Generalized Amalgamation Classes...

... and Limit Models: Implicit Logics

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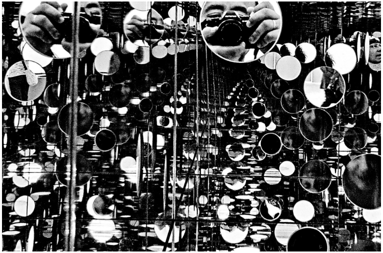

Contents

Classes of Models
- Amalgamation: Fraïssé, Hrushovski
  - Amalgams
- Predimensions and amalgam control / Hrushovski
- Abstract Elementary Classes
- Extracting the Logic: The Presentation Theorem

Coherent classes and limits: sheaves
- Introduction: Why a Model Theory of Sheaves?
- Genericity
- Consequences: Kripke, Forcing, etc.

Metrics / Continuity
- Sheaves of Hilbert Spaces
- Mixtures: Metrical Fibers / Topological Fibers

The Who

- **Samson Abramsky, Adam Brandenburger**: The Sheaf-Theoretic Structure of Non-Locality and Contextuality, New Journal of Physics 13 (2011).
- **Xavier Caicedo**: Lógica de los haces de estructuras, Revista de la Academia Colombiana de Ciencias Exactas, Físicas y Naturales, XIX, no. 74, (1995) 569-585.
- **Angus Macintyre**, Model-completeness for sheaves of structures, Fund. Math. 81 (1973), pp. 73–89. Model Theory: Geometrical and Set-Theoretical Aspects and Prospects, Bull. Symb. Logic, Vol. 9, No. 2 (June 2003), pp. 197-212.
- **Maicol Ochoa, Andrés Villaveces**: Sheaves of Metric Structures.. WOLLIC 2016.
- **Gabriel Padilla, Andrés Villaveces**: Non Standard Cohomology For Equivariant Sheaves: The Role Of Generic Models.

Tomorrow / The Why(s)

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So, stay tuned (tomorrow, 15:30) if you want to listen to an "essay" on coherence, limit structures, with examples and questions.

Today / The How, the What ...

We will wander in this tutorial through the definitions of

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... and through the problem of how the syntax emerges from these classes of models.

The age of a structure

Fix L a countable language, M an L-structure.

Definition

The age of M is the set $\mathcal{K}$ of all finitely generated L-structures of M (mod isomorphism).

This satisfies:

- (HP=Hereditary Property) $A \in \mathcal{K}$, $B \subset A$ fin. gen. $\Rightarrow \exists B \in \mathcal{K}$.
- (JEP=Joint Embedding Property)
  $A, B \in \mathcal{K} \Rightarrow \exists C \in \mathcal{K}, \exists$ embeddings $A \hookrightarrow C, B \hookrightarrow C$.

The age of a structure

Point: reconstruct an L-structure from its finitely generated substructures OR…

The age of a structure

Point: reconstruct an L-structure from its finitely generated substructures OR…

build a new L-structure from some given set of finitely generated structures!

Canonical?

Lemma

Any countable $\mathcal{K}$ consisting of fin. gen. L-structures with HP and JEP is the age of some countable structure.

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!!! No uniqueness: for example, $(\mathbb{Q}, <)$ and $(\omega, \in)$ have the same age

The fix to the lack of uniqueness is **homogeneity**. A countable M is $\aleph_0$-homogeneous iff for any fin. gen. substructures A_1, A_2 of M, any isomorphism $\sigma : A_1 \to A_2$ extends to an automorphism of M.

AP

$\mathcal{K}$ has (AP=the Amalgamation Property) iff for any $A, B_1, B_2 \in \mathcal{K}$ and embeddings $A \hookrightarrow B_i$ we can complete the square

$$\begin{tikzcd}
B_1 & & B_2 \\
& A \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}$$

The Fraïssé Amalgamation Theorem

Theorem

Let L be a countable language and $\mathcal{K}$ a nonempty countable set of fin.gen. L-structures satisfying HP, JEP and AP. Then there is a unique countable homogeneous L-structure M_∞ with age $\mathcal{K}$.

Proof: the uniqueness is by back and forth. For existence, as before, but being more careful in the choice of the order.

Example

The random graph

Refinining the class - different limits

Side remark: axiomatizing M_∞, in general, is difficult.

Example

If we now take $\mathcal{K}$ the set of finite linear orders but **only allow** embeddings ($A \hookrightarrow B$) whenever

$$a, c \in A, b \in B \Rightarrow (B \models a < b < c \rightarrow b \in A).$$

The amalgam is now $(\mathbb{Z}, <)$ instead of $(\mathbb{Q}, <)$.

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Need to specify, in addition to $\mathcal{K}$, the 'allowed' embeddings.

A more general framework - Göbel,Droste

Theorem

In a countable language L, if $\mathcal{K}$ is a class of countable L-structures (with chosen embeddings) such that $\mathcal{K}_0$ (the fin. gen. substructures from $\mathcal{K}$) satisfies

- *$\mathcal{K}$ contains the unions of the chains of $\mathcal{K}_0$*
- *All $A \in \mathcal{K}$ are unions of chains from $\mathcal{K}_0$*
- $|\mathcal{K}_0/\approx| = \aleph_0$

Then $\mathcal{K}$ has a $\mathcal{K}$-universal and $\mathcal{K}_0$-homogeneous object iff $\mathcal{K}_0$ has JEP and AP.

Predimension and Amalgamation

Hrushovski's construction of a new s.m. set can be regarded as arising in two steps:

1. Amalgamation (obtaining a M_∞ from a class)
2. Collapsing (to pull down the Morley rank)

A careful use of predimensions (δ) and multiplicity function (μ) is the key to

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$$(\mathcal{K}, \delta)$$

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EXAMPLE 0 - ACYCLIC GRAPHS = FINITE TREES

Example

Consider graphs (R a symmetric binary relation). Pick $\delta(G, R) = |G| - |R|$.
Let $\mathcal{F} = \{G | \forall X \subset_{fin} G(\delta(X) \geq 1)\}$.
Then $\mathcal{F}$ consists of acyclic graphs.
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Let $\mathcal{F} = \{G | \forall X \subset_{fin} G(\delta(X) \geq 1)\}$.
Then $\mathcal{F}$ consists of acyclic graphs.
This fails AP!!! However, if we choose STRONG embeddings
($G \lhd H$ if $\forall X \subset_{fin} H(\delta(X \cap G) \leq \delta(X)))$.
If we refine to these embeddings, we get (in this easy case) the
“random forest" as the limit M_∞.

AECs: focus on "fitting", "embedding" (ἁρμόττον)

	Concept	Model Theory	Emphasis
Elementary	tò lógos	$M \models \varphi$	Linguistic control
Nonelementary	tò harmótton	$M \prec_{\mathcal{K}} N$	Embedding

Earlier classes included Fraïssé classes, Jónsson classes. Definition of AEC and related properties (AP, Galois type, JEP, tameness). What model theory could be developed with no compactness? With apparently so few tools to grasp meaningful properties?

Change of Focus: from formulas to embeddings

φ, T, …	
formulas, theories	

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$\varphi, T, \ldots$	$\prec_K$
formulas, theories	embeddings, encasings …

A World of Pure Phenomena...

... without precise descriptions, apparently, but with a strong notion of how pieces fit within one another—ἁρμόττον

The name for that in contemporary model theory is "strong extension" $M \prec_{\mathcal{K}} N$.

Roughly: all small configurations/problems from M that have a solution in N also have another solution in M.

Abstract Elementary Classes

The move from control of a class of structures by the logic (satisfaction of formulas—tò lógos) to "strong embeddings" between models (tò harmótton—encasing[1]) is strongly in the direction of formalism freeness.

[1] The connection between these concepts and model theory will be further explored in forthcoming work with J. Kennedy. See Patočka for a discussion on the emergence and role of tò harmótton in Greek Aesthetics.

THE DEFINITION (PART 1)

Definition (Abstract Elementary Class)

Fix a language L. A class $\mathcal{K}$ of L-structures, together with a binary relation $\prec_{\mathcal{K}}$ on $\mathcal{K}$ is an abstract elementary class (for short, AEC) if:

1. Both $\mathcal{K}$ and $\prec_{\mathcal{K}}$ are closed under isomorphism. This means two things: first, if $M' \approx M \in \mathcal{K}$ then $M' \in \mathcal{K}$; second, if M', N' are L-structures with $M' \subset N'$, $M' \approx M$, $N' \approx N$ and $M \prec_{\mathcal{K}} N$ then $M' \prec_{\mathcal{K}} N'$.
2. If $M, N \in \mathcal{K}$, $M \prec_{\mathcal{K}} N$ then $M \subset N$,
3. $\prec_{\mathcal{K}}$ is a partial order,
4. (**Coherence**) If $M \subset N \prec_{\mathcal{K}} N'$ and $M \prec_{\mathcal{K}} N'$ then $M \prec_{\mathcal{K}} N$,

THE DEFINITION (PART 2)

Definition

1. (**LS**) There is a cardinal (called "the Löwenheim-Skolem number" of the class) $\kappa = LS(\mathcal{K}) \geq \aleph_0$ such that if $M \in \mathcal{K}$ and $A \subset |M|$, then there is $N \prec_{\mathcal{K}} M$ with $A \subset |N|$ and $|N| \leq |A| + LS(\mathcal{K})$,
2. (**Unions of $\prec_{\mathcal{K}}$-chains**) If $(M_i)_{i<\delta}$ is a $\prec_{\mathcal{K}}$-increasing chain of length δ (δ a limit ordinal), then
   - $\bigcup_{i<\delta}(M_i)_{i<\delta} \in \mathcal{K}$,
   - for each $j < \delta$, $M_j \prec_{\mathcal{K}} \bigcup_{i<\delta} M_i$,
   - if for each $i < \delta$, $M_i \prec_{\mathcal{K}} N \in \mathcal{K}$ then $\bigcup_{i<\delta} M_i \prec_{\mathcal{K}} N$ (**smoothness**)

At this point, we have the following situation:

- So far, no control on possible axiomatization of the class $\mathcal{K}$. The emphasis is placed on its being closed under the constructions specified in the axioms. However, there is logical control of these classes.
- These are not necessarily amalgamation classes: there is no amalgamation axiom. However, many AECs do satisfy the amalgamation property. Furthermore, the model theory will depend on the kind of amalgamation possible in the class.

Extracting the Logic: The Presentation Theorem

Although the definition places no emphasis whatsoever on formulas or theories, there is a general Presentation Theorem that enables in some cases to extract properties from first order logic, or from various infinitary logics, and translate them into properties of the class $\mathcal{K}$.

Extracting the Logic: The Presentation Theorem

Theorem

Let $(\mathcal{K}, \prec_{\mathcal{K}})$ be an AEC in a language L. Then there exist

- *A language $L' \supset L$, with size $LS(\mathcal{K})$,*
- *a (first order) theory T' in L' and*
- *a set of T'-types, Γ', such that*

$$\mathcal{K} = PC(L, T', \Gamma') := \{M' \upharpoonright L \mid M' \models T', M' \text{ omits } \Gamma'\}.$$

Moreover, if $M', N' \models T'$, they both omit Γ', $M = M' \upharpoonright L$ and $N = N' \upharpoonright L$, then

$$M' \subset N' \Leftrightarrow M \prec_{\mathcal{K}} N.$$

Extended Objects / Variable Objects

Objects in the world present themselves as extended in time (or in other classical (or non-classical) "categories"):

- Physical objects, individuals, etc.

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- Particles, even neutrinos (for some particles, order of 10^{-20} seconds, yet still "time")
- Concepts? Thoughts? Ideas? Visualizations? Perceptions?

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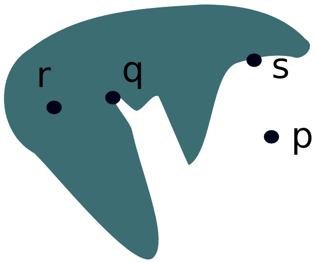
Yet logic (at the limit) is "too rough"

(Really, classical logic.)



- For p and r the predicate "is in the green zone" is clear - classical logic "agrees" with perception.

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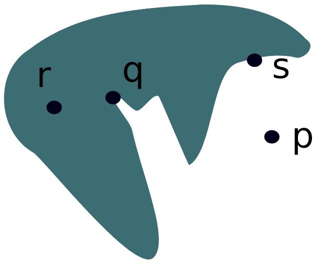

- For p and r the predicate "is in the green zone" is clear - classical logic "agrees" with perception.
- For q and s (at "limit situations") classical logic forces one to make a decision (open, closed green zone, etc.).

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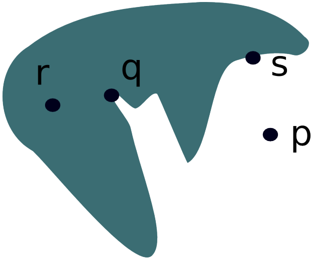

- Perception does not follow classical logic.

Physics, geometry, and "limit" phenomena

As we know since the late 1920's, Physics (wave models, quantum phenomena of "undecidability" or "uncertainty", noncommutativity of operators corresponding to formalizations of observability, etc.) has the kind of "limit phenomena" that may call for a logic of variable entities.

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Algebraic geometry of the postwar period (Leray, Cartan, Weil, and then Grothendieck reflects this same "shift of perspective": sheaves, sites, topoi.)

Instant velocity / Paradigm change

Instant velocity has exactly the same behavior as “the color of point”: it really is an abstraction of a property of neighborhoods.
Excluded middle may be dropped!
The strong paradigm becomes Truth Continuity.

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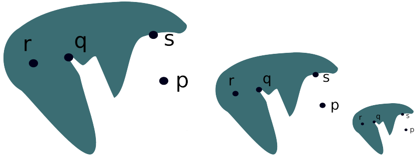
Excluded middle ~~may~~ HAS to be dropped!
The strong paradigm becomes Truth Continuity.

Truth Continuity

If an individual (an entity, a particle, etc.) has some property on some point of its domain of extension, there has to be a neighborhood of this point in this domain in which this property holds of all points.



Sheaves over topological spaces

Fix X a topological space. The pair (E, p) is a sheaf over X if and only if E is a topological space and $p : E \to X$ is a surjective local homeomorphism.

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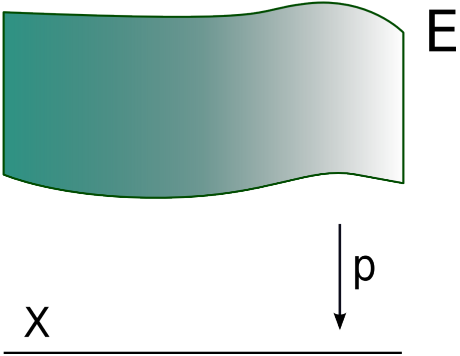

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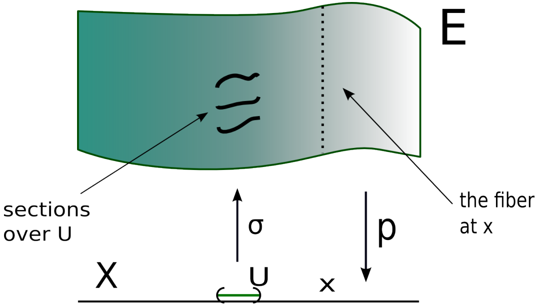
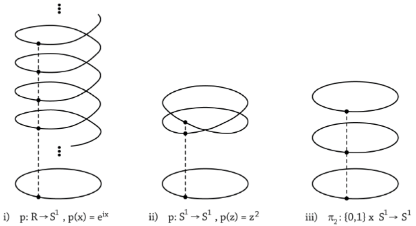
- The (images of) sections σ form a basis for the topology of E (a section is a continuous partial inverse of p defined on an open set $U \subset X$),
- If two sections σ, τ coincide at a point a then there exists an open set $U \ni a$ such that $\sigma \restriction U = \tau \restriction U$

Sections - objects



Classical and not



(from Caicedo's monograph)

Cildo Meireles - Fontes

A little history

Sheaves over topological spaces go back to **H. Weyl** (1913), in his work on Riemann surfaces.
They "reappear" strongly in **Cartan**'s seminar (1948-1952) and then catch flight with the French Algebraic Geometry School of the Postwar (**Serre**, **Leray**, etc.).
Weil: Séminaire de géométrie algébrique: study of the zeta function on finite fields.
Finally, **Grothendieck** generalizes further the frame (to sites = small categories endowed with "Grothendieck topologies"). **Deligne** then proves Weil's conjectures.

Sheaves of Structures

A sheaf of structures $\mathfrak{A}$ over X consists of:

1. A sheaf (E, p) over X,
2. On every fiber $p^{-1}(a)$ ($a \in X$), a structure

$$\mathfrak{A}_a = (E_a, (R_i^a)_i, (f_j^a)_j, (c_k^a)_k,)$$

   such that $E_a = p^{-1}(a)$, and

   - For every i, $R_i^{\mathfrak{A}} = \bigcup_{x \in X} R_i^{\mathfrak{A}_x}$ is open
   - For every j, $f_j^{\mathfrak{A}} = \bigcup_{x \in X} f_j^{\mathfrak{A}_x}$ is continuous
   - For every k, $c_k^{\mathfrak{A}} : X \to E$ such that $x \mapsto c_k^{\mathfrak{A}_x}$ is a continuous global section

Truth Continuity?

Fact

For all atomic formulas $\varphi(v)$ we have that

$$\mathfrak{A}_x \models \varphi(\sigma(x)) \textit{ iff } \exists U \ni x \forall y \in U \Big(\mathfrak{A}_y \models \varphi(\sigma(y)) \Big)$$

This also holds for positive Boolean combinations of atomic formulas.

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However, this fails for negations!
The solution to this failure is to switch to an emphasis on forcing.

Satisfaction and forcing (pointwise and local)

Three notions: satisfaction at each fiber, forcing at a point $x \in X$, forcing at a (non-empty) open set $U \subset X$:

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How do we compare them? Before diving into the definitions of the forcing notions, notice that the first one is <u>pointwise</u> while the second one is <u>local</u>. Also notice that satisfaction in $\mathfrak{A}_x$ is about <u>values</u> of sections at x (the $\sigma(x)$) whereas pointwise (over x) or local forcing (over U) are about the <u>whole</u> section σ defined on U.

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Sections are the new objects: formulas $\varphi(v_1, v_2, \cdots)$ will be "evaluated" by "replacing" v_i by a section σ_i or by its value at an element x of X, $\sigma_i(x)$.

Pointwise forcing

- For atomic φ and $t_1, \cdots, t_n$ terms,
  $\mathfrak{A} \Vdash_x (t_1 = t_2)[\vec{\sigma}] \Leftrightarrow t_1^{\mathfrak{A}_x}[\vec{\sigma}(x)] = t_2^{\mathfrak{A}_x}[\vec{\sigma}(x)]$
  similarly for relation symbols.

Forcing $\neg, \rightarrow, \forall$ at x requires information "around" x. It is an exercise to check Truth Continuity for $\Vdash_x$.

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Truth continuity - II

A semantics can also be defined *directly* over open sets:

$$\mathfrak{A} \Vdash_U \varphi[\sigma],$$

where U is an open set in the domain of σ.

Definition

$\mathfrak{A} \Vdash_U \varphi[\sigma]$ if and only if for every $x \in U$, $\mathfrak{A} \Vdash_x \varphi[\sigma(x)]$.

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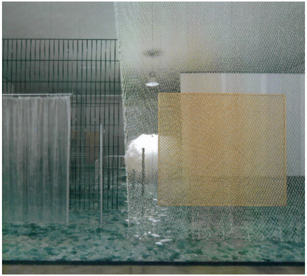
Pointwise versus local



Sheaves		Pointwise = Local
Presheaves		They may differheight

In a way, $\Vdash_U$ is more direct than $\Vdash_x$

In some steps ($\neg, \rightarrow, \forall$) of the definition of $\Vdash_x$ one needs to have access "from x" to information about forcing "around x" - this guarantees in the end the Truth Continuity paradigm.

In the definition of $\Vdash_U$, the nontrivial steps require knowledge of fibers defined over <u>sub</u>open sets of U. As it stands, this is non-trivial knowledge. Notice also that forcing a disjunction of two formulas "spreads" the forcing of each formula to one portion of U - the only requirement being that this may be done while still covering U.

Generic Model Theorem

The Generic Model Theorem is the version of the (Model Theoretic) Forcing Theorem for this notion. Caicedo generalized the Macintyre version to sheaves of arbitrary First Order structures. Further generalizations (adaptations) are due to Caicedo, Ochoa and V. (later!).

Generic Filters

Definition

Given $\mathfrak{A}$ a sheaf of structures over X, a generic filter $\mathbb{F}$ for $\mathfrak{A}$ is a filter of open sets of X such that

- for every $\varphi(\sigma)$ and every σ defined on $U \in \mathbb{F}$, there is some $W \in \mathbb{F}$ such that $\mathfrak{A} \Vdash_W \varphi(\sigma)$ or $\mathfrak{A} \Vdash_W \neg\varphi(\sigma)$
- for every σ defined on $U \in \mathbb{F}$, for every $\varphi(u, \sigma)$, if $\mathfrak{A} \Vdash_U \exists u \varphi(u, \sigma)$, then there exists $W \in \mathbb{F}$ and μ defined on W such that $\mathfrak{A} \Vdash_W \varphi(\mu, \sigma)$

For some topological spaces, this definition of genericity of a filter may be made more purely topological/geometrical (and less dependent on formulas and forcing). However, in the general case, this is not necessarily possible - and we must rely on this logical definition.

Existence - generic models

Fact

Generic filters exist.

Definition (Generic Models)

Given a generic filter $\mathbb{F}$ and $\mathfrak{A}(U) = \{\sigma | \mathrm{dom}(\sigma) = U\}$, let

$$\mathfrak{A}[\mathbb{F}] = \lim_{U \in \mathbb{F}} \mathfrak{A}(U) = \bigsqcup_{U \in \mathbb{F}} \mathfrak{A}(U) / \sim_{\mathbb{F}}$$

where $\sigma \sim_{\mathbb{F}} \mu$ iff there exists $W \in \mathbb{F}$ such that $\sigma \restriction W = \mu \restriction W$. Also,

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- $(\sigma_1 / \sim_{\mathbb{F}}, \dots, \sigma_n / \sim_{\mathbb{F}}) \in R^{\mathfrak{A}[\mathbb{F}]} \Leftrightarrow \exists U \in \mathbb{F} (\sigma_1, \dots, \sigma_n) \in R^{\mathfrak{A}(U)}$
- $f^{\mathfrak{A}[\mathbb{F}]}(\sigma_1 / \sim_{\mathbb{F}}, \dots, \sigma_n / \sim_{\mathbb{F}}) = f^{\mathfrak{A}(U)}(\sigma_1, \dots, \sigma_n) / \sim_{\mathbb{F}}$

Finally, the theorem...

Theorem (Generic Model Theorem)

Let $\mathbb{F}$ be a generic filter for a sheaf of topological structures $\mathfrak{A}$ over X. Then

$$\begin{aligned}\mathfrak{A}[\mathbb{F}] \models \varphi(\sigma/\sim_{\mathbb{F}}) \quad &\iff \quad \{x \in X | \mathfrak{A} \Vdash_x \varphi^G(\sigma(x))\} \in \mathbb{F} \\ &\iff \quad \exists U \in \mathbb{F} \textit{ such that } \mathfrak{A} \Vdash_U \varphi^G(\sigma).\end{aligned}$$

Here, φ^G is a formula equivalent classically to φ, but not necessarily in an intuitionistic framework! (The formula φ^G is sometimes called the Gödel translation of φ - in 1925, Kolmogorov had independently defined an equivalent translation.)

Caicedo's Theorem

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More on the Generic Model Theorem

Cohen's construction of generic models for set theory is the first published result along these lines. Later, Robinson, Barwise and Keisler used generic model theorems to get Omitting Types Theorems in various logics, generalized by Caicedo. Ellerman's "ultrastalk theorem" (1976) is a GMTh for maximal filters. Miraglia also proves a similar result for Heyting-valued models.

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$$\sigma \mapsto \sigma^* = \sigma \cup \{(\infty, [\sigma]_{\sim_{\mathbb{F}}})\}.$$

The fiber "at infinity"

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$$\sigma \mapsto \sigma^* = \sigma \cup \{(\infty, [\sigma]_{\sim_{\mathbb{F}}})\}.$$

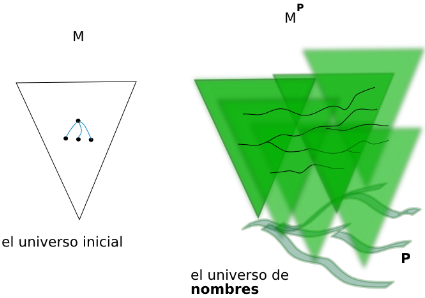
Then, the GMTh just means that in the new sheaf $\mathfrak{A}^\infty$ this fiber is classic:

$$\mathfrak{A}^\infty \Vdash_\infty \varphi(\sigma_1^*, \cdots, \sigma_n^*) \Leftrightarrow \mathfrak{A}[\mathbb{F}] \models \varphi([\sigma_1^*], \cdots, [\sigma_n^*])$$

Łoś as a first consequence

The Łoś theorem is clearly a special case of the Generic Model Theorem, corresponding to endowing X with the discrete topology. Therefore, the Model Theory of sheaves has a twisted form of compactness - of course relative to a context with no excluded middle.

The forcing theorem



The forcing theorem of Set Theory is another special case: take a partially ordered set $\mathbb{P}$, endowed with the order topology (basic open sets are downward closed sets). The Generic Model Theorem provides a model of set theory, where satisfaction is given by forcing on points. BUT in this kind of topological spaces, forcing over an open set is reducible to forcing over a point.

In general... (?)

Most topological spaces, however, do not arise from partially ordered sets. A natural question (fairly unexplored) is what other kinds of models of set theory may be obtained by forcing with such topological spaces.

Other applications of the GMTh

- Kripke models - generalized semantics
- Set-theoretic forcing
- Robinson's Joint Consistency Theorem (=Amalgamation over Models)
- Various Omitting Types Theorems (Caicedo, Brunner-Miraglia)
- Control over new kinds of limit models

Sheaves of Hilbert Spaces

Why?

1. Hilbert Spaces are (still) a crucial tool for formalization of concepts and objects in Physics and in Chemistry
2. In Physics: really algebras of operators acting on Hilbert spaces.
3. In Chemistry: really predicates on Hilbert spaces.
4. In both, the dynamical properties of evolution of a system are relevant.

The problem of a model theory for continuous (metric) structures

So, we want to be able to put Hilbert spaces (and more structure on top of them, such as predicates for reactions, or operators for observables) on fibers.
We could in principle do that as we have seen so far, but immediately we get the problem that we may get lots of non-standard Hilbert spaces (infinitesimals, etc.).
Moreover, we want the logic to "keep track" of (say) the distance to a projection $p(v)$, the convergence of a sequence in H, isometric isomorphism, $(1+\varepsilon)$-isomorphism, etc. etc.
Finally, we need to be able to take limits of Cauchy sequences at will in our structures: metric completeness is crucial.

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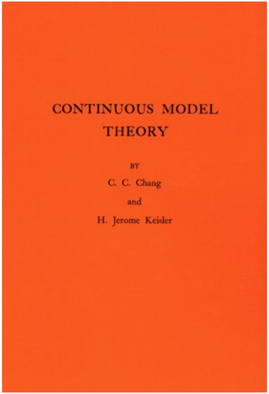
That is the rôle of Continuous Model Theory.

Continuous Model Theory - Origins



Although the origins of CMTh go back to Chang & Keisler (1966), and in some (restricted) ways to von Neumann's Continuous Geometryrecent takes on Continuous Model Theory are based on formulations due to Ben Yaacov, Usvyatsov and Berenstein of Henson and Iovino's Logic for Banach Spaces.

Continuous predicates and functions

Definition

Fix (M, d) a bounded metric space. A continuous n-ary predicate is a uniformly continuous function

$$P : M^n \to [0, 1].$$

A continuous n-ary function is a uniformly continuous function

$$f : M^n \to M.$$

Metric structures

Therefore, metric structures are of the form

$$\mathcal{M} = \left(M, d, (f_i)_{i \in I}, (R_j)_{j \in \mathcal{J}}, (a_k)_{k \in K}\right)$$

Each function, relation must be endowed with a modulus of uniform continuity.

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$$\mathcal{M} = \left(M, d, (f_i)_{i \in I}, (R_j)_{j \in \mathcal{J}}, (a_k)_{k \in K}\right)$$

where the R_i and the f_j are (uniformly) continuous functions with values in $[0, 1]$, the a_k are distinguished elements of M.
Remember: M is a bounded metric space.
Each function, relation must be endowed with a modulus of uniform continuity.

Examples of FO metric structures

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- Any FO structure, endowed with the discrete metric.

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- For a probability space $(\Omega, \mathcal{B}, \mu)$, construct a metric structure $\mathcal{M}$ based on the usual measure algebra of $(\Omega, \mathcal{B}, \mu)$.
- Representations of C^*-algebras (Argoty, Berenstein, Ben Yaacov, V.).
- Valued fields.

The syntax

1. Terms: as usual.
2. Atomic formulas: $d(t_1, t_n)$ and $R(t_1, \cdots, t_n)$, if the t_i are terms. Formulas are then interpreted as functions into $[0, 1]$.
3. Connectives: continuous functions from $[0, 1]^n \to [0, 1]$. Therefore, applying connectives to formulas gives new formulas.
4. Quantifiers: $\sup_x \varphi(x)$ (universal) and $\inf_x \varphi(x)$ (existential).

Interpretation

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Notice also that the set of connectives is too large, but it may be "densely" and uniformly generated by $0, 1, x/2, \dot{-}$: for every ε, for every connective $f(t_1, \cdots, t_n)$ there exists a connective $g(t_1, \cdots, t_n)$ generated by these four by composition such that $|f(\vec{t}) - g(\vec{t})| < \varepsilon$.

Stability Theory

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- ω-stability,
- Dependent theories (Ben Yaacov),

- Keisler measures, NIP (Hrushovski, Pillay, etc.).

Stability Theory

- Stability (Ben Yaacov, Iovino, etc.),
- Categoricity for countable languages (Ben Yaacov),
- ω-stability,
- Dependent theories (Ben Yaacov),
- Not much geometric stability theory: no analog to Baldwin-Lachlan (no minimality, except some openings by Usvyatsov and Shelah in the context of $\aleph_1$-categorical Banach spaces),
- NO simplicity!!! (Berenstein, Hyttinen, V.),
- Keisler measures, NIP (Hrushovski, Pillay, etc.).

"Continuous Model Theory" beyond First Order

Several contexts, some unexplored so far.

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3. Sheaves of (metric) structures. Our work with Ochoa, motivated by problems originally in Chemistry. NEXT!

Tomorrow: Metric Sheaves - the Why?

We'll start by their construction (quickly) and will focus on

1. Limit Models (Approximation Structures in Physics - Zilber and independently Hirvonen-Hyttinen)
2. Hilbert Spaces are (still) a crucial tool for formalization of concepts in Physics and in Chemistry
3. In Physics: really algebras of operators acting on Hilbert spaces. (Zilber, Cruz, etc.)
4. In Chemistry: really predicates on Hilbert spaces. (Ochoa, V., forthcoming).
5. In Arithmetic Geometry: sheaves for j-mappings (V., Zilber)
6. Abramsky Sheaves for Entanglement and Kochen-Specker.

Thanks, and a question for you

¡Muchas gracias por su atención!

Where else?