# Sheaves of Metric Structures

Andrés Villaveces
*Universidad Nacional de Colombia - Bogotá*
(joint with Maicol A. Ochoa - *University of Pennsylvania*)

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# Contents

# Ideal (limit) models in Physics - Goals

Various questions (classical and recently posed or revisited) in Physics point towards the need of various kinds of "ideal structures", and tools of contrast between "real structures" and those ideal (limit) structures.

I plan to illustrate three of these questions, examine some of their answers and contrast with the tools of generic models. I will also provide some side questions for discussion, coming from more model-theoretic considerations.

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# Zilber: Structural approximation

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Implicit knowledge by the physicist of the structure of his model, not yet available to mathematicians? (Rabin, Rieffel, Zeidler)

# Model Theory's advantages

1. **Arbitrary** structures.
2. Hierarchy of types of structures (or of theories): stability theory.
3. At the "top" of the hierarchy: Hrushovski-Zilber's Zariski Structures - generalized algebraic varieties over ACF, with relations = Zariski-closed sets.
4. One-dimensional objects of Zariski structures are exactly finite covers of algebraic curves - these correspond to "nonclassical" structures coming from non-commutativity phenomena.
5. With Model Theory on Sheaves: strong ways of controlling limit models.
6. May even go "beyond logic-dependence" and get several of the previous (Abstract Elementary Classes).

# Limit and ideal models, à la Zilber

Quoting Zilber's [Zil]:

The process of understanding the physical reality by working in an ideal model can be interpreted as follows. We assume that the ideal model $\mathbb{M}_{\text{ideal}}$ is being chosen from a class of "nice" structures, which allows a good theory. We suppose that the real structure $\mathbb{M}_{\text{real}}$ is "very similar" to $\mathbb{M}_{\text{ideal}}$ (...) approximated by a sequence $\mathbb{M}_i$ of structures and $\mathbb{M}_{\text{real}}$ is one of these, $\mathbb{M}_i = \mathbb{M}_{\text{real}}$ sufficiently close to $\mathbb{M}_{\text{ideal}}$. The notion of approximation must also contain both logical and topological ingredients. (...)

# This goes on...

... the reason that we wouldn't distinguish two points in the ideal model $\mathbb{M}_{\text{ideal}}$ is that the corresponding points are very close in the real world $\mathbb{M}_{\text{real}}$ so that we do not see the difference (using the tools available). In the limit of the $\mathbb{M}_i$'s this sort of difference will manifest itself as an infinitesimal. In other words, the limit passage from the sequence $\mathbb{M}_i$ to the ideal model $\mathbb{M}_{\text{ideal}}$ must happen by killing the infinitesimal differences. (...) This corresponds to taking a specialization (...) from an ultraproduct $\prod_D \mathbb{M}_i$ to $\mathbb{M}_{\text{ideal}}$.

His examples of structural approximation include no less than the Gromov-Hausdorff limit of metric spaces and deformation of algebraic varieties.

BUT...

... We note that the scheme is quite delicate regarding metric issues. In principle we may have a well-defined metric (...) on the ideal structure only. Existence of a metric, especially the one that gives rise to a structure of a differentiable manifold, is one of the key reasons of why we regard some structures as "nice" or "tame". The problem of whether and when a metric on $\mathbb{M}$ can be passed to approximating structures $\mathbb{M}_i$ might be difficult, indeed we don't know how to answer this problem in some interesting cases.

# Zilber's approach to structures for Physics

In a nutshell... Zariski Geometries:

$$\mathbb{M} = (M, \mathcal{C})$$

where $M$ is a set and $\mathcal{C}$ is a collection of basic predicates. $\mathcal{C}$ is a basis of closed sets for a topology on each $M^n$ such that

- Projections are $pr : M^n \to M^k$ are continuous.
- Closed sets "are linear, surfaces"... there is a dimension $\dim R$ of every closed set such that if $R$ is irreducible

$$\dim R = \dim pr(R) + \dim(gen.fiber)$$

- (Presmoothness) $U$ irred. is presmooth if for every irred. rel. closed subsets $S_1, S_2 \subset U$ and any irreducible component $S_0$ of $S_1 \cap S_2$

$$\dim S_0 \geq \dim S_1 + \dim S_2 - dimU.$$

# Hrushovski-Zilber's theorem

Theorem (Classification Theorem - Hrushovski-Zilber)

*Any one-dimensional Zariski geometry* $\mathbb{M}$ *that is "non-linear" is associated to a smooth algebraic curve* $C$ *over an algebraically closed field* $F$ *through a surjective map* $p : M \to C(F)$*, definable in* $\mathbb{M}$ *in such a way that the fibres are all of some finite size* $N$*.*

So, Zariski geometry is "almost" algebraic geometry, but the structure of the finite fibers has been studied by Zilber and found to contain "jewels" of information.

There are "not enough" definable coordinate functions $M \to F$ to encode all the structure of $\mathbb{M}$ - the usual coordinate algebra gives just $C(\mathbb{F})$. In [Zil2]

# Zilber's Structural Approximation

Given a topological structure $\mathbb{M}$ and a family of structures $\mathbb{M}_i$, $i \in I$, in the same language, $\mathbb{M}$ is approximated by $\mathbb{M}_i$ along an ultrafilter $D$ on $I$ if for some elementary extension $M^D \succ \prod \mathbb{M}_i / D$ of the ultraproduct there is a surjective homomorphism

$$\lim_D : \mathbb{M}^D \to \mathbb{M}.$$

# Examples

These include:

1. The Gromov-Hausdorff limit of metric spaces along a non-principal ultrafilter $D$.
2. Structural approximation of a quantum torus at $q$ by quantum tori at roots of unity.

# Döring-Isham's more radical take

On the continuum in Physics, in Döring and Isham's [DorIsh]:

`opinions differ greatly as to whether radical revision is necessary at the beginning or will emerge "along the way" - they take the iconoclastic view that "a radical step is needed at the very outset"`

problematic a priori continuum (category error?):

# Problems with abandoning the continuum

Hilbert spaces or C*-algebras

Geometric quantization

Probability functions on a non-distributive quantum logic

Deformation quantization

Formal path-integrals (cf. Denef, Loeser, Hrushovski)

# Realism in Physics

In Döring and Isham's description, the three tenets of "realism" in Physics are:
Properties are meaningful (cf. observables = formulas)
Propositions about the system are Boolean valued
There is a space of states (microstates) (=types) that encodes "the way things are"

# Replacing the continuum - topoi

In classical Physics: to each physical quantity $A$ is associated a real-valued function $A : S \to \mathbb{R}$ - where $S$ is a space of states. They replace this idea by choosing an appropriate topos $\tau_\varphi$:

- Two special objects of the topos $\tau_\varphi$: state-object $\Sigma_\varphi$ and quantity-value object $R_\varphi$.
- A physical quantity is an ARROW of the topos: $A_\varphi : \Sigma_\varphi \to R_\varphi$.
- Logic is a Heyting algebra (needs strengthening!), propositions take values in the topos. Types - cf. Caicedo-Montoya.

# Kochen-Specker's Impossibility Theorem

The common sense belief that "every physical quantity must have a value even if we do not know what it is" is challenged in Quantum Physics at the level of the formalism itself: Kochen and Specker proved in 1967 the impossibility of assigning values to all physical quantities while preserving the functional relations between them. This has a sheaf "model theoretical" flavor that was first noticed by Domenech, Freytes and De Ronde, who built a first sheaf theoretic analysis of the theorem.

Döring and Isham have constructed a sheaf "spectral presheaf" that, within the topos-theoretic realm, captures Kochen-Specker as the non-existence of global sections for those spectral presheaves, when the Hilbert space has dimension $\geq 2$.

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$$\mathcal{D}\mathcal{E}(U') \to \mathcal{D}\mathcal{E}(U) :: d \mapsto d \restriction U$$
- The existence of a **global section** for such a sheaf ("empirical model") implies the existence of a local deterministic hidden-variable model.

# Limits

Theorem (A classical Generic Model Theorem)

*Let $\mathbb{F}$ be a generic filter for a sheaf of topological structures $\mathfrak{A}$ over $X$. Then*

$$\begin{aligned} \mathfrak{A}[\mathbb{F}] \models \varphi(\sigma/\sim_{\mathbb{F}}) \quad &\Longleftrightarrow \quad \{x \in X | \mathfrak{A} \Vdash_x \varphi^G(\sigma(x))\} \in \mathbb{F} \\ &\Longleftrightarrow \quad \exists U \in \mathbb{F} \textit{ such that } \mathfrak{A} \Vdash_U \varphi^G(\sigma). \end{aligned}$$

Here, $\varphi^G$ is a formula equivalent classically to $\varphi$, but not necessarily in an intuitionistic framework! (The formula $\varphi^G$ is sometimes called the Gödel translation of $\varphi$ - in 1925, Kolmogorov had independently defined an equivalent translation.)

# More on the Generic Model Theorem

Cohen's construction of generic models for set theory is the first published result along these lines. Later, Robinson, Barwise and Keisler used generic model theorems to get Omitting Types Theorems in various logics, generalized by Caicedo. Ellerman's "ultrastalk theorem" (1976) is a GMTh for maximal filters. Miraglia also proves a similar result for Heyting-valued models.

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$$\sigma \mapsto \sigma^* = \sigma \cup \{(\infty, [\sigma]_{\sim_{\mathbb{F}}})\}.$$

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Then, the GMTh just means that in the new sheaf $\mathfrak{A}^\infty$ this fiber is classic:

$$\mathfrak{A}^\infty \Vdash_\infty \varphi(\sigma_1^*, \cdots, \sigma_n^*) \Leftrightarrow \mathfrak{A}[\mathbb{F}] \models \varphi([\sigma_1^*], \cdots, [\sigma_n^*])$$

# Łoś as a first consequence

The Łoś theorem is clearly a special case of the Generic Model Theorem, corresponding to endowing $X$ with the discrete topology. Therefore, the Model Theory of sheaves has a twisted form of compactness - of course relative to a context with no excluded middle.

# The forcing theorem

The forcing theorem of Set Theory is another special case: take a partially ordered set $\mathbb{P}$, endowed with the order topology (basic open sets are downward closed sets). The Generic Model Theorem provides a model of set theory, where satisfaction is given by forcing on points. BUT in this kind of topological spaces, forcing over an open set is reducible to forcing over a point.
Most topological spaces, however, do not arise from partially ordered sets. A natural question (fairly unexplored) is what other kinds of models of set theory may be obtained by forcing with such topological spaces.

# Other applications of the GMTh

- Kripke models - generalized semantics
- Set-theoretic forcing
- Robinson's Joint Consistency Theorem (=Amalgamation over Models)
- Various Omitting Types Theorems
- Control over new kinds of limit models

# Sheaves of Hilbert Spaces

Why?

1. Hilbert Spaces are (still) a crucial tool for formalization of concepts and objects in Physics and in Chemistry
2. In Physics: really algebras of operators acting on Hilbert spaces.
3. In Chemistry: really predicates on Hilbert spaces.
4. In both, the dynamical properties of evolution of a system are relevant.

In the case of Chemistry, the current treatment is unsatisfactory: capturing the relevant predicates (chemical structure, chemical reaction) has depended on physics to a degree that some theoretical chemists consider excessive.

# The problem of a model theory for Hilbert Spaces

So, we want to be able to put Hilbert spaces (and more structure on top of them, such as predicates for reactions, or operators for observables) on fibers.
We could in principle do that as we have seen so far, but immediately we get the problem that we may get lots of non-standard Hilbert spaces (infinitesimals, etc.).
Moreover, we want the logic to "keep track" of (say) the distance to a projection $p(v)$, the convergence of a sequence in $H$, isometric isomorphism, $(1+\varepsilon)$-isomorphism, etc. etc.
Finally, we need to be able to take limits of Cauchy sequences at will in our structures: metric completeness is crucial.

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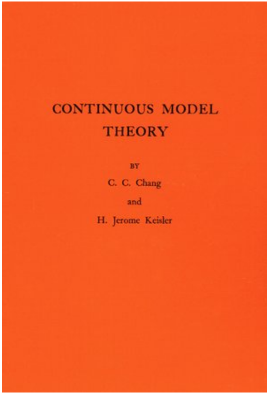
That is the rôle of Continuous Model Theory.

# Continuous Model Theory - Origins




Although the origins of CMTh go way back (von Neumann, Chang & Keisler (1966), and in some (restricted) ways to von Neumann's Continuous Geometryrecent takes on Continuous Model Theory are based on formulations due to Ben Yaacov, Usvyatsov and Berenstein of Henson and Iovino's Logic for Banach Spaces.

# Continuous predicates and functions

Definition

Fix $(M, d)$ a bounded metric space. A continuous $n$-ary predicate is a uniformly continuous function

$$P : M^n \to [0, 1].$$

A continuous $n$-ary function is a uniformly continuous function

$$f : M^n \to M.$$

# Metric structures

Therefore, metric structures are of the form

$$\mathcal{M} = \left(M, d, (f_i)_{i\in I}, (R_j)_{j\in \mathcal{J}}, (a_k)_{k\in K}\right)$$

Each function, relation must be endowed with a modulus of uniform continuity.

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$$\mathcal{M} = \left(M, d, (f_i)_{i \in I}, (R_j)_{j \in \mathcal{J}}, (a_k)_{k \in K}\right)$$

where the $R_i$ and the $f_j$ are (uniformly) continuous functions with values in $[0, 1]$, the $a_k$ are distinguished elements of $M$.
Remember: $M$ is a bounded metric space.
Each function, relation must be endowed with a modulus of uniform continuity.

# Examples of FO metric structures

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- Any FO structure, endowed with the discrete metric.
- Banach algebras (bounding them).
- Hilbert spaces with inner product as a binary predicate.
- For a probability space $(\Omega, \mathcal{B}, \mu)$, construct a metric structure $\mathcal{M}$ based on the usual measure algebra of $(\Omega, \mathcal{B}, \mu)$.
- Representations of $C^*$-algebras (Argoty, Berenstein, Ben Yaacov, V.).
- Valued fields.

# The syntax

1. Terms: as usual.
2. Atomic formulas: $d(t_1, t_n)$ and $R(t_1, \cdots, t_n)$, if the $t_i$ are terms. Formulas are then interpreted as functions into $[0, 1]$.
3. Connectives: continuous functions from $[0, 1]^n \to [0, 1]$. Therefore, applying connectives to formulas gives new formulas.
4. Quantifiers: $\sup_x \varphi(x)$ (universal) and $\inf_x \varphi(x)$ (existential).

# Interpretation

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Notice also that the set of connectives is too large, but it may be "densely" and uniformly generated by $0, 1, x/2, \dot{-}$: for every $\varepsilon$, for every connective $f(t_1, \cdots, t_n)$ there exists a connective $g(t_1, \cdots, t_n)$ generated by these four by composition such that $|f(\vec{t}) - g(\vec{t})| < \varepsilon$.

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- Stability (Ben Yaacov, Iovino, etc.),
- Categoricity for countable languages (Ben Yaacov),
- $\omega$-stability,
- Dependent theories (Ben Yaacov),
- Not much geometric stability theory: no analog to Baldwin-Lachlan (no minimality, except some openings by Usvyatsov and Shelah in the context of $\aleph_1$-categorical Banach spaces),
- NO simplicity!!! (Berenstein, Hyttinen, V.),
- Keisler measures, NIP (Hrushovski, Pillay, etc.).

# "Continuous Model Theory" beyond First Order

Several contexts, some unexplored so far.

1. Metric Abstract Elementary Classes (Hirvonen, Hyttinen - $\omega$-stability, V. Zambrano - superstability, domination, notions of independence): an amalgam of the power of Abstract Elementary Classes with metric ideas.

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3. Sheaves of (metric) structures. Our work with Ochoa, motivated by problems originally in Chemistry. NEXT!

# Sheaves of Hilbert Spaces

Why?

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A sheaf of metric structures $\mathfrak{A}$ over $X$ consists of:

1. A sheaf $(E, p)$ over $X$,
2. On every fiber $p^{-1}(x)$ $(x \in X)$, a metric structure

$$(\mathfrak{A}_x, d_x) = (E_x, (R_i^x)_i, (f_j^x)_j, (c_k^x)_k, d_x, [0, 1])$$

such that $E_x = p^{-1}(x)$, $(E_x, d_x)$ is a complete bounded metric space of diameter 1, and

- For every $i$, $R_i^{\mathfrak{A}} = \bigcup_{x \in X} R_i^x$ is open
- For every $j$, $f_j^{\mathfrak{A}} = \bigcup_{x \in X} f_j^x$ is continuous
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- For every $k$, $c_k^{\mathfrak{A}} : X \to E$ such that $x \mapsto c_k^x$ is a continuous global section
- The premetric $d^{\mathfrak{A}} := \bigcup_{x \in X} d_x : \bigcup_{x \in X} E_x^2 \to [0,1]$ is a continuous function.

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# Truth Continuity - adapted to metric

Truth Continuity is still the guiding paradigm. Remember in the "discrete" case, negation was the first stumbling block - the first place where forcing was needed in a non-trivial way.
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Negation in continuous, metric logic, is weak: the semantics really treats $\leq$ and $\geq$ as "negations" of each other...

Truth continuity happens without the need of forcing in two basic cases:

- Formulas $\varphi$ composed of max, min, $\dot{-}$ and inf: $\mathfrak{A}_x \models \varphi(x) < \varepsilon$ if and only if this happens at all $y$ near $x$
- Similarly for $\varphi > \varepsilon$ when $\varphi$ is built of max, min, $\dot{-}$ and sup.

# Pointwise forcing

With Ochoa, we define $\mathfrak{A} \Vdash_x \varphi < \varepsilon$ and $\mathfrak{A} \Vdash_x \varphi > \varepsilon$, for $x \in X$:

- Atomic: $\mathfrak{A} \Vdash_x d(t_1, t_2) < \varepsilon \Leftrightarrow d_x(t_1^{\mathfrak{A}_x}, t_2^{\mathfrak{A}_x}) < \varepsilon$
  $\mathfrak{A} \Vdash_x d(t_1, t_2) > \varepsilon \Leftrightarrow d_x(t_1^{\mathfrak{A}_x}, t_2^{\mathfrak{A}_x}) > \varepsilon$
  $\mathfrak{A} \Vdash_x R(t_1, \cdots, t_n) < \varepsilon \Leftrightarrow R^{\mathfrak{A}_x}(t_1^{\mathfrak{A}_x}, t_2^{\mathfrak{A}_x}) < \varepsilon$
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- ...

# Pointwise forcing

With Ochoa, we define $\mathfrak{A} \Vdash_x \varphi < \varepsilon$ and $\mathfrak{A} \Vdash_x \varphi > \varepsilon$, for $x \in X$:

- Atomic: $\mathfrak{A} \Vdash_x d(t_1, t_2) < \varepsilon \Leftrightarrow d_x(t_1^{\mathfrak{A}_x}, t_2^{\mathfrak{A}_x}) < \varepsilon$
  $\mathfrak{A} \Vdash_x d(t_1, t_2) > \varepsilon \Leftrightarrow d_x(t_1^{\mathfrak{A}_x}, t_2^{\mathfrak{A}_x}) > \varepsilon$
  $\mathfrak{A} \Vdash_x R(t_1, \cdots, t_n) < \varepsilon \Leftrightarrow R^{\mathfrak{A}_x}(t_1^{\mathfrak{A}_x}, t_2^{\mathfrak{A}_x}) < \varepsilon$
  $\mathfrak{A} \Vdash_x R(t_1, \cdots, t_n) > \varepsilon \Leftrightarrow R^{\mathfrak{A}_x}(t_1^{\mathfrak{A}_x}, t_2^{\mathfrak{A}_x}) > \varepsilon$
- $\mathfrak{A} \Vdash_x \max(\varphi, \psi) < \varepsilon \Leftrightarrow \mathfrak{A} \Vdash_x \varphi < \varepsilon$ and $\mathfrak{A} \Vdash_x \psi < \varepsilon$. Sim. for $>$.
- $\mathfrak{A} \Vdash_x \min(\varphi, \psi) \Leftrightarrow \mathfrak{A} \Vdash_x \varphi$ or $\mathfrak{A} \Vdash_x \psi$. Sim. for $>$.
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- $\mathfrak{A} \Vdash_x \max(\varphi, \psi) < \varepsilon \Leftrightarrow \mathfrak{A} \Vdash_x \varphi < \varepsilon$ and $\mathfrak{A} \Vdash_x \psi < \varepsilon$. Sim. for $>$.
- $\mathfrak{A} \Vdash_x \min(\varphi, \psi) \Leftrightarrow \mathfrak{A} \Vdash_x \varphi$ or $\mathfrak{A} \Vdash_x \psi$. Sim. for $>$.
- $\mathfrak{A} \Vdash_x 1 \dot{-} \varphi < \varepsilon \Leftrightarrow \mathfrak{A} \Vdash_x \varphi > 1 \dot{-} \varepsilon$. Sim. for $>$.
- $\mathfrak{A} \Vdash_x \varphi \dot{-} \psi < \varepsilon$ iff and only if one of the following holds:
  - $\mathfrak{A} \Vdash_x \varphi < \psi$
  - $\mathfrak{A} \nVdash_x \varphi < \psi$ and $\mathfrak{A} \nVdash_x \varphi > \psi$
  - $\mathfrak{A} \Vdash_x \varphi > \psi$ and $\mathfrak{A} \Vdash_x \varphi < \psi + \epsilon$.
- $\mathfrak{A} \Vdash_x \varphi \dot{-} \psi > \varepsilon$ iff $\mathfrak{A} \Vdash_x \varphi > \psi + \varepsilon$
- $\dots$

# Pointwise forcing - continued

Quantifiers:

- $\mathfrak{A} \Vdash_x \inf_{s \in A_x} \varphi(s) < \varepsilon$ iff there exists a section $\sigma$ such that $\mathfrak{A} \Vdash_x \varphi(\sigma) < \varepsilon$.

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- $\mathfrak{A} \Vdash_x \inf_s \varphi(s) > \varepsilon$ iff there exists an open set $U \ni x$ and a real number $\delta_x > 0$ such that for every $y \in U$ and every section $\sigma$ defined on $y$, $\mathfrak{A} \Vdash_y \varphi(\sigma) > \varepsilon + \delta_x$

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- $\mathfrak{A} \Vdash_x \sup_s \varphi(s) < \varepsilon$ iff there exists an open set $U \ni x$ and a real number $\delta_x > 0$ such that for every $y \in U$ and every section $\sigma$ defined on $y$, $\mathfrak{A} \Vdash_y \varphi(\sigma) < \varepsilon - \delta_x$
- $\mathfrak{A} \Vdash_x \inf_{s \in A_x} \varphi(s) > \varepsilon$ iff there exists a section $\sigma$ such that $\mathfrak{A} \Vdash_x \varphi(\sigma) > \varepsilon$.

# Around truth continuity

- We have $\mathfrak{A} \Vdash_x \inf_s(1 \dot{-} \varphi) > 1 \dot{-} \varepsilon$ if and only if $\mathfrak{A} \Vdash_x \sup_s \varphi < \varepsilon$.

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- We can also define $\mathfrak{A} \Vdash_x \varphi(s) \leq \varepsilon$ iff $\mathfrak{A} \nVdash_x \varphi(s) > \varepsilon$ and dually for $\geq$.
- With this, for $0 < \varepsilon' < \varepsilon$, if $\mathfrak{A} \Vdash_x \varphi(s) \leq \varepsilon'$ then $\mathfrak{A} \Vdash_x \varphi(s) < \varepsilon'$

# A metric on sections? (Not yet)

So far so good, but we have (for the time being) lost the metric on the sections (so, the corresponding presheaves $\mathfrak{A}(U)$ are still missing the "metric" feature - they do not live in the correct category yet).

- Sections have different domains
- Triangle inequality is tricky
- Restrict to sections with domains in a filter of open sets
- But the ultralimit (even in that case) could fail to be complete!

# Rather… a pseudometric

Fix $F$ a filter of open sets of $X$. For all sections $\sigma$ and $\mu$ with domain in $F$ define

$$F_{\sigma\mu} = \{U \cap \mathrm{dom}(\sigma) \cap \mathrm{dom}(\mu) | U \in F\}.$$

Then the function

$$\rho_F(\sigma, \mu) = \inf_{U \in F_{\sigma\mu}} \sup_{x \in U} d_x(\sigma(x), \mu(x))$$

is a pseudometric on the set of sections with domain in $F$.

# Completeness of the induced metric

**Theorem (Ochoa)**

*Let $\mathfrak{A}$ be a sheaf of metric structures defined over a regular topological space $X$. Let $F$ be an ultrafilter of regular open sets. Then, the metric induced by $\rho_F$ on $\mathfrak{A}[F]$ is complete.*

# Local Forcing for Metric Structures

Forcing over an open set is somewhat more tricky in this case. We have the following definition.

Definition

Let $\mathfrak{A}$ be a sheaf of metric structures defined on $X$, $\varepsilon > 0$, $U$ open in $X$, $\sigma_1, \cdots, \sigma_n$ sections defined on $U$. Then

- $\mathfrak{A} \Vdash_U \varphi(\sigma) < \varepsilon \iff \exists \delta < \varepsilon \forall x \in U(\mathfrak{A} \Vdash_x \varphi(\sigma) < \delta)$
- $\mathfrak{A} \Vdash_U \varphi(\sigma) > \delta \iff \exists \varepsilon > \delta \forall x \in U(\mathfrak{A} \Vdash_x \varphi(\sigma))$

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There is an involved, equivalent, inductive definition. We also have $\mathfrak{A} \Vdash_U \inf_\sigma(1 \dot{-} \varphi(\sigma) > 1 \dot{-} \varepsilon \iff \mathfrak{A} \Vdash_U \sup_U \varphi(\sigma) < \varepsilon$, and a maximal principal principle (existence of witnesses of sections).

# Metric Generic Model and the Theorem

For the appropriate notion of genericity, we build the generic model as in the discrete case. The definition of genericity guarantees the completeness of $\mathfrak{A}[F]$.

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Theorem (Metric GMTh)

*Let $F$ be a generic filter on $X$, $\mathfrak{A}$ a sheaf of metric structures on $X$ and $\sigma_1, \cdots, \sigma_n$ sections. Then*

1. $\mathfrak{A}[F] \models \varphi([\sigma_1]/\sim_F, \cdots, [\sigma_n]/\sim_F) < \varepsilon \iff \exists U \in F$ *such that* $\mathfrak{A} \Vdash_U \varphi(\sigma_1, \cdots, \sigma_n) < \varepsilon$
2. $\mathfrak{A}[F] \models \varphi([\sigma_1]/\sim_F, \cdots, [\sigma_n]/\sim_F) > \varepsilon \iff \exists U \in F$ *such that* $\mathfrak{A} \Vdash_U \varphi(\sigma_1, \cdots, \sigma_n) > \varepsilon$

# Zilber's Approximation of Structures

Recall

## Definition (Zilber)

Given a structure $M$ in a topological language and structures $M_i$ in the same language, $M$ is approximated by the $M_i$ along an ultrafilter $D$ if for some $M^D \succ \prod_D M_i$ there is a surjective homomorphism

$$\lim : M^D \to M.$$

In the Cuernavaca 2010 Meeting on Physics and Model Theory, the discussion of comparing carefully the two constructions arose. Sheaves of models seem to incorporate better the limit construction (topological), but Zilber's constructions lends itself more quickly to a stability-theoretical analysis.

# Hrushovski: Weil, Shelah - schemes, ?

Hrushovski discusses in his paper on Galois Theory for differential algebra [Hru] the dichotomy between monster models and schemes in classical model theory.

| | | |
|---|---|---|
| Alg Geom | Weil's Univ Dom | Schemes |
| Mod Th | Shelah's Monsters $\mathbb{C}^{eq}$ | Sheaves? |

However, this still is very unbalanced. Why?

# Schemes in Model Theory

Four approaches for building algebraic theories:

1. Proof theoretic: objects are formulas (ideals, equations, etc.)
2. Opposite extreme: taking seriously abstract algebraic structures (fields, rings, etc.) almost to the point of ignoring formulas.
3. Representable functors. "Formulas" no longer viewed syntactically, but as functors taking "structures to solution sets".
4. Universal domains ("monster models"). The functor is replaced by its value in a single structure - the "amalgam of all structures".

Algebra: (1) $\to$ (4) $\to$ (2) $\to$ (4) $\to$ (1)+(3)
Model Theory: (1) $\to$ (2) $\to$ (3) – Shelah introduced (4)

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