Categoricity in Non-Elementary contexts - The Role of Large Cardinals

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Shelah's Categoricity Conjecture

- A classical problem in the model theory of AECs has been to find versions of Morley's Theorem (the Łoś Conjecture) for AECs - Transferring Categoricity.
- "Semantic versions" of the model theory of $L_{\lambda^+,\omega}(Q)$.

Conjecture (Shelah - around 1980)

For every λ, there exists μ_λ such that if $\mathcal{K}$ is an AEC with $LS(\mathcal{K}) = \lambda$, categorical in some cardinality $\geq \mu_\lambda$, then $\mathcal{K}$ is categorical in all cardinalities greater than μ_λ.

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Proving categoricity transfer not only reveals a strong form of "semantic completeness" of the class $\mathcal{K}$ but also involves understanding deeply how models are embedded into one another and how types p are controlled by small "projections" $p \restriction M$.

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- Boney (2013:) consistency of the full conjecture, under a proper class of strongly compact cardinals. More partial results by Vasey.

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- Other dividing lines from FO Model Theory "extend" to a (more tenuous, but more structural) "Classification Theory for AECs": NIP for example is really connected to the Genericity Pair Conjecture, a statement on the behaviour of a large groupoid of partial isomorphims of homogeneous structures in a class.

Grossberg-VanDieren: tameness isolated

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Their proof built on a previous proof of the "downward" transfer by Shelah but has a crucial element: isolating the notion of tameness ("buried" in Shelah's proof of the downward part - fleshing out the notion allows Grossberg/VanDieren to prove the upward categoricity).

Localizing difference

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- we want: to localize this to checking that there is some $M_0 \in \mathcal{P}^*_\kappa(M)$ and $X_0 \in \mathcal{P}_\kappa(N_0)$ such that

$$\mathrm{gatp}(X_0/M_0) \neq \mathrm{gatp}(f(X_0)/M_0)$$

Tameness and type-shortness

Definition ((κ, λ)-tameness for μ, type shortness)

Let $\kappa < \lambda$. An aec $\mathcal{K}$ with AP and $LS(\mathcal{K}) \leq \kappa$ is

- (κ, λ)-tame for sequences of length μ if for every $M \in \mathcal{K}$ of size λ, if $p_1 \neq p_2$ are Galois types over M then there exists $M_0 \prec_{\mathcal{K}} M$ with $|M_0| \leq \kappa$ such that

$$p_1 \upharpoonright M_0 \neq p_2 \upharpoonright M_0$$

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- (κ, λ)-typeshort over models of cardinality μ if for every $M \in \mathcal{K}$ of size μ, if $p_1 \neq p_2$ are Galois types over M and $p_i = \text{gatp}(X_i/M)$ where $X_i = (x_{i,\alpha})_{\alpha<\lambda}$, there exists $I \subset \lambda$ of cardinality $\leq \kappa$ such that $p_1^I \neq p_2^I$:

$$gatp((x_{1,\alpha})_{\alpha\in I}/M) \neq gatp((x_{2,\alpha})_{\alpha\in I}/M).$$

Dual notions - stability

The two notions are clearly dual (parameters/realizations):

- In tameness, a narrow orbit (fixing large models) is controlled by the thicker orbits that approximate it (parameter locality),

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The two notions are clearly dual (parameters/realizations):

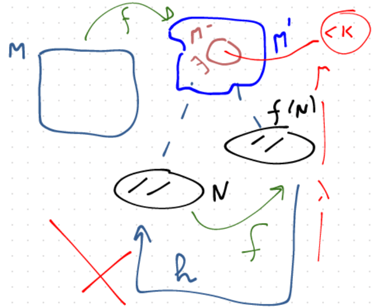
- In tameness, a narrow orbit (fixing large models) is controlled by the thicker orbits that approximate it (parameter locality),
- In type shortness, the orbit of a long sequence is controlled by the narrower orbits of its subsequences (realization locality)...

These dualities are equivalences under stability conditions. In general, they are not.

Duality under categoricity - "Heirs/coheirs"

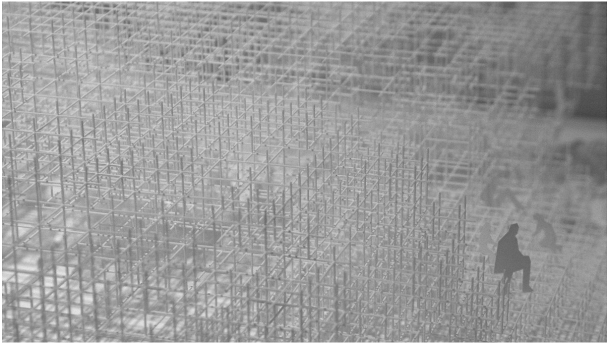
Theorem (Boney)

If a $\mathcal{K}$ (with a monster) is categorical in μ and is $(<\kappa, \mu)$-tame for λ-length types, then $\mathcal{K}$ is $(<\kappa, \mu)$-short for types over λ-sized domains.



Let M, M' of size μ, N of size λ such that $\mathrm{gatp}(M/N) \neq \mathrm{gatp}(M'/N)$. Use μ-categoricity to get $f \in Aut(\mathbb{C})$ such that $f \upharpoonright M : M \approx M'$.
Now, $\mathrm{gatp}(f(N)/M') \neq \mathrm{gatp}(N/M')$: if equal, there is some $h \in Aut(\mathbb{C}/M')$ so that $h \circ f(N) = N$ - so $h \circ f(M) = h(M') = M'$ so $\mathrm{gatp}(M/N) = \mathrm{gatp}(M'/N)$. Now we use the $(<\kappa, \mu)$-tameness: get $M^- \in \mathcal{P}^*_\kappa(M')$ such that $\mathrm{gatp}(f(N)/M^-) \neq \mathrm{gatp}(N/M^-)$. Again as before $\mathrm{gatp}(f^{-1}(M^-)/N) \neq \mathrm{gatp}(M^-/N)$. But $f^{-1}(M^-) \in \mathcal{P}^*_\kappa(M)$. $\square$

Large Cardinals & Model Theory



Getting Tameness from Large Cardinals

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Theorem (Boney)

If κ is strongly compact and $\mathcal{K}$ is essentially below κ (i.e. $LS(\mathcal{K}) < \kappa$ or $\mathcal{K} = Mod(\psi)$ for some $L_{\kappa,\omega}$-sentence ψ) then $\mathcal{K}$ is $(< (\kappa + LS(K)^+, \lambda$-tame and $(< \kappa, \lambda)$-typeshort for all λ.

The proof is direct, given the strength of the hypothesis. Boney and Unger proved (2015) that under strong inaccessibility of κ, the $(< \kappa, \kappa)$-tameness of all aecs implies κ's strong compactness.

A strong lemma - maybe an overkill?

Theorem (Łoś Theorem for aecs under strong compact cardinals)

Let $\mathcal{K}$ be an AEC with $LS(\mathcal{K}) < \kappa$, κ a strongly compact cardinal. Suppose there is $N_0 \leq_{\mathcal{K}} N$ and $p \in ga - S^I(N_0)$ with $|N_0 < \kappa$, $|I| < \kappa$ and a κ-complete ultrafilter U over I. Then

$$[h]_U \in \prod N/U \models p \quad \textit{iff} \quad \{i \in I | h(i) \models p\} \in U.$$

(There are theorems with weaker conclusions, starting from measurables, strongly compact, etc.)

The conjecture is consistent

Theorem (Boney)

Let κ be strongly compact and $\mathcal{K}$ an aec essentially below κ. If $\mathcal{K}$ is categorical in a successor $\lambda^+ > LS(\mathcal{K})^+$ then $\mathcal{K}$ is categorical in all $\mu \geq \min\{\lambda^+, \beth_{(2^{Hanf(LS(K))})^+}\}$.

Theorem (Boney)

In models with a proper class of strongly compact cardinals, the Shelah Conjecture (for successors) holds.

A little more...

Theorem

Let κ be a Π^2_1-indescribable cardinal. If $\mathcal{K}$ is an AEC with $LS(K) < \kappa$ and $\mathcal{K}_\kappa$ has a unique limit model, then for every $\lambda < \kappa$, there exists $\mu \in (\lambda, \kappa)$ such that $\mathcal{K}_\mu$ has a unique limit model.

(And similar results using versions of downward reflection, for categoricity transfer, amalgamation, tameness...)

Generalized compactness phenomena

The fact that tameness/type shortness hover around strong compactness/supercompactness is not so surprising after all: they are forms of "generalized compactness".

- κ has the tree property + inaccessibility $\equiv$ Weak Compactness of κ
- κ has the supertree property + inaccessibility $\equiv$ Supercompactness of κ
- Every aec $\mathcal{K}$ is $(<\kappa, \kappa)$-tame + inaccessibility $\equiv$ seems to be rather strong.

Challenges for Set Theory?

Under a proper class of strongly compact cardinals, Boney showed that

$$\text{Every AEC } \mathcal{K} \text{ with arbitrarily large models is tame.} \tag{1}$$

(He gives weaker versions of tameness, obtained from proper classes of measurables and weakly compact cardinals.)
All this seems rather reducible to weaker large cardinals, at least for a lot of model theory!

Lower bounds

Notice that

$$\text{Every AEC } \mathcal{K} \text{ with } LS(\mathcal{K}) < \kappa \text{ is } (<\kappa, \kappa)\text{-tame} \tag{2}$$

already implies $V \neq L$: Baldwin and Shelah constructed a counterexample to $(<\kappa, \kappa)$ starting from an almost free, non-free, non-Whitehead group of cardinality κ. In L this may happen at any κ regular, not strongly compact.
On the other hand, Hart-Shelah's example of an $L_{\omega_1,\omega}$-sentence categorical in $\aleph_0, \aleph_1, \cdots, \aleph_k$ but NOT in $\aleph_{k+2}$ shows that pushing tameness FOR ALL aecs below $\aleph_\omega$ is impossible.

Collapsing and its limitations

Collapsing large cardinals while keeping some of their properties has a long history of interesting results. For instance,

- Mitchell: collapsed a weakly compact to $\aleph_2$ while keeping the tree property. This was later generalized (collapsing much more) in order to get the tree property at all the $\aleph_n$'s and/or in $\aleph_{\omega+1}$ (Magidor, Cummings, Neeman, Fontanella, etc.)

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- The direct collapse of (say) a strongly compact κ where you have $(<\kappa, \kappa)$-tameness to (say) $\aleph_2$ does not work:
- The resulting classes $j(\mathcal{K})$ and (if $\mathcal{K} = PC(L, T', \Gamma')$ the classes $\mathcal{K}^{V[G]} = PC^{V[G]}(L, T', j(\Gamma'))$ exhibit interesting (buy wide open) behavior.

A dichotomic behavior

- Under Weak Diamond:

Theorem (from Sh88)

(Under $2^\kappa < 2^{\kappa^+}$). Every aec $\mathcal{K}$ with $LS(\mathcal{K}) \leq \kappa$, categorical in κ, failing AP for models of size κ has 2^{κ^+} many non-isomorphic models of cardinality κ^+.

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- Example under MA:
  (MA_{ω_1}) There is a class (axiomatizable in $L_{\omega_1,\omega}(Q)$) that is $\aleph_0$-categorical, fails AP in $\aleph_0$ and is also categorical in $\aleph_1$. This can be lifted below continuum.

Forcing isomorphism/categoricity

Theorem (Asperó, V.)

The existence of a weak AEC, categorical in both $\aleph_1$ and $\aleph_2$, failing AP in $\aleph_1$, is consistent with ZFC+CH+$2^{\aleph_1} = 2^{\aleph_2}$.

The result is obtained by an ω_3-iteration over a model of GCH, where we

- Start with GCH in V.
- Build a countable support iteration of length ω_3, where
- at each stage α of the iteration you consider in $V^{\mathbb{P}_\alpha}$ two models $M_0, M_1 \in \mathcal{K}$, $|M_0| = |M_1| = \aleph_2$ (use a bookkeeping function) and
- fix $(M_i^0)_{i<\omega_2}$, $(M_i^1)_{i<\omega_2}$ resolutions of the two models with $M_i^\varepsilon = N_i \cap M_\varepsilon$ where $(N_i)_{i<\omega_2}$ is an $\in$-increasing and $\subset$-continuous of elementary substructures of some $H(\theta)$ of size $\aleph_1$ containing M_0 and M_1...

Forcing isomorphism/categoricity

- at this stage iterate with $\mathbb{Q}_\alpha$ the partial order consisting of countable partial isomorphisms p between M_0 and M_1 such that if $x \in \mathrm{dom}(p)$ and i is the minimum such that $x \in M_i^0$ then $p(x) \in M_i^1$.
- Each stage $\mathbb{Q}_\alpha$ of the iteration, and all the forcing $\mathbb{P}_{\omega_3}$ is σ-closed and $\mathbb{P}_{\omega_3}$ has the $(\aleph_2) - a.c.$ (need CH for the relevant (!) Δ-lemma).

Thank you for your Attention!