Modular Invariants and Model Theory

Some recent interactions

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Contents

Contents

Contents

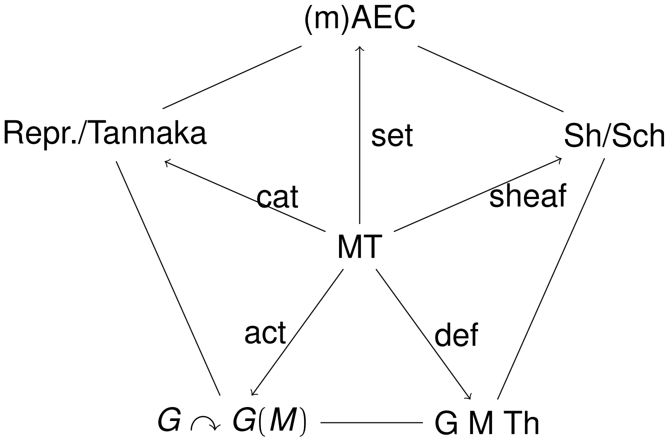
Analogies, geometry, model theory

Aussi nous savons, nous, ce que cerchait à deviner Lagrange, quand il parlait de métaphysique à propos de ses travaux d’algèbre; c’est la théorie de Galois, qu’il touche presque du doigt, à travers un écran qu’il n’arrive pas à percer. Là où Lagrange voyait des analogies, nous voyons des théorèmes. Mais ceux-ci ne peuvent s’énoncer qu’au moyen de notions et de “structures” qui pour Lagrange n’étaient pas encore des objets mathématiques...

(André Weil in De la métaphysique à la mathématique (1960). Quoted by Yves André en Ambiguity Theory, Old and New. Bollettino U.M.I. 2008)

Model Theory, among various areas of Mathematics:

We start with a “map” of Model Theory, at the crossroads of various different areas (or disciplines) in Mathematics:

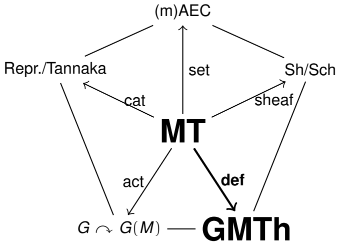


The five directions

The five directions capture apparently very distant aspects that in recent (or older) times connect to Model Theory: set theoretic (AEC, mAEC), sheaf theoretic, internal definability geometry, actions of groups, generalizations of representation theory to categorical (geometric) formalisms.

The five directions are thus an attempt to build a framework to go beyond the classical dichotomy SET vs CAT (or the dichotomy between a set-theoretic approach vs a geometric one). Model Theory is indeed becoming more geometric, as Macintyre claims, but in ways that go beyond his own perspective and beyond the dichotomy.

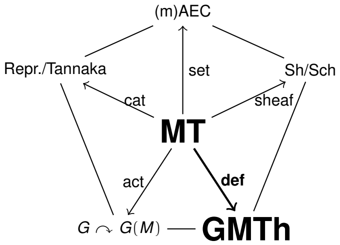
Direct Image Theorems (classical)



Direct Image Theorems: These include the following classical constructions and are at the heart of the interaction between Model Theory (**MT**) and Geometric Model Theory (**GMTh**):

1. Tarski-Chevalley: projections of constructible sets in ACF are constructible (QE).
2. Tarski: same for RCF.
3. QE for *p*-adic analytic structures, Witt vectors, etc.
4. Valued fields (Henselian) and QE.
5. Ax-Kochen-Ersov: For every positive integer d, for cofinitely many primes p, every p-adic homogeneous polynomial of degree d in at least $d^2 + 1$ variables has a nontrivial zero.

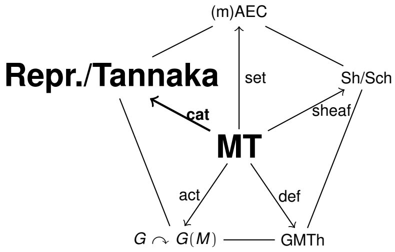
Deeper connections



Deeper connections between Model Theory (**MT**) and Geometric Model Theory (**GMTh**):

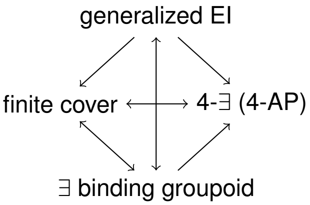
1. Mordell-Lang: proof by Hrushovski in characteristic $p > 0$, using characterizations of one-basedness,
2. André-Oort: proof by Scanlon and Medvedev - use of Zariski Geometries (Hrushovski-Zilber) + analysis of modular definable sets.

Homotopy/homology

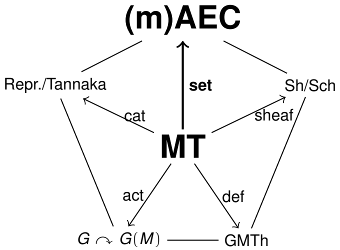


Towards "representation theory"

1. Kamensky: Tannaka formalism in a category theory: Galois theoretical approach to model theory.
2. "Homotopy theory in Model Theory": Goodrick, Kim, Kolesnikov push Hrushovski's theorem into a homotopy of types (under ω-stability, NDOP, etc.)
3. Finite Covers



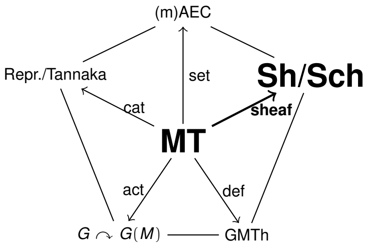
AEC



Model Theory of metric AEC: generalizing Model Theory (**MT**) to non-elementary classes, even in metric contexts (**(m)AEC**):

1. Shelah-Usvyatsov: versions of categoricity transfer: frameworks for Banach space theory, etc.
2. Hirvonen-Hyttinen: categoricity transfer for finitary metric AECs.
3. Grossberg-Lessmann-VanDieren: transfer of categoricity under tameness.
4. Grossberg-VanDieren-V.: study of variants of superstability for AEC,
5. V.-Zambrano: generalization of superstability to metric AEC.

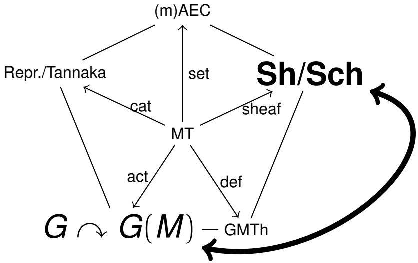
↬ sheaves



Connections between Model Theory (**MT**), Sheaves and Geometry (**Sh/Sch**):

1. Macintyre: uses ideas of Comer (based on Feferman-Vaught and Grothendieck) to show that $Th(\mathrm{Comm.Rings})$ has a model companion: first "transfer to limit" principles ... "extend metamathematical results on fields to the corresponding result for certain regular rings" ... Macintyre (1973)
2. Ellerman: Ultrastalk Theorem (a forcing theorem for regular topological spaces - 1974)
3. Caicedo: the Generic Model Theorem - a generalization for arbitrary topological spaces.

↬ *equivariant* sheaves

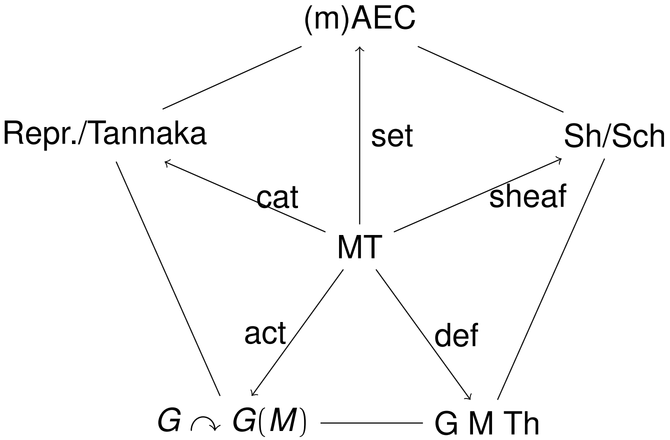


Sheaves, equivariant sheaves generalizing the model theory on sheaves to group actions and to stability theory:

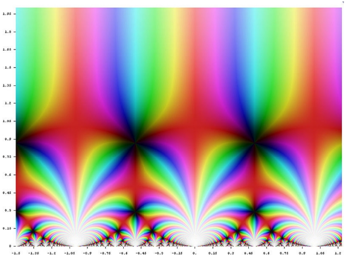
1. Ochoa, V.: Generalizations of model theory on sheaves to metric structures. A metric version of the generic model theorem.
2. Padilla, V.: G-structures (when $G \curvearrowright G(M)$ is a group acting on a structure - equivariance - lifted to coherent and exact actions on G-sheaves (presheaves of G-structures that satisfy G-coherence and G-exactness conditions). Generic Model Theorem).
3. Padilla, V., Zambrano: stability theory of sheaves and G-sheaves.

Model Theory, again

The *j* invariant provides a fantastic test-example for these crisscrossing patterns in Model Theory and its



Classical j invariant



j-invariant on $\mathbb{C}$ (Wikipedia article on j-invariant)

Klein defines the function (we call) “classical j”

$$j : \mathbb{H} \to \mathbb{C}$$

(where $\mathbb{H}$ is the complex upper half-plane)
through the explicit rational formula

$$j(\tau) = 12^3 \cdot \frac{g_2(\tau)^3}{g_2(\tau)^3 - 27 g_3(\tau)^3}$$

with g_2 and g_3 certain rational functions (“of Eisenstein”).

Basic facts about classical j

The function j is a modular invariant of elliptic curves (and classical tori).

- j is analytic, except at ∞
-
$$j(\tau) = j\left(\frac{a\tau + b}{c\tau + d}\right)$$
if $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$.

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More basic facts

The following are equivalent:

1. There exists $s \in SL_2(\mathbb{Z})$ such that $s(\tau) = \tau'$,
2. $\mathbb{T}_\tau \approx \mathbb{T}_{\tau'}$ (elliptic curves — classical tori — isomorphic as Riemann surfaces)
3. $j(\tau) = j(\tau')$...

where $\mathbb{T}_\tau := \mathbb{C}/\Lambda$, and $\Lambda \leq \mathbb{C}$ is a lattice.

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Moreover...

Classical j is therefore an invariant of tori (=elliptic curves) with the following additional advantages:

- It has an explicit formula
- It is a "modular function" of $SL_2(\mathbb{Z})$ - invariant under the action of that group (it captures "isogeny")
- It is associated to the study of the endomorphism group $End(\mathbb{E})$, for an elliptic curve $\mathbb{E}$.
- (Schneider, 1937): if τ is a quadratic irrationality then $j(\tau)$ is algebraic of degree $h_{f,K}$.
- if $e^{2\pi i\tau}$ is algebraic then $j(\tau), \frac{j'(\tau)}{\pi}, \frac{j''(\tau)}{\pi^2}$ are mutually transcendental.

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From $\mathbb{C}$ to $\mathbb{R}$

Or consider Yu. Manin's Altertraum: find the analogue of Complex Multiplication for $\theta \in \mathbb{Q}(\sqrt{D})$ for $D > 0$ square free, by replacing elliptic curves by quantum tori.
[The term "complex multiplication" refers (for an elliptic curve) to having Endomorphism group larger than $\mathbb{Z}$. It can be reduced to having, for every $\mu \in \mathbb{Q}(\sqrt{D})$ (now for $D < 0$ and square free), that $j(\mu)$ is an algebraic integer and generates the classfield $H(\mu)$ of $\mathbb{Q}(\sqrt{D})$.]

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Toward quantum tori: from $\mathbb{C}$ to $\mathbb{R}$

Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$, and let Λ_θ be the pseudo-lattice $\langle 1, \theta \rangle$. The quotient

$$\mathbb{R}/\Lambda_\theta$$

is for many good purposes our "quantum torus", associated to the irrational number θ. It is a one-parameter subgroup of the (classical) torus $\mathbb{T}(i)$.

There are long, detailed descriptions in terms of leaf spaces corresponding to "Kronecker" foliations, etc.

Getting hold of quantum versions of j

Gendron proposes ways of dealing with generalized j-invariants, addressing problems such as

- New definition domain (from $\mathbb{H}$ to $\mathbb{R} \setminus \mathbb{Q}$)
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Castaño-Bernard and Gendron's definition.

Let $\theta \in \mathbb{R}$. The quantum modular invariant $j^{qu}(\theta)$ is a discontinuous, multivalued, analogue of the classical modular invariant:

Let $\Lambda_\varepsilon(\theta) := \{n \in \mathbb{N} \mid \|n\theta\| < \varepsilon\}$, where $\|\cdot\|$ measures the distance to the nearest integer.

This is the "quantum lattice".

The ε-zeta function of θ is given by

$$\zeta_{\theta,\varepsilon}(s) := \sum_{n \in \Lambda_\varepsilon(\theta)} n^{-s}.$$

(The value $2\zeta_{\theta,\varepsilon}(2k)$ is the analogue to the classical Eisenstein series of weight k.) The ε-modular invariant is given by

$$j_\varepsilon(\theta) := \frac{12^3}{1 - J_\varepsilon(\theta)}, \qquad J_\varepsilon(\theta) := \frac{49}{40}\frac{\zeta_{\theta,\varepsilon}(6)^2}{\zeta_{\theta,\varepsilon}(4)^3}.$$

The set of limit points as $\varepsilon \longrightarrow 0$ is

$$j^{qu}(\theta) := \lim_{\varepsilon \longrightarrow 0} j_\varepsilon(\theta) :$$

a $GL_2(\mathbb{Z})$-invariant discontinuous and multivalued function

$$j^{qu} : \mathbb{R} \multimap \mathbb{R} \cup \{\infty\}.$$

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Spectra and non-commutative geometry

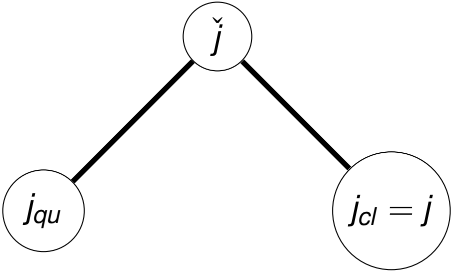
One can regard

$$j^{qu} : \mathbb{R} \multimap \mathbb{R} \cup \{\infty\}$$

as a spectrum of some (yet unknown to us) operator.
However, we follow a path different from more usual ones in non-commutative geometry.

An example of a sheaf construction / universal j

Gendron proposes a detailed construction of a sheaf over a topological space, and a generalization of classical j called "universal j-invariant" - a specific section of a sheaf.



The specific construction of universal *j*

(Castaño-Bernard, Gendron)
Let ${}^*\mathbb{Z} := \mathbb{Z}^{\mathbb{N}}/\mathfrak{u}$ for some nonprincipal ultrafilter $\mathfrak{u}$ on $\mathbb{N}$. Define

$$H := \{[F_i] \subset {}^*\mathbb{Z}^2 \text{ hyperfinite } \}.$$

This set is partially ordered with respect to inclusion so we may consider the Stone space

$$R := Ult(H).$$

For each $\mathfrak{p} \in R$ and $\mu \in \mathbb{H}$ one may define the *j*-invariant

$$j(\mu, \mathfrak{p})$$

as follows:

The construction

The idea: the classical j-invariant is an algebraic expression involving Eisenstein series which is a function of $\mu \in \mathbb{H}$. We can associate to $[F_i] \subset {}^*\mathbb{Z}^2$ a hyperfinite sum modelled on the formula of the classical j-invariant, denoted

$$j(\mu)_{[F_i]} \in {}^*\mathbb{C}.$$

We get a net

$$\{j(\mu)_{[F_i]}\}_{[F_i]\in H} \subset {}^*\mathbb{C}.$$

Consider the sheaf ${}^\diamond\check{\mathbb{C}} \to R$ for which the stalk over $\mathfrak{p}$ is

$${}^\diamond\mathbb{C}_{\mathfrak{p}} := ({}^*\mathbb{C})^H/\mathfrak{p}.$$

Then we may define a section:

$$\check{j} : \mathbb{H} \times R \hookrightarrow {}^\diamond\check{\mathbb{C}}, \quad \check{j}(\mu, \mathfrak{p}) := \{j(\mu)_{[F_i]}\}_{[F_i]\in H}/\mathfrak{p}.$$

Group actions - choosing an irrational angle

What really is at stake in these constructions is the invariance under various group actions.
For each $\theta \in \mathbb{R}$ there is a distinguished subset $R_\theta \subset R$ of ultrafilters which "see" θ:

$$R_\theta = \{\mathfrak{p} |\ \mathfrak{p} \supset \mathfrak{c}_\theta\}$$

where $\mathfrak{c}_\theta$ is the cone filter generated by the cones

$$cone_\theta([F_i]) = \{[F_i]' \supset [F_i] |\ [F_i] \subset [F_i]' \subset {}^*\mathbb{Z}^2(\theta)\}.$$

In the above,

$${}^*\mathbb{Z}^2(\theta) = \{({}^*n^\perp, {}^*n) |\ {}^*n\theta - {}^*n^\perp \simeq 0\}.$$

Restricting to quantum and classical *j*

The quantum *j*-invariant is defined as the restriction:

$$\check{j}^{qu}(\theta) := \check{j}|_{R_\theta}(i, \cdot).$$

If we denote

$$R_{cl} = \{\mathfrak{p} |\ \mathfrak{p} \supset \mathfrak{c}\}$$

where $\mathfrak{c}$ is the filter generated by *all* cones over hyperfinite sets in ${}^*\mathbb{Z}^2$:

$$cone([F_i]) = \{[F_i]' \supset [F_i] |\ [F_i]' \subset {}^*\mathbb{Z}^2\}.$$

Then the restriction

$$\check{j}^{cl} := \check{j}|_{R_{cl}}$$

satisfies

$$\check{j}^{cl}(\mu, \mathfrak{p}) \simeq j(\mu), \quad \forall \mu \in \mathbb{H},$$

where *j* is the usual *j*-invariant.

Duality I

Note the duality in the way of recovering the classical and quantum invariants:

- the classical invariant is recovered along a unique fiber ${}^{\diamond}\check{\mathbb{H}}_{\mathfrak{u}}$ (i.e., a <u>leaf</u> of the quotient of sheaves $\widehat{Mod}$),
- the quantum invariant is obtained by <u>fixing</u> the fiber parameter $i \in \mathbb{H}$ and letting $\mathfrak{u} \in Cone(\theta)$ vary: it therefore arises from a local section defined by i (a <u>transversal</u> of $\widehat{Mod}$).

Conjectures

The main goal is to check that if $\theta \in \mathbb{R} \setminus \mathbb{Q}$ is quadratic, then Hilbert's classfield H_K of $K = \mathbb{Q}(\theta)$ (K's maximal unramified extension) equals

$$K(j(\theta)).$$

This would give a solution to Hilbert's 12th problem for quadratic real extension (unramified case).
Again, the main point is to prove the analog of "complex multiplication" (keypoint: the algebraicity of $j(\mu)$, when $\mu \in \mathbb{Q}(\sqrt{D})$, for $D < 0$ square free - and the fact that $j(\mu)$ essentially generates the Hilbert classfield $H(\mu)$ of $\mathbb{Q}(\sqrt{D})$).
We conjecture (with Gendron) that for $\theta \in \mathbb{R}$ there exists a duality relation between the classical invariant $j(i\theta)$ and the quantum invariant $j(\theta)$.

Duality II

More precisely, we associate to $j(i\theta)$ and $j(\theta)$ two nets

$$\{j(i\theta)_\alpha\} \qquad \text{and} \qquad \{j(\theta)_\alpha\}$$

whose elements are algebraically interdependent. The two nets converge to a common limit. The classical net $\{j(i\theta)_\alpha\}$ lives along a fixed leaf of $^\diamond\widehat{Mod}$; the quantum net$\{j(\theta)_\alpha\}$ lives on a fixed transversal of $^\diamond\widehat{Mod}$.

A través

(Foto: AV [proyecto **moving topoi**], sheaves 3)

Sorts for tori and *j*

Harris and Zilber provide a contrasting view of *j* invariants - directed toward categoricity and generalizations of *j* maps toward higher dimensions (Shimura varieties). The starting point is a view of *j* mappings as axiomatized in $L_{\omega_1,\omega}$:

Standard fibers

Using $L_{\omega_1,\omega}$, Harris and Zilber axiomatize classical j:

Let L be a language for two-sorted structures of the form

$$\mathfrak{A} = \langle\langle H; \{g_i\}_{i\in\mathbb{N}}\rangle, \langle F, +, \cdot, 0, 1\rangle, j : H \to F\rangle$$

where $\langle F, +, \cdot, 0, 1\rangle$ is an algebraically closed field of characteristic 0, $\langle H; \{g_i\}_{i\in\mathbb{N}}\rangle$ is a set together with countably many unary function symbols, and $j : H \to F$. Let then

$$Th_{\omega_1,\omega}(j) := Th(\mathbb{C}_j) \cup \forall x \forall y (j(x) = j(y) \to \bigvee_{\gamma\in SL_2(\mathbb{Z})} x = \gamma(y))$$

for $\mathbb{C}_j$ the "standard model" $(\mathbb{H}, \langle\mathbb{C}, +, \cdot, 0, 1\rangle, j : \mathbb{H} \to \mathbb{C})$. (Standard fibers means "fibers are orbits")

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Categoricity of classical *j*

The theory $Th_{\omega_1,\omega}(j) + \mathrm{trdeg}(F) \geq \aleph_0$ is categorical in all infinite cardinalities.
They use an instance of the adelic Mumford-Tate conjecture for products of elliptic curves to show this. The strategy to build an isomorphism between two models M and M' consists (as expected) in

- Identifying $\mathrm{dcl}^M(\emptyset)$ with $\mathrm{dcl}^{M'}(\emptyset)$ to start a back-and-forth argument.
- Assume we have $\langle \bar{x} \rangle \approx \langle \bar{x}' \rangle$ and take new $y \in M$ — we need to find $y' \in M'$ to extend the partial isomorphism (satisfying the same quantifier free type)
- (Quoting Harris:) we can realize the field type of a finite subset of a Hecke orbit over any parameter set (algebraicity of modular curves),...
- then show that the information in the type is contained in the finite part ("Mumford-Tate" open image theorem) ... every point $\tau \in \mathbb{H}$ corresponds to an elliptic curve E — the type of τ is determined by algebraic relations between torsion points of E... determined by the Galois representation of the Tate module of E.

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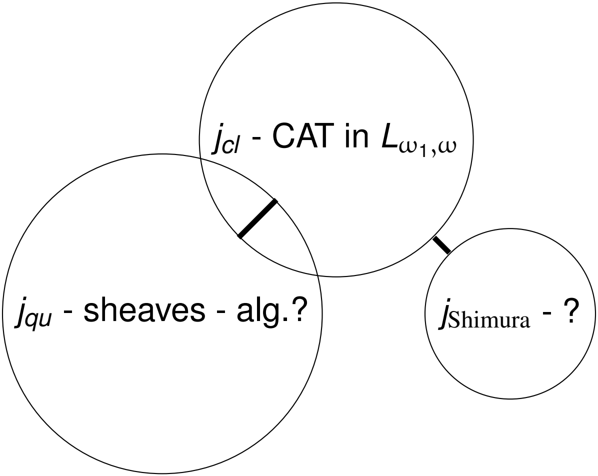
Two directions - Really?

The current model theoretic analysis of *j* looks at two possible extensions:



Two different toolkits

The two directions of generalization (to quantum tori/real multiplication on the one hand, to higher dimension varieties/Shimura on the other hand) of *j* maps calls different aspects of model theory (at the moment, an "amalgam" has started, but is far along the way):

- The model theory of abstract elementary classes (in particular, the theory of excellence - now "old" (1980s) but recently clarified by the "five authors": Bays, Hart, Hyttinen, Kesälä, Kirby - <u>Quasiminimal Structures and Excellence</u>. A kind of cohomological analysis of models (and types), connected to categoricity and "smoothness".
- The model theory of sheaves (remotely based on works by Macintyre, Ellerman), developed by Caicedo and further extended by Ochoa, Padilla, V. to metric and equivariant sheaves.

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Model Theory for sheaves

Caicedo's Generic Model Theorem is a "topological version" of the Łoś theorem, adapted to sheaves. It generalizes the Forcing Theorem of set theory.

Theorem (Caicedo)

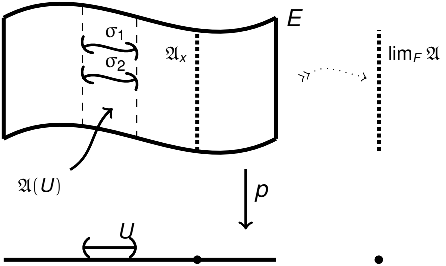
Fix a first order vocabulary τ. Let X be a topological space, $\mathfrak{A}$ a sheaf of τ-structures over X, F a filter of open sets generic for $\mathfrak{A}$, and $\varphi(v_1, \cdots, v_n)$ a τ-formula. Then, given sections $\sigma_1, \cdots, \sigma_n$ of the sheaf (defined on some open set in F), we have

$$\mathfrak{A}^X/F \models \varphi(\sigma_1/_{\sim F}, \cdots, \sigma_n/_{\sim F}) \quad \Leftrightarrow \quad \exists U \in F, \quad \mathfrak{A} \Vdash_U \varphi^G(\sigma_1, \cdots, \sigma_n).$$

A formula holds at a "limit" of sheaf of structures if and only if it is forced by some open set in the generic filter.
Just as in forcing, open sets are approximations of ideal points!

A topological representation

A topological representation of our sheaf: let $E \overset{p}{\twoheadrightarrow} X$ be a local homeomorphism. We call fibers (or stalks) the preimages $p^{-1}(x)$. They are always discrete subspaces of E. (Continuous) sections σ (the elements of the structures $\mathfrak{A}(U)$ over every open set U are partial inverses of p: $p \circ \sigma = id_U$. As usual, we identify sections σ with their images; these images form a basis for the topology of E.



Generic Model Theorem for equivariant sheaves

In recent work, we have extended Caicedo's Generic Model Theorem in two directions:

- to metric sheaves (with Ochoa) - over regular topological spaces
- to equivariant sheaves (with Padilla) - a group *G* acting on the sheaf, conditions on construction of *G*-sheaves (coherence and exactness not just at the level of the presheaf but also at the level of the action).

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Analogies, geometry, model theory

(As Manin would say...)

> For some reasons reflecting the nature of our kind of living matter (e.g., the fact that we are build of massive particles), we tend to project the adèlic nature onto its real side. We can equally well spiritually project it upon its non-Archimedean side and calculate most important things arithmetically.
>
> The relation between "real" and "arithmetical" pictures of the world is that of complementarity, like the relation between conjugate observables in quantum mechanics.
>
> *(Yu. Manin in Reflections of Arithmetical Physics, in the volume Mathematics as Metaphor.)*

Thank you for your attention!