MODEL THEORY OF ABSTRACT ELEMENTARY CLASSES: SOME RECENT TRENDS

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Thanks to the IPM in Tehran for the invitation to do research on Model Theory and to give a minicourse on Abstract Elementary Classes.

Introduction

These notes correspond to a short minicourse (five 90 minute sessions) given at the IPM in Tehran in November 2015. In spite of the title, this minicourse also covers basics of the model theory of Abstract Elementary Classes and is rather self-contained. A few recent lines of research appear mentioned in context - but without the details of the development:

- canonicity of non-forking independence,
- connections with large cardinals,
- examples arising from the "Oxford school".

The aim is to provide a rather contemporary - albeit very quick overview of a selection of topics where research is active in the world of Abstract Elementary Classes.

An inspiration for these notes came from Rami Grossberg's [4], originally also a minicourse he gave in 2001 at Bilgi University in Istanbul. There the treatment of the subject includes more detail on the development of basic stability theory of AECs - I try to respond here with less stability theory and more of the aforementioned connections. In many ways, the current minicourse should be read *together with* [4]; many topics mentioned briefly here are expounded in detail there, at least the parts corresponding to Days 1 and 2 here.

The model theory of Abstract Elementary Classes (AECs) started with Shelah's attempt, in the early 1980s, to generalize his earlier results to infinitary logics. **Categoricity Transfer** was the primary driving force for the development of this model theory and it continues to be, more than three decades on, perhaps the main force behind some of the sharper developments in the area.

better word?

The initial defining feature in the model theory of AECs is a steady shift from an emphasis on syntax to an emphasis on semantics, a reduction of the rôle of definability in favor of more focus on "strong" embeddings between models and automorphism groups of large models.

1. Day 1: The early days of AECs.

1.1. **The Origins / The Basics.** One of the questions that started the process was the problem of proving Categoricity Transfer, a Morley-like theorem, for the infinitary logic $L_{\omega_1,\omega}$. Namely, is it true that if an $L_{\omega_1,\omega}$-sentence ψ is categorical in some uncountable cardinal, then it is categorical in all uncountable cardinals?

More generally, what is the behavior of the function $I(\psi,\lambda) := |\{M \models \psi \mid |M| = \lambda\}/\approx|$, for a sentence ψ of the logic $L_{\omega_1,\omega}$?

Another early origin of Abstract Elementary Classes, complementary to the Categoricity problem, was Shelah's idea of (as expressed in his paper *The Lazy Model-Theoretician's Guide to Stability Theory* [8]) speaking mainly to "those who are interested in **algebraically-minded model theory,** i.e., generic models, the class of e-closed models and universal-homogeneous models **rather than elementary classes** and saturated models. These were his words in 1975. He continues: "our main point is that though stability theory was developed for the latter context, almost everything goes through in the wider context (with suitable changes in the definitions).

This declaration (the "almost everything goes through") entailed more than it could seem at first sight: in many ways it is true but it took a long time to build up the right notions of stability, of types, of independence.

Replacing formulas by an abstract notion of "strong embedding" between L-structures is the first important point. In the definition of AECs we do **not** declare membership in the class by satisfying some sentence or some axiomatic system. The relation $\models$, basic in First Order logic, takes a back seat here, and the main relation $\leqslant_K$ (a generalization of the elementary submodel relation $\prec$ of first order) now leads the game.

Definition 1.1 (Abstract Elementary Class). Fix a language L. A class $\mathcal{K}$ of L-structures, together with a binary relation $\leqslant_K$ on $\mathcal{K}$ is an *abstract elementary class* (for short, AEC) if:

(1) Both $\mathcal{K}$ and $\leqslant_K$ are closed under isomorphism. This means two things: first, if $M' \approx M \in \mathcal{K}$ then $M' \in \mathcal{K}$; second, if M', N' are L-structures with $M' \subset N'$, $M' \approx M$, $N' \approx N$ and $M \leqslant_K N$ then $M' \leqslant_K N'$.
(2) If $M, N \in \mathcal{K}$, $M \leqslant_K N$ then $M \subset N$,
(3) $\leqslant_K$ is a partial order,
(4) (**Coherence**) If $M \subset N \leqslant_K N'$ and $M \leqslant_K N'$ then $M \leqslant_K N$,
(5) (**LS**) There is a cardinal (called "the Löwenheim-Skolem number" of the class) $\kappa = LS(\mathcal{K}) \geqslant \aleph_0$ such that if $M \in \mathcal{K}$ and $A \subset |M|$, then there is $N \leqslant_K M$ with $A \subset |N|$ and $|N| \leqslant |A| + LS(\mathcal{K})$,
(6) (**Unions of $\leqslant_K$-chains**) If $(M_i)_{i<\delta}$ is a $\leqslant_K$-increasing chain of length δ (δ a limit ordinal), then
   - $\bigcup_{i<\delta}(M_i)_{i<\delta} \in \mathcal{K}$,
   - for each $j < \delta$, $M_j \leqslant_K \bigcup_{i<\delta} M_i$,
   - if for each $i < \delta$, $M_i \leqslant_K N \in \mathcal{K}$ then $\bigcup_{i<\delta} M_i \leqslant_K N$.

Remark 1.2. The **unions axiom** may always be strengthened to unions of directed systems. This is proved by induction on the cofinality of the directed system.

At this point, we have the following situation:

- So far, no control on possible axiomatization of the class $\mathcal{K}$. The emphasis is placed on its being closed under the constructions specified in the axioms. However, later (in subsection 2.1) we focus on the logical control of these classes. Remember Shelah's "algebraically-minded model theory".
- These are not necessarily amalgamation classes: there is no amalgamation axiom. However, many AECs do satisfy the amalgamation property. Furthermore, the model theory will depend on the kind of amalgamation possible in the class.

1.2. **First examples.** Here are some examples of AECs:

- Let T be a complete first order theory. Then $(\mathrm{Mod}(T), \prec)$ is an AEC. If T is countable, then $LS(\mathrm{Mod}(T)) = \aleph_0$.
- For ψ a sentence of $L_{\omega_1,\omega}$ and $\mathcal{F}$ a countable fragment of $L_{\omega_1,\omega}$ containing ψ, $(\mathrm{Mod}(\psi), \prec_{\mathcal{F}}^{TV})$ is an AEC with Löwenheim-Skolem number $\aleph_0$. $M \prec_{\mathcal{F}}^{TV} N$ iff $M \subset N$ and they satisfy a "Tarski-Vaught test" with respect to sentences of the fragment $\mathcal{F}$.
- Analogously, for ψ a sentence of $L_{\kappa^+,\omega}$ and $\mathcal{F}$ a fragment of $L_{\kappa^+,\omega}$ of cardinality $\leqslant \kappa$ containing ψ, $(\mathrm{Mod}(\psi), \prec_{\mathcal{F}}^{TV})$ is an AEC with Löwenheim-Skolem number κ.
- The class of locally finite groups, with the usual subgroup relation, is an AEC.
- Various interesting AECs consisting of abelian groups can be obtained by using the "pure subgroup" relation, etc.
- The class of countable models of arithmetic, together with ω-like models of size $\aleph_1$, with the end elementary extension relation, is an AEC.
- Various classes axiomatizable with cardinality quantifiers, are indeed AECs.
- The class of all atomic models of a complete FO theory T, with $\prec$, is an AEC.
- The class of all e.c. models of a complete FO theory T, with $\prec$, is an AEC.

1.3. **Categoricity Transfer: the Main Problem.** The primal driving force behind AECs was in the early days Shelah's *Categoricity Transfer Conjecture*: to find versions of Morley's Theorem for AECs:

A very wide version of the Categoricity Conjecture:

Conjecture 1.3. For every cardinal λ, there exists a cardinal μ_λ such that if $\mathcal{K}$ is an AEC with $LS(\mathcal{K}) = \lambda$ categorical in a cardinal $\geqslant \mu_\lambda$, then $\mathcal{K}$ is categorical in every cardinal greater than or equal to μ_λ.

It is an understatement to say that this problem has led the main developments not only of model theory of AECs but of the whole of model theory for the past half-century. Morley's proof triggered the development of alternate proofs (Shelah, Baldwin-Lachlan) that in the early 1970s helped find the main structural features of model classes (the stability hierarchy, the role of strongly minimal sets and later the role of regular types). Later on, with the development of the model theory of AECs this story continued, and naturally increased in complexity (even taking as a standard the already very complex book Classification Theory, the new model theory of Abstract Elementary Classes ushered new degrees of complexity that are still far from being mapped out).

In attempting to prove a version of the categoricity conjecture one is led (forced?) to understand most of the following:

- the structure of types
- versions of saturation (here, in the next section, Galois-saturation is one of the main options)
- the stability spectrum - usually guaranteeing that categoricity in κ implies enough stability below κ
- some form of locality of "forking-independence", to use a type omitted by a model in some cardinality and finding a *small* set over which it is independent
- a version of transfer of omitting types (e.g. with primary models over sets of indiscernibles),
- in some cases, minimal types - or regular types
- alternatively, Vaughtian pair techniques
- alternatively, building *dimension* into the proof, as in Baldwin-Lachlan
- ...

The point here is that in AECs the situation with the categoricity conjecture also follows this pattern: attempting to prove it (or proving particular cases) has forced model theorists to develop parallels to the concepts appearing in the previous list. This development has so far taken more than three decades and is bound to produce structural knowledge that (as happened in first order) later entangles with the rest of mathematics.

One good example of the previous phenomenon is the concept of an *excellent class*. Excellence was originally discovered by Shelah to analyze models of sentences of $L_{\omega_1,\omega}$ in connection with the categoricity conjecture. Much later, at least two connections with other parts of mathematics were discovered, related to excellence:

- Zilber based his early analysis of pseudo-exponential fields on what was originally called "quasiminimal excellent classes" - recently, the more combinatorially complicated excellence has been shown to be a consequence of the quasiminimality.

- The Hrushovski analysis of abstract elimination of imaginaries is partially connected to a variant of excellence (and also to binding groupoids and to properties of covers).

1.4. **Types in AECs -Local and global properties in AECs.** We now define the notion of a type in AECs.

Definition 1.4. Fix an aec $(\mathcal{K}, \leqslant_{\mathcal{K}})$, $M_\ell \prec_{\mathcal{K}} N_\ell$ models in $\mathcal{K}$, $a_\ell \in N$, for $\ell = 0, 1$. Then $(M_0, N_0, a_0) \sim (M_1, N_1, a_1)$ if and only if

- $M_0 = M_1$,
- there exist $\tilde{N}$ and $\mathcal{K}$-embeddings $f_\ell : N_\ell \to \tilde{N}$ such that $f_0(a_0) = f_1(a_1)$ and $f_0 \restriction M_0 = f_1 \restriction M_1$.

The previous definition does not necessarily yield an equivalence relation; however, if the aec $\mathcal{K}$ satisfies the amalgamation property (AP) next defined, $\sim$ is indeed an equivalence relation (exercise!).

Definition 1.5. An aec $\mathcal{K}$ satisfies the amalgamation property (AP) iff for every triple M_0, M_1, M_2 such that $M_0 \prec_{\mathcal{K}} M_1$ and $M_0 \prec_{\mathcal{K}} M_2$ there exist a model M_3 and maps $g_\ell : M_\ell \to M_3$ $(\ell = 1, 2)$ such that $f_1 \restriction M_0 = f_2 \restriction M_0$.

Under the Amalgamation Property, $\sim$ is an equivalence relation between triples. The equivalence class of (M, N, a) is called the *Galois type* of a over M in N - and sometimes just the "type" of a over M in N. A common notation for this equivalence class is $\mathrm{gatp}(a/M, N)$.

- Usual first order (syntactic) types are an example of these. Therefore, this notion generalizes the usual one. (Exercise: if $a, b \in M$, for a saturated model M of a first order theory T, $\mathrm{tp}(a/M_0) = \mathrm{tp}(b/M_0)$ iff $\mathrm{gatp}(a/M_0, M) = \mathrm{gatp}(b/M_0, M)$, when $M_0 \prec M$.)
- This notion also refines the notion of a syntactic type (consistent set of formulas in one variable) in $L_{\omega_1,\omega}$: $\mathrm{gatp}(a/M, N) = \mathrm{gatp}(b/M, N)$ implies that $\mathrm{tp}_{L_{\omega_1,\omega}}(a/M) = \mathrm{tp}_{L_{\omega_1,\omega}}(b/M)$. The converse usually does not hold: Galois types are usually "finer" than syntactic types; they detect differences that may be unavailable to syntactic types.

1.4.1. *Monster model.* When the class $\mathcal{K}$ satisfies the amalgamation property (AP), the Joint Embedding Property (JEP) and has no maximal models, one can easily show that it has a **monster model** - that is a model $\mathbb{M}$ that is

- strongly homogeneous: every isomorphism $f : M \to N$ with $M, N \leqslant_{\mathcal{K}} \mathbb{M}$ and $|M| = |N| < |\mathbb{M}|$ can be extended to $\hat{f} \in \mathrm{Aut}(\mathbb{M})$.

- Galois-saturated: every Galois type of the form (M, N, a) for $M \leqslant_{\mathcal{K}} \mathbb{M}$ can be realized in $\mathbb{M}$ - i.e., there exists $b \in \mathbb{M}$ such that $(M, N, a) \sim (M, \mathbb{M}, b)$.

When there is a monster model $\mathbb{M}$, the following holds (exercise):

Proposition 1.6. For any $a, b \in \mathbb{M}$ and $M \leqslant_{\mathcal{K}} \mathbb{M}$, the following two are equivalent:

- $\mathrm{gatp}(a/M, \mathbb{M}) = \mathrm{gatp}(b/M, \mathbb{M})$
- there exists $f \in Aut(\mathbb{M}/M)$ such that $f(a) = b$.

This justifies the usual identification "Galois types = orbital types": Galois types are really orbits under the automorphism group of the monster, fixing the base of the type. In many sources in AECs this is the most common way to treat Galois types - but in situations with no monster model or no amalgamation one has to use the relation $\sim$ between triples.

1.4.2. *Counting types.* The number of types is, as in first order, fundamental. An AEC $\mathcal{K}$ is Galois-stable in λ if and only if –as expected– the number of Galois types over any model of size λ is λ. This value is the minimum possible, by the following lemma (exercise); $ga - S(M)$ denotes the set of all Galois types over M.

Lemma 1.7. For any model M of size λ, $\lambda \leqslant |ga - S(M)| \leqslant 2^{\lambda}$.

Later in these notes, we will sketch a proof of stability below a categoricity cardinal in appropriate AECs.

2. Day 2: Stability Theory of AECs.

2.1. **The Presentation Theorem.** We follow Lessmann's simplication of the exposition of the proof[7].

Theorem 2.1 (Shelah). Let $(\mathcal{K}, \leqslant_K)$ be an AEC in a language L. Then there exist

- A language $L' \supset L$, with size $LS(\mathcal{K})$,
- A (first order) theory T' in L' and
- A set of T'-types, Γ', such that

$$\mathcal{K} = PC(L, T', \Gamma') := \{M' \restriction L \mid M' \models T', M' \text{ omits } \Gamma'\}.$$

Moreover, if $M', N' \models T'$, they both omit Γ', $M = M' \restriction L$ and $N = N' \restriction L$,

$$M' \subset N' \Leftrightarrow M \leqslant_K N.$$

Proof Let $L' = L \cup \{F^n_i \mid i < LS(\mathcal{K}), n < \omega\}$ and let T' be the theory consisting of the following axioms:

- $\exists x(x = x)$,

- $\forall x_0 \cdots \forall x_{n-1} F^n_i(x_0, \cdots, x_{n-1}) = x_i$, for $i < n$.

Notice that this does not determine the values of F^n_i for $i \geqslant n$.

Now let $M' \models T'$ and $a \in M$ a tuple. For each subtuple b of a let

$$U_b := \{F^m_i(b) \mid i < LS(\mathcal{K})\}, \qquad \text{with } \ell(b) = m.$$

Then $U_b \subset M'$. Whether U_b is or not the universe of a submodel of $M' \restriction L$ is determined by the quantifier-free type of a over the empty set,

$$\mathtt{qftp}_{L'}(a/\emptyset, M').$$

Now, if U_b is indeed the universe of a submodel of $M \restriction L$ then this submodel is "generated" (entirely determined) by b and M' - call it M_b.

The isomorphism type of M_b, and its membership in $\mathcal{K}$, are also determined by $\mathtt{qftp}_{L'}(a/\emptyset, M')$, as well as whether $M_b \leqslant_K M_a$:

Fact 2.2. If $\mathtt{qftp}_{L'}(a_1/\emptyset, M') = \mathtt{qftp}_{L'}(a_2/\emptyset, M')$ then

- $M_{b_1} \approx M_{b_2}$ whenever b_1 is a subtuple of a_1, b_2 is the corresponding subtuple of a_2 and U_{b_1}, U_{b_2} are the universes of a submodel of $M' \restriction L$.
- In this case, $M_{b_1} \in \mathcal{K}$ iff $M_{b_2} \in \mathcal{K}$.
- $M_{b_1} \leqslant_K M_{a_1}$ iff $M_{b_2} \leqslant_K M_{a_2}$.

Let now Γ' be the set of all qf types of the form $\mathtt{qftp}_{L'}(a/\emptyset, M')$ for $M' \models T'$ and a a tuple in M' that *do not* satisfy the two conditions:

- For every subtuple b of a the set U_b is the universe of a submodel M_b of $M' \restriction L$ and $M_b \in \mathcal{K}$.
- For every subtuple b of a, $M_b \leqslant_K M_a$.

Fact 2.3. $\mathcal{K} = PC(L, T', \Gamma')$.

Proof of 2.3. If $M \in \mathcal{K}$, define the expansion M' of M by induction on $n < \omega$ (on the values of the functions F^n_i for $i < LS(\mathcal{K})$) in such a way that for each $a \in M$ of length n, the set $U_a = \{F^n_i(a) \mid i < LS(\mathcal{K})\}$ is a $\mathcal{K}$-submodel M_a of M, and the assignment is consistent with T'.

<u>$n = 0$:</u> Let $M_\emptyset \leqslant_K M$ of size $LS(\mathcal{K})$ and $U_\emptyset = \{F^0_i \mid i < LS(\mathcal{K})\}$ an enumeration of its universe.

<u>$n \to n+1$:</u> let $M_a \leqslant_K M$ of size $LS(\mathcal{K})$ extend the union of the U_b's for b a subtuple of a. Now enumerate U_a by $\{F^n_i(a) \mid i < LS(\mathcal{K})\}$, in such a way that $F^n_i(a) = a_i$ for $i < n+1$.

Check that $M' \models T'$ and omits Γ'.

Now, if $M \in PC(L, T', \Gamma')$ then let $M' \models T'$ omitting Γ' be such that $M' \restriction L = M$. By the omission of Γ', all sets U_a, for each $a \in M$, are the universe of a submodel $M_a \subset M$, and $M_a \in \mathcal{K}$. Now the $\mathcal{K}$-system $(M_a \mid a \in M)$ is clearly directed since $M_a, M_b \leqslant_K M_{ab}$. Then we have that $\bigcup_{a\in M} M_a \in \mathcal{K}$. Now, by the axiomatization of T', we have $a \in U_a$, hence $\bigcup_{a\in M} M_a = M$. Therefore $M \in \mathcal{K}$. □

Corollary 2.4 ("Hanf" number of an AEC). If an AEC $\mathcal{K}$ has a model of cardinality $\geqslant \beth_{(2^{LS(\mathcal{K})})^+}$ then it has arbitrarily large models.

Proof Use the Hanf number for PC classes (this uses the undefinability of well orders). □

Theorem 2.5 (Shelah). Let $(\mathcal{K}, \leqslant_K)$ be an AEC with amalgamation and arbitrarily large models. If $\mathcal{K}$ is categorical in $\lambda > LS(\mathcal{K})$ then it is μ-galois-stable for each cardinal $\mu \in [LS(\mathcal{K}), \lambda)$.

Proof Let $\mathcal{K}$ be as in the statement and assume that the conclusion fails for some minimal $\mu \in [LS(\mathcal{K}), \lambda)$. Let $M \in \mathcal{K}_\mu$ be such that $|ga{-}S(M)| > \mu$. Now amalgamate over M all models containing realizations of types in $|ga - S(M)|$. By AP and NMM we can without loss of generality assume this amalgam is of size $\geqslant \lambda$, by Downward Löwenheim-Skolem, we can find $N_1 \in \mathcal{K}_\lambda$ with $> \mu$ realizations of types in $|ga - S(M)|$.

We restrict ourselves to the case λ regular - the proof can be extended to arbitrary λ.

Now let $\mathcal{K} = PC(L, T', \Gamma')$. We can assume wlog that L' has Skolem functions. Since $\mathcal{K}$ has arbitrarily large models, it has Ehrenfeucht-Mostowski models. Let $N_2 := EM(\lambda) \restriction L$, $N_2 \in \mathcal{K}$ and $|N_2| = \lambda$.

Claim 2.6. N_2 realizes only μ many types of $ga - S(M')$ for each $M' \leqslant_K N_2$ of size μ.

Proof of claim (sketch). Given $M' \leqslant_K N_2$ of size μ, let $J \subset \lambda$ with $|J| = \mu$ and $M' \subset EM(J) \restriction L$. Since $(\lambda, <)$ is well-ordered, it can only realize μ many $\{<\}$-types over J. Also, if a and b are sequences in λ realizing the same cut over J, there exists $(I', <')$ extending $(\lambda, <)$ such that a and b are automorphic over J in I' (e.g. $I' = {}^{\omega>}\lambda$). Then,

$$gatp(\tau(a)/EM(J) \restriction L; N') = gatp(\tau(b)/EM(J) \restriction L; N')$$

hence

$$gatp(\tau(a)/M'; N') = gatp(\tau(b)/M'; N')$$

(for any term τ of the Skolem language) Then N_1 and N_2 are two non-isomorphic models in $\mathcal{K}$, of size λ. This contradicts the hypothesis of categoricity in λ. □

2.2. **Tameness and locality.**

Idea: "localizing" the condition of... extending a map f that fixes a model M in an aec $\mathcal{K}$ to a $\mathcal{K}$-embedding:

- if no embedding f of the class that fixes M sends some N_0 to some N_1 then

$$gatp(N_0/M) \neq gatp(N_1/M)$$

- *we want*: to localize this to checking that there is some $M_0 \in \mathcal{P}^*_\kappa(M)$ and $X_0 \in \mathcal{P}_\kappa(N_0)$ such that

$$\mathrm{gatp}(X_0/M_0) \neq \mathrm{gatp}(f(X_0)/M_0)$$

Definition 2.7 ((κ,λ)-tameness for μ, type shortness). Let $\kappa < \lambda$. An aec $\mathcal{K}$ with AP and $LS(\mathcal{K}) \leqslant \kappa$ is

- (κ,λ)-tame for sequences of length μ if for every $M \in \mathcal{K}$ of size λ, if $p_1 \neq p_2$ are Galois types over M then there exists $M_0 \prec_{\mathcal{K}} M$ with $|M_0| \leqslant \kappa$ such that

$$p_1 \restriction M_0 \neq p_2 \restriction M_0$$

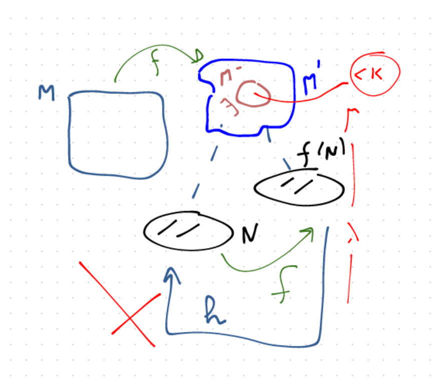
  (where $p_i = \mathrm{gatp}(X_i/M)$, X_i ordered in length μ, $i = 1, 2$)
- (κ,λ)-typeshort over models of cardinality μ if for every $M \in \mathcal{K}$ of size μ, if $p_1 \neq p_2$ are Galois types over M and $p_i = \mathrm{gatp}(X_i/M)$ where $X_i = (x_{i,\alpha})_{\alpha<\lambda}$, there exists $I \subset \lambda$ of cardinality $\leqslant \kappa$ such that $p_1^I \neq p_2^I$:

$$\mathrm{gatp}((x_{1,\alpha})_{\alpha\in I}/M) \neq \mathrm{gatp}((x_{2,\alpha})_{\alpha\in I}/M).$$

The two notions are clearly dual (parameters/realizations):

- In tameness, a narrow orbit (fixing large models) is controlled by the thicker orbits that approximate it (parameter locality),
- In type shortness, the orbit of a long sequence is controlled by the narrower orbits of its subsequences (realization locality)...

Theorem 2.8 (Boney). If a $\mathcal{K}$ (with a monster) is categorical in μ and is $(<\kappa,\mu)$-tame for λ-length types, then $\mathcal{K}$ is $(<\kappa,\mu)$-short for types over λ-sized domains.



Let M, M' of size μ, N of size λ such that $\mathrm{gatp}(M/N) \neq \mathrm{gatp}(M'/N)$.
Use μ-categoricity to get $f \in \mathrm{Aut}(\mathbb{C})$ such that $f \restriction M : M \approx M'$.
Now, $\mathrm{gatp}(f(N)/M') \neq \mathrm{gatp}(N/M')$: if equal, there is some $h \in \mathrm{Aut}(\mathbb{C}/M')$ so that $h \circ f(N) = N$ - so $h \circ f(M) = h(M') = M'$ so $\mathrm{gatp}(M/N) =$

gatp(M'/N). Now we use the $(<\kappa,\mu)$-tameness: get $M^- \in \mathcal{P}^*_\kappa(M')$ such that gatp$(f(N)/M^-) \neq$ gatp(N/M^-). Again as before gatp$(f^{-1}(M^-)/N) \neq$ gatp(M^-/N). But $f^{-1}(M^-) \in \mathcal{P}^*_\kappa(M)$. □

3. Day 3: Stability Theory for AECs (II).

We already have that (under AP and NMM) the categoricity of a class $\mathcal{K}$ in a cardinal $\lambda > LS(\mathcal{K})$ implies its galois-stability below λ. This marks a beginning of stability theory for AECs. In this lecture we will explore two notions of independence (definition and basic properties) and more consequences of Galois-stability. I will describe in particular a recent (2015) result of Boney, Grossberg, Kolesnikov and Vasey: canonicity of forking independence (2015) [1]. Finally, I will discuss notions of superstability for abstract elementary classes - results of myself with Grossberg and VanDieren [5], from around 2008, appearing soon in Math. Log. Quarterly.

3.1. Independence Notions for Galois Types: Splitting, Forking, etc.

The rôle of forking independence in the development of first order stability theory is perhaps the most central in that theory. The definition comes from 1970, but later Lascar (1974), Harnik and Harrington (1984) and finally Kim and Pillay (1997) showed in various stages the canonicity of an abstract independence notion connected to forking.

Boney and Grossberg,

abstract indep notion

I, M, B (base monotonicity), $(C)_\kappa$, T (left transitivity), T_* (right transitivity), S, U, $E = E_0 + E_1$, L, E_+ (strong extension)

Definition 3.1.
- $\downarrow$ has $(C)_\kappa$ (**continuity**) if whenever $A\downarrow_M N$ there exists $A^- \subset A$, $B^- \subset N$ of size $<\kappa$ such that for all $N_0 \geqslant_K M$ containing B^-, $A^-\downarrow_M N_0$.
- $\downarrow$ has T (**left transitivity**) if $M_1\downarrow_{M_0} N$ and $M_2\downarrow_{M_1} N$ with $M_0 \leqslant_K M_1 \leqslant_K M_2$, then $M_2\downarrow_{M_0} N$.
- $\downarrow$ has T_* (**right transitivity**) if $A\downarrow_{M_0} M_1$ and $A\downarrow_{M_1} M_2$ with $M_0 \leqslant_K M_1 \leqslant_K M_2$, then $A\downarrow_{M_0} M_2$.
- (E) consists of the following two properties:
  - E_0 (**Existence**) - for all A, $A\downarrow_M M$,
  - E_1 (**Extension**) - given a set A and $M \leqslant_K N \leqslant_K N'$, if $A\downarrow_M N$ then there is $A' \equiv_N A$ such that $A'\downarrow_M N'$.
- (L) **Local character**: $\kappa_\alpha(\downarrow) < \infty$ for all α, where $\kappa_\alpha(\downarrow) :=$ min$\{\lambda \in REG \cup \{\infty\} \mid$ for all $\mu = \text{cf}\ \mu \geqslant \lambda$, all increasing, continuous chains $\langle M_i \mid i \leqslant \mu\rangle$ and all sets A of size α, there is some $i_0 < \mu$ such that $A\downarrow_{M_{i_0}} M_\mu$.

- (U) **Uniqueness**: If $A \mathop{\smile\!\!\!\!\!\!|}_M N$, $A' \mathop{\smile\!\!\!\!\!\!|}_M N$ and $f : A \equiv_M A'$ then $g : A \equiv_N A'$ for some g so that $g \restriction A = f \restriction A$.
- (S) **Symmetry**: If $A \mathop{\smile\!\!\!\!\!\!|}_M N$ then there is $M' \geqslant_K M$ with $A \subset M'$ such that $N \mathop{\smile\!\!\!\!\!\!|}_M M'$. If A is a model extending M one can take $M' = A$.

nonsplitting

Definition 3.2 (Forking - Boney-Grossberg). (Also called coheir independence) - Fix a cardinal $\kappa > LS(\mathcal{K})$. "Small" here means "size $< \kappa$". For $M \prec N$ let

$$A \mathop{\smile\!\!\!\!\!\!|}^{ch}_M N \quad \Leftrightarrow \quad \text{for every small} A^- \subset A \text{ and } N^- \leqslant_K N,$$
$$\text{there is } B^- \subset M \text{ such that } B^- \equiv_{N^-} A^-.$$

Fact 3.3 (Properties of coheir). Let $\kappa > LS(\mathcal{K})$

- $\mathop{\smile\!\!\!\!\!\!|}^{ch}$ has continuity for κ $(C)_\kappa$ and transitivity (T).
- If M is κ-saturated, then $\mathop{\smile\!\!\!\!\!\!|}^{ch}$ has (E_0) over M.
- If κ is furthermore regular and $\mathcal{K}$ is fully $(< \kappa)$-tame, fully $(< \kappa)$-type short, has no weak $(< \kappa)$-order property and $\mathop{\smile\!\!\!\!\!\!|}^{ch}$ has existence and extension (E) then $\mathop{\smile\!\!\!\!\!\!|}^{ch}$ has uniqueness (U) and symmetry (S).

The hardest work is to prove symmetry.

Theorem 3.4 (Additional properties of coheir). Right transitivity (T_*) can be deduced from either symmetry and transitivity or from uniqueness (see [1, 5.9,5.11]). Local character follows from symmetry (see [1, 6.4]).

3.2. **Canonicity of Forking Independence.** This generalizes in some sense Lascar/Harrington/Harnik - but is restricted to tame classes and requires assuming the extension property.

Theorem 3.5 (Canonicity of coheir). Assume $\mathcal{K}$ is fully $(< \kappa)$-tame, fully $(< \kappa)$-type short and weakly κ-Galois stable. Assume $\mathop{\smile\!\!\!\!\!\!|}^{ch}$ has (E). Then

- $\mathop{\smile\!\!\!\!\!\!|}^{ch}$ has $(C)_\kappa$, (T), (T_*), (S), (U) and (L).
- Any independence notion satisfying (E) and (U) must be $\mathop{\smile\!\!\!\!\!\!|}^{ch}$ for base models in $\mathcal{K}_{\geqslant\kappa}$.

3.3. **Superstability, frames and limits.** Uniqueness of Limit Models (GVV).

4. Day 4: Connections with Set Theory.

4.1. **Strongly compact cardinals and tameness.** Back to strongly compact cardinals and tameness. Boney's result, interaction between AECs and large cardinals. Do directly with embeddings rather than ultraproducts?

Theorem 4.1 (Boney). If κ is strongly compact and $\mathcal{K}$ is essentially below κ (i.e. $\mathrm{LS}(\mathcal{K}) < \kappa$ or $\mathcal{K} = \mathrm{Mod}(\psi)$ for some $L_{\kappa,\omega}$-sentence ψ) then $\mathcal{K}$ is $(< (\kappa + \mathrm{LS}(K)^+, \lambda$-tame and $(< \kappa, \lambda)$-typeshort for all λ.

The proof is direct, given the strength of the hypothesis. Boney and Unger have proved (March 2015) that under strong inaccessibility of κ, the $(< \kappa, \kappa)$-tameness of all aecs implies κ's strong compactness. (?)

Notice that

$$\text{Every AEC } \mathcal{K} \text{ with } \mathrm{LS}(\mathcal{K}) < \kappa \text{ is } (< \kappa, \kappa)\text{-tame} \tag{1}$$

already implies $V \neq L$: Baldwin and Shelah constructed a counterexample to $(< \kappa, \kappa)$ starting from an almost free, non-free, non-Whitehead group of cardinality κ. In L this may happen at any κ regular, not strongly compact.

On the other hand, Hart-Shelah's example of an $L_{\omega_1,\omega}$-sentence categorical in $\aleph_0, \aleph_1, \cdots, \aleph_k$ but NOT in $\aleph_{k+2}$ shows that pushing tameness FOR ALL aecs below $\aleph_\omega$ is impossible.

4.2. **Collapse / Other properties.** Collapse. Tree properties.

4.3. **Set Theoretic Dichotomies.** Local theory.

Set theoretical dichotomies. Statement, sketch of the proof.

5. Appendix: more examples

Note: the topics in this lecture will be explored (in a different style) during the Colloquium lecture at IPM, on Wednesday 26.11.

5.1. **Early examples.**

5.2. **The Oxford group - Quasiminimal AECs, covers.**

5.3. **Modular invariants.** Various lines of interaction between Model Theory and subareas of Geometry have evolved in recent years (based on earlier interactions centered around the emergence of geometric stability theory). One of those lines, originally centered at Oxford around Boris Zilber and his group of collaborators, has evolved from early exploration of "pseudo-analytic" structures (the primal examples being fields with pseudo-exponentiation - the so-called *Zilber field*, various analytic covers, various other "Zariski geometries") to more recent variants. Among them stand modular invariants: the classical j-mapping and higher-dimensional variants. More recently, Christopher Daw and Adam Harris (see [3] and [6]) studied the categoricity of structures capturing the j-mapping and several generalizations, and showed that for many of them categoricity is equivalent to a condition on their Galois representations present in the Mumford-Tate Conjecture. For the one-dimensional case Serre proved the conjecture; for higher dimensional cases the equivalence between categoricity of an associated structure (built as an Abstract Elementary Class, axiomatized in the logic $L_{\omega_1,\omega}$) and (as yet unproven) versions of the Mumford-Tate Conjecture also hold.

These results are not a singularity: interaction between the model theoretic properties of the j-function and their arithmetic geometry have been revealed at the level of *definability, quasi-minimality* and *categoricity.*

Interaction between Model Theory and Geometry thus happens at the level of various automorphic functions, beyond the classical j (further connections between coefficients of Fourier expansions of the j-function and certain group representations).

In addition to these connections, there is also the development of the Real Multiplication program (started by Manin) - one crucial step involves a good definition of a *quantum* version of the j-function. There have been various attempts in this direction; one of them, due to Gendron, is rooted in another part of model theory: non-standard methods (see [2]).

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