrm{Aut}_A(M) \subset H$.

The SIP allows us to recover the topological structure of $\mathrm{Aut}(M)$ from its pure group structure:

Open neighborhoods of id in pointwise convergence topology are sets containing pointwise stabilizers $\mathrm{Aut}_A(M)$ for some finite A.

- SIP holds for random graph, infinite set, DLO, vector spaces over finite fields, generic relational structures, $\aleph_0$-categorical $\aleph_0$-stable structures, etc.
- It fails e.g. for $M \models ACF_0$ with ∞ transc. degree.

Back to Reconstruction (à la Lascar et al.)

- Every automorphism of M extends uniquely to an automorphism of M^{eq}; therefore, $\mathrm{Aut}(M) \approx \mathrm{Aut}(M^{eq})$ canonically.
- Having that $M^{eq} \approx N^{eq}$ implies that M and N are bi-interpretable.
- If M is $\aleph_0$-categorical, any open subgroup of $\mathrm{Aut}(M)$ is a stabilizer $\mathrm{Aut}_\alpha(M)$ for some imaginary α. Also $\mathrm{Aut}(M) \curvearrowright \{H \leqslant \mathrm{Aut}(M) \mid H \text{ open}\}$ (conjugation).
- The action $\mathrm{Aut}(M) \curvearrowright$ is (almost) $\approx$ to $\mathrm{Aut}(M) \curvearrowright M^{eq}$.

So, we have recovered the action of $\mathrm{Aut}(M)$ on M^{eq} from the topology of $\mathrm{Aut}(M)$... so, if M, N are countable $\aleph_0$-categorical structures, TFAE:

- There is a bicontinuous isomorphism from $\mathrm{Aut}(M)$ onto $\mathrm{Aut}(N)$
- M and N are bi-interpretable.

Therefore, at least in the $\aleph_0$-categorical case, the theory of M, $Th(M)$, may be recovered up to biinterpretability from the pure algebraic structure of the group $\mathrm{Aut}(M)$.

Lascar points to the following simple limitations as to the recovery of the structure:

(1) There exist two saturated countable structures, with only one of them $\aleph_0$-categorical, having topologically isomorphic automorphism groups: M the trivial infinite set, N the same as M but with a named infinite co-infinite set of constants.
(2) There exist two countable structures M and N with M saturated, N not saturated, such that $\mathrm{Aut}(M) \approx_{\text{top}} \mathrm{Aut}(N)$: M is the countable saturated model of the theory of infinitely many disjoint unary predicates, N is the model of an equivalence relation with infinitely many infinite classes, together with a named constant in all but exactly one class.

(Exercise!)

6. Strong Automorphisms - Imaginaries

The following example, due to Cherlin and Hrushovski independently, shows a bad failure of the SIP:

$L = \{R_n \mid 0 < n < \omega\}$, each symbol R_n of arity $2n$.

<u>Axioms</u> of T_0: each R_n is an equivalence relation on the n-element subsets of the model, with exactly two classes.

Then we let T be the model completion of T_0: for each n we add the axiom

"if $x_1, \dots, x_n$ are different elements of the model, then for each $k \leqslant n$ and each subset $s \subset \{1, 2, \dots, n\}$ of cardinality k and each class y_s modulo R_{k+1}, there exists an element z differnt from all the x_i's and such that $\{x_i \mid i \in s\} \cup \{z\}$ is in the class y_s."

The theory has EQ and is $\aleph_0$-categorical. Let M be its unique countable model and let $\mathcal{U}$ be a non-principal ultrafilter over ω. Then the subgroup H of $\mathrm{Aut}(M)$ given by

$$H = \{f \in \mathrm{Aut}(M) \mid \{n \in \mathbb{N} \mid f \text{ fixes both classes mod } R_n\} \in \mathcal{U}\}$$

has index 2 and is not open (exercise!).

7. The Uncountable Case

We now switch focus to the uncountable, first order, case.

Fix $\lambda = \lambda^{<\lambda}$ an uncountable cardinal, and fix M a saturated model of cardinality λ.

We now use the topology $\mathcal{T}^\lambda$ on $\mathrm{Aut}(M)$, whose basic open sets around 1_M are stabilizers of subsets of size $< \lambda$ - as before $\mathrm{Aut}_A(M)$ but now $A \subset M$ with $|A| < \lambda$.

$\mathrm{Aut}(M)$ with this topology is of course no longer a Polish space. The techniques from Descriptive Set Theory that have been used for the countable case need to be replaced (Friedman, Hyttinen and Kulikov have a start of Descriptive Set Theory for some uncountable cardinalities, however).

Theorem 10. *[Lascar-Shelah: Uncountable saturated models have the SIP] Let M be saturated, of cardinality $\lambda = \lambda^{<\lambda}$ and let G be a subgroup of $\mathrm{Aut}(M)$ such that $[\mathrm{Aut}(M) : G] < 2^\lambda$. Then there exists $A \subset M$ with $|A| < \lambda$ such that $\mathrm{Aut}_A(M) \subset G$.*

The proof consists of building directly (assuming that G does not contain any open set $\mathrm{Aut}_A(M)$ around the identity) a **binary tree** of height λ of automorphisms of M in such a way that every two of them are not conjugate. This is enough but requires two crucial notions: **generic** and **existentially closed (sequences of) automorphisms**. These are obtained by assuming that G is not open.

8. Sketch of a proof of the SIP for some nonelementary classes

Although results on the reconstruction problem, so far have been stated and *proved* for saturated models in first order theories, the scope of the matter can go far beyond:

- Abstract Elementary Classes with well-behaved closure notions, and the particular case:
- Quasiminimal (qm excellent) Classes.

8.1. **The setting: strong amalgamation classes.** A setting for homogeneity: let $(\mathcal{K}, \prec_{\mathcal{K}})$ be an AEC, with $LS(\mathcal{K}) \leqslant \lambda$, $|M| = \kappa > \lambda$, $\kappa^{<\kappa} = \kappa$.

- Let $\mathcal{K}^{<}(M) := \{N : N \preccurlyeq_{\mathcal{K}} M, |N| < \kappa\}$ and fix $M \in \mathcal{K}$ homogeneous.
- The topology τ^{cl}: base of open neighborhoods given by sets of the form $\mathrm{Aut}_X(M)$ where $X \in \mathcal{C}$, where

$\mathcal{C} := \left\{\mathrm{cl}^M(A) : A \subseteq M \text{ such that } |A| < \kappa\right\}$ and the "closure operator" is $\mathrm{cl}^M(A) := \bigcap_{A \subset N \prec_{\mathcal{K}} M} A$.

This class of cl^M-closed sets has enough structure for the proof of SIP.

Theorem 11. *[SIP for $(\mathrm{Aut}(M), \mathcal{T}^{cl})$ - Ghadernezhad, V.] Let M be a homogeneous model in an AEC $(\mathcal{K}, \prec_{\mathcal{K}})$, with $|M| = \lambda = \lambda^{<\lambda} > LS(\mathcal{K})$, such that $\mathcal{K}^{<\lambda}$ is a strong amalgamation class. Let $G \leqslant \mathrm{Aut}(M)$ with $[\mathrm{Aut}(M) : G] \leqslant \lambda$ (this is, G has small index in $\mathrm{Aut}(M)$). Then there exists $X \in \mathcal{C}$ such that $\mathrm{Aut}_X(M) \leqslant G$ (i.e., G is open in $(\mathrm{Aut}(M), \mathcal{T}^{cl})$).*

8.2. **Genericity and... Amalgamation Bases.** We need a technique to get a large number of conjugates to an automorphism f. We construct a tree.

Proof. (rough sketch): suppose G has small index in $Aut(M)$ but is not open (does not contain any basic $Aut_X(M)$ for $X \in \mathcal{C}$.

We have enough tools (generic sequences and strong amalgamation bases) to build a "Lascar-Shelah tree" to reach a contradiction (2^λ many branches giving automorphisms of M, g_σ for $\sigma \in 2^\lambda$ such that if $\sigma \neq \tau \in 2^\lambda$ then $g_\sigma^{-1} \circ g_\tau \notin G$). □

Of course, the possibility of getting these 2^λ-many automorphisms requires using the non-openness of G to get the construction going.

A λ-Lascar-Shelah tree for M and $G \leqslant Aut(M)$ is a binary tree of height λ with, for each $s \in 2^{<\lambda}$, a model $M_s \in \mathcal{K}^<(M)$, $g_s \in Aut(M_s)$, $h_s, k_s \in Aut_{M_s}(M)$ such that

- $h_{s,0} \in G$ and $h_{s,1} \notin G$ for all $s \in \mathcal{S}$;
- $k_{s,0} = k_{s,1}$ for all $s \in \mathcal{S}$;
- for $s \in \mathcal{S}$ and all $t \in \mathcal{S}$ such that $t \leqslant s$:$h_t[M_s] = M_s$ (i.e. $h_t \in Aut_{\{M_s\}}(M)$) and ...;
- for $s \in \mathcal{S}$ and all $t \in \mathcal{S}$ such that $t \leqslant s$: $g_s \cdot (h_t \upharpoonright M_s) \cdot g_s^{-1} = k_t \upharpoonright M_s$;
- for $s \in \mathcal{S}$ and $\beta <$ length(s): $a_s \in M_s$;
- for all s, the families $\{h_t : t \leqslant s, t \in \mathcal{S}\}$ and $\{k_t : t \leqslant s, t \in \mathcal{S}\}$ are elements of $\mathcal{F}$ (i.e. they are generic).

8.3. **Generic Sequences and Strong Amalgamation Bases.** The main technical tools in the construction of a LS tree are

- Guaranteeing generic sequences of automorphisms ($g \in Aut(M)$ is generic if $\forall N \in \mathcal{K}^<(M)$ such that $g \upharpoonright N \in Aut(N)$, $\forall N_1 \succ_{\mathcal{K}} N$, $N_1 \in \mathcal{K}^<(M)$, $\forall h \supset g \upharpoonright N, h \in Aut(N_1)$, $\exists \alpha \in Aut_N(M)$ such that $g \supset \alpha \circ h \circ \alpha^{-1}$),
- showing they are unique up to conjugation,
- getting a generic sequence $\mathcal{F} = (g_i : i \in I)$ such that
  - the set $\{i \in I : g_i \upharpoonright M_0 = h$ and $g_i \notin G\}$ has cardinality κ for all $M_0 \in \mathcal{K}^<(M)$ and $h \in Aut(M_0)$;
  - the set $\{i \in I : g_i \in G\}$ has cardinality κ.

Another way to get generics is via "aut-independence"

Definition 12. Let $A, B, C \in \mathcal{C}$. Define $A \mathop{\smile\!\!\!\!\!\mid}^a_B C$ if for all $f_1 \in Aut(A)$ and all $f_2 \in Aut(C)$ and for all $h_i \in \mathcal{O}_{f_i}$ $(i = 1, 2)$ such that $h_1 \upharpoonright A \cap C = h_2 \upharpoonright A \cap C$ and $h_1 \upharpoonright B = h_2 \upharpoonright B$ then $\mathcal{O}_g \neq \emptyset$ where $g := f_1 \cup f_2 \cup h_1 \upharpoonright B$.

where $\mathcal{O}_f := \left\{\hat{f} \in Aut(M) \mid \hat{f} \supset f\right\}$, whenever f is an automorphism of a subset of M.

Definition 13. Let $A, B, C \in \mathcal{C}$. Define $A \mathop{\smile\!\!\!\!\!\mid}^{a-s}_B C$ if and only if $A' \mathop{\smile\!\!\!\!\!\mid}^a_B C'$ for all $A' \subseteq A$ and $B' \subseteq B$ with $A', B' \in \mathcal{C}$.

Fact 14. $\mathop{\smile\!\!\!\!\!\mid}^{a-s}$ *satisfies symmetry, monotonicity and invariance.*

8.4. **Free $\mathop{\smile\!\!\!\!\!\mid}^{a-s}_B$ C-amalgamation.** The class $\mathcal{C}$ has the *free* $\mathop{\smile\!\!\!\!\!\mid}^{a-s}$*-amalgamation property* if for all $A, B, C \in \mathcal{C}$ with $A \cap B = C$ there exists $B' \in \mathcal{C}$ such that $ga - tp(B'/C) = ga - tp(B/C)$ (or there exists $g \in Aut_C(M)$ that $g[B] = B'$) and $A \mathop{\smile\!\!\!\!\!\mid}^{a-s}_C B'$.

Fact 15. *Suppose* $\mathcal{C}$ *has the free* $\mathop{\smile\!\!\!\!\!\mid}^{a-s}$*-amalgamation property. Then generic automorphisms exist.*

8.5. **Quasiminimal pregeometry classes - Zilber field.** In a language L, a *quasiminimal pregeometry* class $\mathcal{Q}$ is a class of pairs $\langle H, \mathrm{cl}_H \rangle$ where H is an L-structure, cl_H is a pregeometry operator on H such that the following conditions hold:

(1) Closed under isomorphisms,
(2) For each $\langle H, \mathrm{cl}_H \rangle \in \mathcal{Q}$, the closure of any finite set is countable.
(3) If $\langle H, \mathrm{cl}_H \rangle \in \mathcal{Q}$ and $X \subseteq H$, then $\langle \mathrm{cl}_H (X), \mathrm{cl}_H \restriction \mathrm{cl}_H (X) \rangle \in \mathcal{Q}$.
(4) If $\langle H, \mathrm{cl}_H \rangle, \langle H', \mathrm{cl}_{H'} \rangle \in \mathcal{Q}$, $X \subseteq H$, $y \in H$ and $f : H \to H'$ is a partial embedding defined on $X \cup \{y\}$, then $y \in \mathrm{cl}_H (X)$ if and only if $f(y) \in \mathrm{cl}_{H'} (f(X))$.
(5) Homogeneity over countable models.

Moreover...

(1) $\mathcal{Q}$ quasiminimal pregeometry class. $M \in \mathcal{Q}$ of size $\aleph_1$, $\mathcal{C} = \{cl(A) \mid A \subset M, A$ small$\}$ then $\mathcal{C}$ has the free aut-independence amalgamation property.
(2) $\mathcal{Q}$ qm pregeom. class $\to$ for every model M of $\mathcal{Q}$, $Aut(M)$ has SIP,
(3) The "Zilber field" has SIP.
(4) Reconstruction?

Part 3. Categories, AECs, Interpolation

9. Comparing, Blending, Pulling Back AECs?

> [From the Abstract: this last construction intends to compare different AECs with tools analogous to those one uses to compare theories in classical settings - and a little bit of Abstract Model Theory (beginning). We start building some functors extracted from underlying languages and then use the Presentation Theorem to lift these functors to other functors between various AECs. This work explores a line different from that of generalizing AECs to concrete categories: we rather use a bit of category theory to understand some intereactions between classical AECs. This is joint work with H. Mariano and P. Zambrano.]

The following few paragraphs are from the introduction to our forthcoming paper (Mariano, V., Zambrano) *A Global Approach to Abstract Elementary Classes.*

Comparing different Abstract Elementary Classes (AEC) in a systematic way has so far been a "secondary" problem: the main focus in the study of the model theory of AECs has been Stability-theoretical in nature: Categoricity Transfer, the Shelah Categoricity Conjecture and the enormous amount of deep results aiming at framing that conjecture, the development of techniques and tools to understand locality, tameness, superstability-like phenomena at the level of AECs, and more recently the emergence of good frames and the development of forking notions for AECs. No analog of Abstract Model Theory for AECs has so far been developed, as far as we may see.

There are, however, several beginnings of a systematic study of a comparison of how different AECs interact with one another. For instance, starting from an AEC $\mathcal{K}$, the work of Shelah and various others systematically extracts a sub-AEC $\mathcal{K}'$ consisting of "saturated" enough models in $\mathcal{K}$ or other subclasses derived from different constructions (Ehrenfeucht-Mostowski models of a class may form a subclass, or limit models, etc.). However, we propose here techniques for comparing AECs pretty much in the same way one one compares theories, or even the way one compares *logics* in Abstract Model Theory, or even in the way one compares different families of objects in general in mathematics (through various moduli spaces, various parametrized families, etc.). Furthermore, we study here constructions derived from these comparisons: *blending* different AECs into one, building *limits* of AECs, studying *interpolation* of AECs (generalizing interpolation of theories in Abstract Model Theory), studying Robinson properties and Beth internal vs external definability in the context of AECs.

9.1. **Categories arising from AECs.** Kirby and independently Lieberman started with the following categories extracted from AECs:

Let $\mathrm{cat}(\mathcal{K})$ be the category with $\mathcal{K}$ as the class of objects and $\mathcal{K}$-embeddings as arrows. This is well defined, as $\prec_{\mathcal{K}}$ is reflexive and compositions of arrows are in the category (by the closure under isomorphism):

$$A \xrightarrow{\cong} A' \xhookrightarrow{\preceq} B \xrightarrow{\cong} B' \xhookrightarrow{\preceq} C,$$

and

$$A' \xrightarrow{\cong} A'' \xhookrightarrow{\preceq} B'.$$

Moreover, the category $\mathrm{cat}(\mathcal{K})$ is closed under all (upward) directed colimits.

The work of Adámek and Rosický, Lieberman and, and Beke and Rosický had already established that $\mathrm{cat}(\mathcal{K})$ is an accessible category—a fact further expanded in the more recent by Lieberman and Rosický.

We will denote by $\mathrm{cat}(\mathcal{K})_{\mathrm{strict}}$ the category with $\mathcal{K}$ as the class of objects and $\prec_{\mathcal{K}}$-inclusions as arrows. Notice this is a subcategory of $\mathrm{cat}(\mathcal{K})$.

AECs, in spite of their seemingly loose linguistic anchor and their definitely semantic bent, have strong symbiosis with logics - the fundamental connection is given by Shelah's Presentation Theorem.

This is also the first essential instance of a "change of languages" in the theory of AECs.

In the model theory of AECs this theorem has crucial consequences: the possibility of showing that under sufficient (and quite weak) conditions, AECs may have a "Hanf number" (a cardinality depending on the Löwenheim-Skolem number of the class, such that if the AEC $\mathcal{K}$ has a model of that cardinality then it has arbitrarily large models) and even admit Ehrenfeucht-Mostowski models (with all the consequences: Morley Omitting Types, etc.).

10. Going global: The Category of all AECs

We now study the first version of a category consisting of AECs - our aim is twofold:

- providing a setting that will enable us to *compare* different AECs through the use of various categorical constructions, and
- building new categories as limits of classes of categories.

The following definition of a category of all AECs is akin to Grothendieck universes:

Definition 16. [The category of all AECs] The category $\mathcal{AEC}$ is given by:

- Objects: all AECs (in all languages L),
- Morphisms: for $\mathcal{K}$ an L-AEC and $\mathcal{K}'$ an L′-AEC, and for each *morphism of languages* $\alpha : L \to L'$, $\alpha^* : (\mathcal{K}', \prec_{\mathcal{K}'}) \to (\mathcal{K}, \prec_{\mathcal{K}})$ is a morphism in $\mathcal{AEC}(\mathcal{K}', \mathcal{K})$ if

$$\begin{array}{ccc} L'-\text{struct.} & \xrightarrow{\alpha^*} & L-\text{struct.} \\ \uparrow{\scriptstyle i'} & & \uparrow{\scriptstyle i} \\ \mathcal{K}' & \xrightarrow{\alpha^*\restriction} & \mathcal{K} \end{array}$$

The following constructions appear in our paper in process and were discussed during the lecture:

- The functoriality of α^*
- The definition of *complete theory of an AEC* $\mathcal{K}$*, the notion of* $\equiv_{\mathcal{K}}$*, the notion of maximal consistent theory of an AEC.*
- The way these interplay with the category $\mathcal{AEC}$.
- New constructions (fusions of AECs, etc.) within this category.
- Robinson, Craig, etc.

References

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