QUANTUM J-MAPPINGS (MODEL THEORY AND SHEAVES) OXFORD MODEL THEORY SEMINAR

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Abstract. Finding a "non-commutative limit" of the j-invariant (to real numbers, in a way that captures reasonably well the connection with extensions of number fields) has prompted several approaches (Manin-Marcolli, Castaño-Gendron). I will describe one of these approaches in a brief way, and I will make some connections to the model theory of sheaves.

Contents

1. From Classical j to Quantum j

1.1. Classical j. Recall a definition of classical j:
if $\mu \in \mathbb{H}$, $\Lambda(\mu) = \mathbb{Z} + \mathbb{Z}\mu$ is the μ-lattice, and the classical torus associated to μ is

$$\mathbb{T}(\mu) := \mathbb{C}/\Lambda(\mu).$$

(This is also a Riemann surface.)

Now, $\mathbb{T}(\mu)$ is equivalent to the elliptic curve $\mathbb{E}(\mu)$ given by $Y^2 = X^3 - g_2(\mu)X - g_3(\mu)$.
Here, let

$$G_k(\mu) := \sum_{0 \neq \gamma \in \Lambda(\mu)} \gamma^{-2k} \qquad k \geqslant 2$$

(the so-called *Eisenstein series*), then

$$g_2(\mu) = 60 \cdot G_2(\mu)$$

$$g_3(\mu) = 140 \cdot G_3(\mu).$$

j *is really a power series...* The classical transition from $\mathbb{T}(\mu)$ to $\mathbb{E}(\mu)$ is

$$z \mapsto (\wp_\mu(z), \wp'_\mu(z))$$

where

$$\wp_\mu(z) = \frac{1}{z^2} + \sum_{0 \neq \gamma \in \Lambda(\mu)} \frac{1}{(\gamma + z)^2} - \frac{1}{\gamma^2}.$$

(Again a series!)

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1.2. **Toward quantum tori: from $\mathbb{C}$ to $\mathbb{R}$.** Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$, and let Λ_θ be the pseudo-lattice $\langle 1, \theta \rangle$ (the subgroup of $\mathbb{R}$ given as $\Lambda_\theta := \mathbb{Z} + \mathbb{Z} \cdot \theta$). The quotient

$$\mathbb{T}(\theta) := \mathbb{R}/\Lambda_\theta$$

is the "quantum torus", associated to the irrational number θ. It is a one-parameter subgroup of the (classical) torus $\mathbb{T}(i)$... and also a Riemann surface.

There are long, detailed descriptions in terms of leaf spaces corresponding to "Kronecker" foliations, etc.

Getting hold of quantum versions of j. The problem:

- New definition domain (from $\mathbb{H}$ to $\mathbb{R} \setminus \mathbb{Q}$)
- Topological issues resulting from the much more chaotic behavior of $\mathbb{R}$ - continuity lost in first approximations
- Rational expressions (multivalued functions now - perhaps the average of the (finite?) set of values is *the* robust invariant).

1.3. **Castaño-Bernard and Gendron's definition.** Let $\theta \in \mathbb{R}$. The *quantum modular invariant* $j^{qu}(\theta)$ is a discontinuous, multivalued, analogue of the classical modular invariant:

Let $\Lambda_\varepsilon(\theta) := \{n \in \mathbb{N} \mid \|n\theta\| < \varepsilon\}$, where $\| \cdot \|$ measures the distance to the nearest integer.

This is the "quantum lattice".

The ε-zeta function of θ is given by

$$\zeta_{\theta,\varepsilon}(s) := \sum_{n \in \Lambda_\varepsilon(\theta)} n^{-s}.$$

(The value $2\zeta_{\theta,\varepsilon}(2k)$ is the analogue to the classical Eisenstein series of weight k.)

The ε-modular invariant is given by

$$j_\varepsilon(\theta) := \frac{12^3}{1 - J_\varepsilon(\theta)}, \qquad J_\varepsilon(\theta) := \frac{49}{40} \frac{\zeta_{\theta,\varepsilon}(6)^2}{\zeta_{\theta,\varepsilon}(4)^3}.$$

The set of limit points as $\varepsilon \to 0$ is

$$j^{qu}(\theta) := \lim_{\varepsilon \to 0} j_\varepsilon(\theta).$$

This is a $GL_2(\mathbb{Z})$-invariant, discontinuous and multivalued function

$$j^{qu} : \mathbb{R} \multimap \mathbb{R} \cup \{\infty\}.$$

Spectra and NCG. One can regard

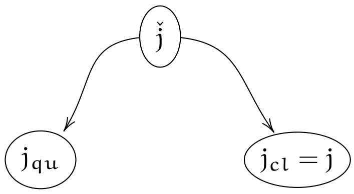
$$j^{qu} : \mathbb{R} \multimap \mathbb{R} \cup \{\infty\}$$

as a spectrum of some (yet unknown to us) operator.

However, we follow a path different from more usual ones in non-commutative geometry.

1.4. **An example of a sheaf construction / universal j.**

Gendron proposes a detailed construction of a sheaf over a topological space, and a generalization of classical j called "universal j-invariant" - a specific *section* of a sheaf.



Let ${}^*\mathbb{Z} := \mathbb{Z}^\omega/\mathfrak{u}$ for some nonprincipal ultrafilter $\mathfrak{u}$ on $\supsetneq$. Define

$$\mathcal{H} := \{[F_i] \subset^* \mathbb{Z}^2 \text{ hyperfinite}\}.$$

This set is partially ordered with respect to inclusion so we may consider the Stone space

$$\mathcal{R} := \mathsf{Ult}(\mathcal{H}).$$

For each $\mathfrak{p} \in \mathcal{R}$ and $\mu \in \mathbb{H}$ one may define the j-invariant

$$j(\mu, \mathfrak{p})$$

as follows.

The idea. The classical j-invariant is an algebraic expression involving Eisenstein series which is a function of $\mu \in \mathbb{H}$. We can associate to $[F_i] \subset^* \mathbb{Z}^2$ a hyperfinite sum modelled on the formula of the classical j-invariant, denoted

$$j(\mu)_{[F_i]} \in^* \mathbb{C}.$$

We get a net

$$\{j(\mu)_{[F_i]}\}_{[F_i]\in\mathcal{H}} \subset^* \mathbb{C}.$$

Consider the sheaf ${}^\diamond\check{\mathbb{C}} \to \mathcal{R}$ for which the stalk over $\mathfrak{p}$ is

$${}^\diamond\mathbb{C}_\mathfrak{p} := ({}^*\mathbb{C})/\mathfrak{p}.$$

Then we may define a section:

$$\check{j} : \mathbb{H} \times \mathcal{R} \longrightarrow, {}^\diamond\check{\mathbb{C}} \qquad \check{j}(\mu, \mathfrak{p}) := \{j(\mu)_{[F_i]}\}_{[F_i]\in\mathcal{H}}/\mathfrak{p}.$$

Group actions - choosing an irrational angle. What really is at stake in these constructions is the invariance under various group actions.

For each $\theta \in \mathbb{R}$ there is a distinguished subset $\mathcal{R}_\theta \subset \mathcal{R}$ of ultrafilters which "see" θ:

$$\mathcal{R}_\theta = \{\mathfrak{p} | \ \mathfrak{p} \supset \mathfrak{c}_\theta\}$$

where $\mathfrak{c}_\theta$ is the "cone filter" generated by the cones

$$\mathsf{cone}_\theta([F_i]) = \{[F_i]' \supset [F_i] | \ [F_i] \subset [F_i]' \subset^* \mathbb{Z}^2(\theta)\}.$$

In the above,

$$\mathbb{Z}^2(\theta) = \{({}^*n^\perp, {}^*n) | \ {}^*n\theta -^* n^\perp \simeq 0\}.$$

Restricting to quantum and classical j. The quantum j-invariant is defined as the restriction:

$$\check{\mathrm{j}}^{\mathrm{sf}}(\theta) := \check{\mathrm{j}}|_{\mathcal{R}_\theta}(\mathfrak{i}, \cdot).$$

If we denote

$$\mathcal{R}_{\mathrm{cl}} = \{\mathfrak{p} |\ \mathfrak{p} \supset \mathfrak{c}\}$$

where $\mathfrak{c}$ is the filter generated by *all* cones over hyperfinite sets in ${}^*\mathbb{Z}^2$:

$$\mathrm{cone}([\mathrm{F}_i]) = \{[\mathrm{F}_i]' \supset [\mathrm{F}_i] |\ [\mathrm{F}_i]' \subset^* \mathbb{Z}^2\}.$$

Then the restriction

$$\check{\mathrm{j}}^{\mathrm{cl}} := \check{\mathrm{j}}|_{\mathcal{R}_{\mathrm{cl}}}$$

satisfies

$$\check{\mathrm{j}}^{\mathrm{cl}}(\mu, \mathfrak{p}) \simeq \mathrm{j}(\mu), \quad \forall \mu \in \mathbb{H},$$

where j is the usual j-invariant.

2. New Directions

2.1. **Duality.** Note the duality in the way of recovering the classical and quantum invariants:

- the classical invariant is recovered along a unique fiber ${}^\diamond\check{\mathbb{H}}_u$ (i.e., a *leaf* of the quotient of sheaves $\widehat{\mathrm{Mod}}$),
- the quantum invariant is obtained by *fixing* the fiber parameter $i \in \mathbb{H}$ and letting $u \in \mathrm{Cone}(\theta)$ vary: it therefore arises from a local section defined by i (a *transversal* of $\widehat{\mathrm{Mod}}$).

2.2. **Conjectures.** The main goal is to check that if $\theta \in \mathbb{R} \setminus \mathbb{Q}$ is quadratic, then Hilbert's classfield H_K of $K = \mathbb{Q}(\theta)$ (K's maximal unramified extension) equals

$$K(j(\theta)).$$

This would give a solution to Hilbert's 12th problem for quadratic real extension (unramified case).

Again, the main point is to prove the analog of "complex multiplication" (keypoint: the algebraicity of $j(\mu)$, when $\mu \in \mathbb{Q}(\sqrt{D})$, for $D < 0$ square free - and the fact that $j(\mu)$ essentially generates the Hilbert classfield $H(\mu)$ of $\mathbb{Q}(\sqrt{D})$).

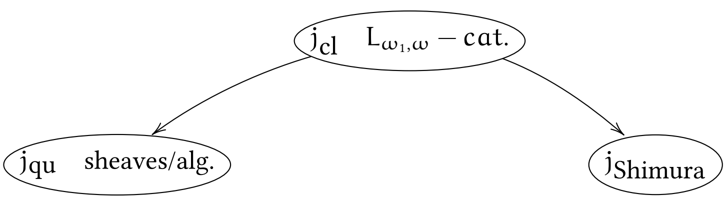
We conjecture (with Gendron) that for $\theta \in \mathbb{R}$ there exists a *duality* relation between the classical invariant $j(i\theta)$ and the quantum invariant $j(\theta)$.

More precisely, we associate to $j(i\theta)$ and $j(\theta)$ two nets

$$\{j(i\theta)_\alpha\} \qquad \text{and} \qquad \{j(\theta)_\alpha\}$$

whose elements are algebraically interdependent. The two nets converge to a common limit. The classical net $\{j(i\theta)_\alpha\}$ lives along a fixed leaf of ${}^\diamond\widehat{\mathrm{Mod}}$; the quantum net $\{j(\theta)_\alpha\}$ lives on a fixed transversal of ${}^\diamond\widehat{\mathrm{Mod}}$.

The current model theoretic analysis of j looks at two possible extensions:



Two different toolkits. The two directions of generalization (to quantum tori/real multiplication on the one hand, to higher dimension varieties/Shimura on the other hand) of j-maps calls different aspects of model theory (at the moment, an "amalgam" has started, but is far along the way):

- The model theory of abstract elementary classes (in particular, the theory of excellence - now "old" (1980s) but recently (2013) streamlined by Bays, Hart, Hyttinen, Kesälä, Kirby - *Quasiminimal Structures and Excellence.* A kind of cohomological analysis of models (and types), connected to categoricity and "smoothness". Zilber field, now j!
- The model theory of sheaves. A tool for blue (topological) "limit" structures.

2.3. **Placing models on sheaves.** Caicedo's Generic Model Theorem is a "topological version" of the Łoś theorem, adapted to sheaves. It generalizes the Forcing Theorem of set theory. Viale uses sheaves to study generic absoluteness.

Theorem 1. *[X. Caicedo] Fix a first order vocabulary* τ*. Let* X *be a topological space,* $\mathfrak{A}$ *a sheaf of* τ*-structures over* X*,* $\mathcal{F}$ *a filter of open sets generic for* $\mathfrak{A}$*, and* $\varphi(\nu_1, \cdots, \nu_n)$ *a* τ*-formula. Then, given sections* $\sigma_1, \cdots, \sigma_n$ *of the sheaf (defined on some open set in* $\mathcal{F}$*), we have*

$$\mathfrak{A}^X/\mathcal{F} \models \varphi(\sigma_1/_{\sim_\mathcal{F}}, \cdots, \sigma_n/_{\sim_\mathcal{F}}) \quad \Leftrightarrow \quad \exists U \in \mathcal{F}, \quad \mathfrak{A} \Vdash_U \varphi^G(\sigma_1, \cdots, \sigma_n).$$

A formula holds at a "limit" of sheaf of structures if and only if it is forced by some open set in the generic filter.

Just as in forcing, open sets are approximations of ideal points!

Generic Model Theorem for equivariant sheaves. We have extended X. Caicedo's Generic Model Theorem in two directions:

- to metric sheaves (with Ochoa) - over *regular* topological spaces - Make the sheaf aware of continuous (metric) model theory at the level of fibers!
- to *equivariant sheaves* (with Padilla) - a group G acting on the sheaf, conditions on construction of G-sheaves (coherence and exactness not just at the level of the presheaf but also at the level of the action).

Model Theory for equivariant sheaves.

Theorem 2. *[Padilla, V., extending X. Caicedo] Fix a first order vocabulary* τ*. Let* X *be a topological space,* $\mathfrak{A}$ *a sheaf of* τ*-structures over* X*,* $\mathcal{F}$ *a filter of open sets generic for* $\mathfrak{A}$*, and* $\varphi(\nu_1, \cdots, \nu_n)$ *a* τ*-formula. Then, given sections* $\sigma_1, \cdots, \sigma_n$ *of the sheaf (defined on some open set in* $\mathcal{F}$*), we have:*

If a group G *acts* equivariantly *on fibers (coherently) then (requiring that* $\mathcal{F}$ *is* G*-invariant)*

$$\mathfrak{A}^X/\mathcal{F}/G \models \varphi(\sigma_1^G/_{\sim_\mathcal{F}}, \cdots, \sigma_n^G/_{\sim_\mathcal{F}}) \quad \Leftrightarrow \quad \exists U \in \mathcal{F}, \quad \mathfrak{A} \Vdash_U \varphi(\sigma_1, \cdots, \sigma_n).$$

Stability theory for quotients of sheaves (foliations, etc.)

References

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