Tree Partition Properties / natural logics for AEC

(A Tale of Two Cities)

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Symbiotic Relations
- Two Logic Groups: Caracas and Bogotá
- Combinatorial Set Theory / Model Theory

Set Theoretic Combinatorics and Definability of AECs
- Some partition relations
- Reflection Classes - AECs
- "Algebraically minded model theory" - Really?
- The Presentation Theorem
- Internal/Natural logic for an AEC

Canonical Trees
- Capturing the AEC with a sentence derived from the canonical tree
- The combinatorics behind

Symbiosis

Two (symbiotic) trees in Honda, Colombia

Symbiosis

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Today, two symbiotic relations: one, between TWO logic groups, the other one, between TWO parts of mathematical logic.

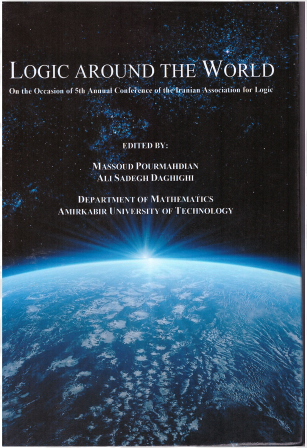
Two Logic Groups - A Tale of Two Cities

It was the best of times, it was the worst of times, it was the age of wisdom, it was the age of foolishness, it was the epoch of belief, it was the epoch of incredulity, it was the season of Light, it was the season of Darkness, it was the spring of hope, it was the winter of despair, we had everything before us, we had nothing before us, we were all going direct to Heaven, we were all going direct the other way…

C. Dickens - A Tale of Two Cities

Writing the history of the Bogotá Logic group led to...

...making explicit how symbiotic the growth of the two logic groups (Caracas and Bogotá) had been:



From the book - the Bogotá logic group and...

mathematical logic. The beginnings were slow (there was no research institute as such[1], and research activities were barely organized in a systematic way back then) but sure-footed: after a few years, Caicedo had formed several Master's students who went on to continue their doctoral formation abroad, and there was an active logic seminar.

The role of the Latin American Mathematical Logic Symposia (known as SLALM by their initials in Spanish) was crucial in the consolidation of the group. Bogotá hosted (at Universidad de los Andes) the 5th SLALM in 1981 and in many ways this event may be regarded as marking the end of the beginning years and the opening a new phase.

It is worth mentioning, anyway, that this first period is marked by both a beginning in isolation and a strong pull against this isolation: although research conditions were scarce (not only in terms of money but first and foremost in terms of organization) and logicians like Caicedo back then had to work in many simultaneous fronts combining research, teaching and administration, the existence of a genuine network of Latin American logicians provided a very interesting support even in those early days[2].

On a more personal note, some very solid friendships of logicians formed back then, at a Latin American level. In this regard, the role of set theorist Carlos Di Prisco was crucial for the consolidation of the group in Bogotá[3]. He was roughly at the same time developing the

1. Even in 2017, there is no independent research institute doing Mathematics in Colombia. All research in mathematical logic is conducted at the universities, by members of the faculty who must combine teaching basic courses, advanced courses, and doing research.

2. The network has continued, as we shall see later.

3. Di Prisco visited Bogotá as a mathematical logician first in 1981. Since then he visited many other times the city, giving lectures, minicourses, and in general bringing his interest in Combinatorial Set Theory to Bogotá. In more recent years he has been living in Bogotá in a more or less permanent basis, still doing logic and being part of the Logic Group, at Universidad de los Andes.

logic group in Caracas, the capital of our neighbour Venezuela, a his personal closeness to Caicedo and in general to the Logic Group Bogotá enabled the growth and blossoming of both groups. T history of both logic groups became intertwined, to this very day. T effects of specific economic and political situations of the t countries have become part of the history of the two groups and th interactions. Other such friendships with logicians in other countr have also triggered similar developments, but the Caracas case particularly close as an example of two different groups growing parallel, in a symbiotic way.

1.2. The group takes off and blossoms: 1980 to now.

During the 1980s the group starts really becoming something m similar to what we have now: Caicedo guides students toward th Master's programs[1], and later some of these students return Colombia (after their own doctoral studies abroad) and join the facu at local universities[2].

The main venues of mathematical logic for a long time w Universidad de los Andes and Universidad Nacional. Caicedo was many years professor at both universities. He split his resea between both places. Until recently, the seminar was joint between two universities—the number of students was growing but still sm for a long time.

Slowly, some of us returned to Bogotá and joined the facult there, and later on started directing doctoral theses as well. T

1. It is worth mentioning that doctoral programs in Mathematics only startec Colombia in 1995, first at Universidad Nacional, and a few years later also at o universities including Universidad de los Andes. This is why all those early these Mathematical Logic were done in the context of Master's studies.

2. Among them, Sergio Fajardo finished his Master's Thesis around 1979 and t went to the University of Wisconsin, where he worked with Keisler toward doctoral thesis; he almost immediately returned to Colombia and was involvec research[5] and teaching until the late 1990s, when he became a promir politician, now of national relevance.

From the book - the Bogotá logic group and...

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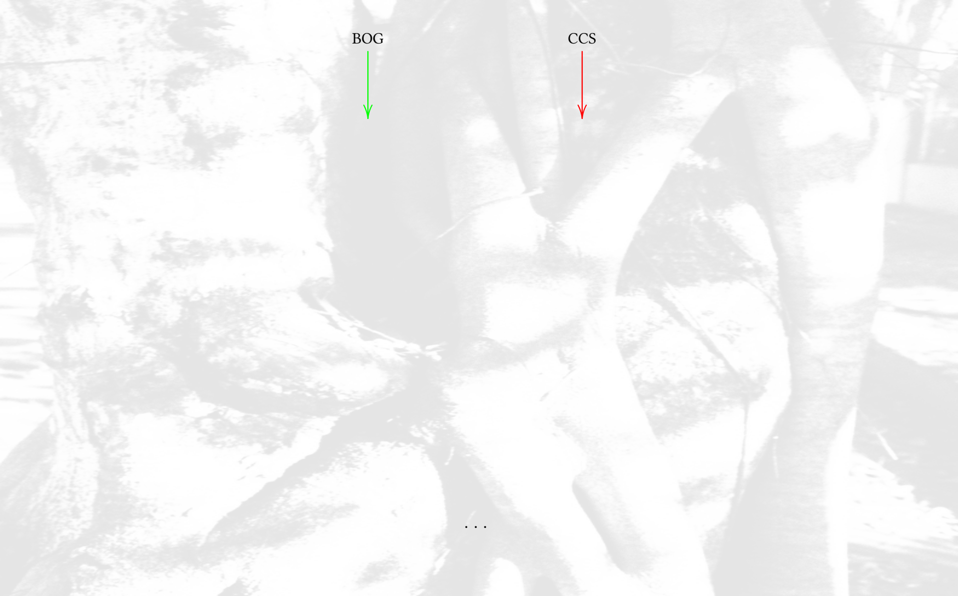
Xavier and Carlos

Symbiosis, slowly unfolding



BOG CCS

...

Symbiosis, slowly unfolding

BOG → Abstract Mod Th / Inf Logic (F,C,I) →

CCS → Comb Set Th / Large Cards (Ll) →

. . .

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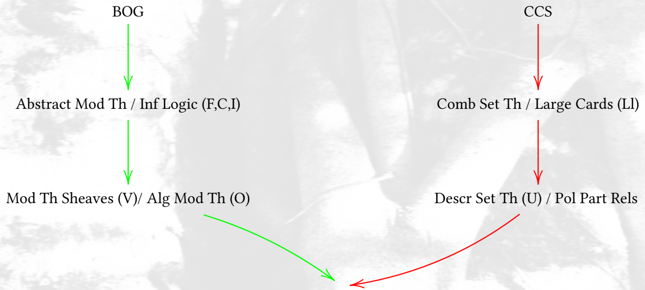

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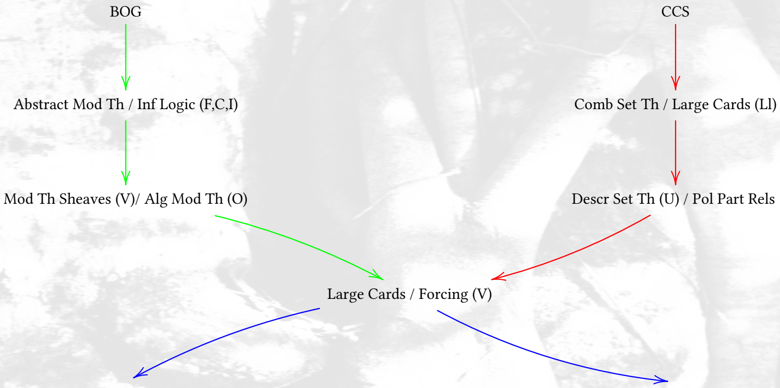

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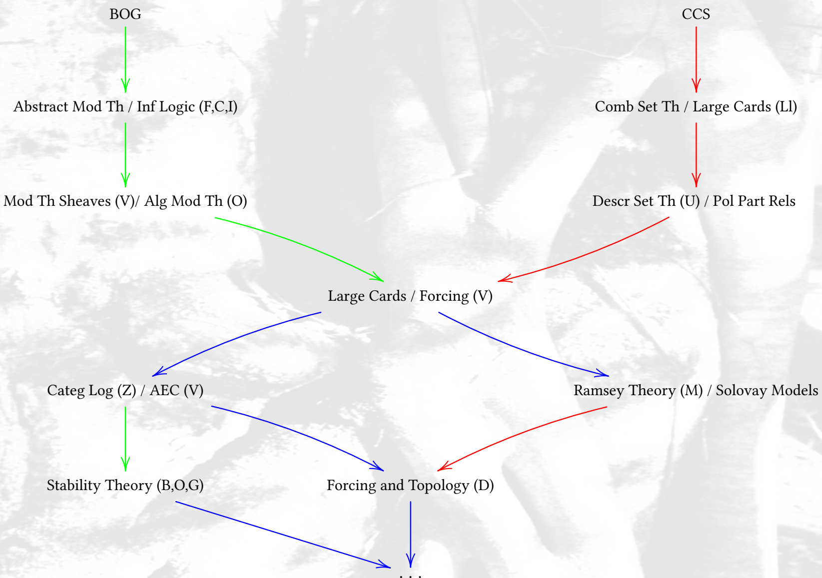
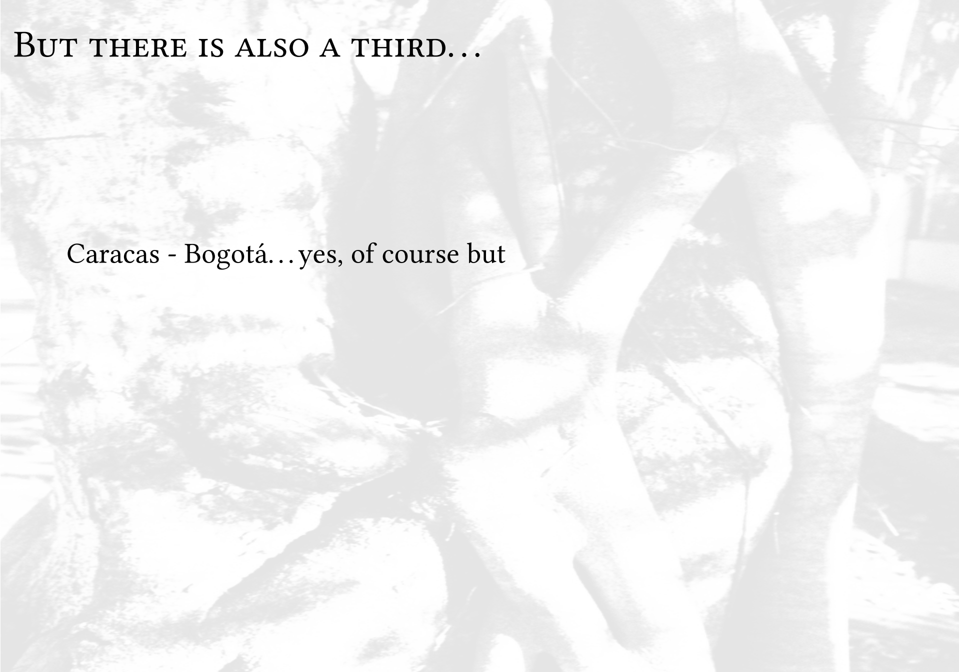


But there is also a third…



Caracas - Bogotá…yes, of course but

But there is also a third…

Caracas - Bogotá…yes, of course but
The beautiful city of Mérida, tucked in the high Andes of Venezuela, was central to the symbiosis of the group (EVM 1991)…Goyo Mijares was also there (I was also a student in the DP + U course in Descriptive Set Theory) and…

Other symbiotic relations

Set Theory / Model Theory

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- Proper forcing (Shelah's anecdote)
- A lot more, of course, in AEC…

Plan

Partition relations for scattered order types (and for trees)

Here is a combinatorial lemma by Komjath and Shelah[1]:
For any α let $\mathsf{FS}(\alpha)$ be the tree of all descending sequences of elements of α. We use $\mathsf{len}(s)$ to denote the length of $s \in \mathsf{FS}(\alpha)$.

Lemma (Komjath-Shelah 2003)

Assume that α is an ordinal and I a set. Set $\lambda = (|\alpha|^{|I|^{\aleph_0}})^{++}$. Suppose $T = \mathsf{FS}(\lambda)$ and $F : T \to I$. Then there is a subtree $T^* = \{(\delta^s_0, \dots, \delta^s_n) : s = (s_0, \dots, s_n) \in \mathsf{FS}(\alpha)\}$ of T and a function $c : \omega \to I$ such that for all $s \in T^*$ we have $F(s) = c(\mathsf{len}(n))$.

[1] A partition theorem for scattered order types. Combin. Probab. Comput. 12 (2003), no. 5-6, 621-626.

Origins: infinitary logic

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More generally, what is the behavior of the function $\mathsf{I}(\psi,\lambda) := |\{\mathsf{M} \models \psi \mid |\mathsf{M}| = \lambda\}/\approx|$, for a sentence ψ of the logic $\mathsf{L}_{\omega_1,\omega}$?

Long story short

After many attempts, the analysis of that primal question ran off from the syntactic extreme (infinitary logic(s)) to a more semantic "extreme".

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The attempts:

- (Keisler) Use "sequentially homogeneous" models. But sequential homogeneity is a consequence of categoricity...
- (Shelah) The role of models of size $\aleph_n$ $(n < \omega)$ in the decomposition of large models, the role of dimension-like obstructions.
- (Shelah) Forcing-like approach to types that would eventually become "Galois types".

"Algebraically-minded model theory" - Really?

Another early origin of Abstract Elementary Classes, complementary to the Categoricity problem, was Shelah's idea of (as expressed in his paper The Lazy Model-Theoretician's Guide to Stability Theory 1973)

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Another early origin of Abstract Elementary Classes, complementary to the Categoricity problem, was Shelah's idea of (as expressed in his paper The Lazy Model-Theoretician's Guide to Stability Theory 1973) speaking mainly to "those who are interested in **algebraically-minded model theory,** i.e., generic models, the class of e-closed models and universal-homogeneous models **rather than elementary classes** and saturated models. These were his words in 1975. He continues: "our main point is that though stability theory was developed for the latter context, almost everything goes through in the wider context (with suitable changes in the definitions)."

What goes through, really?

This declaration (the "almost everything goes through") entailed more than it could seem at first sight: in many ways it is true but it took a long time to build up the right notions of stability, of types, of independence.

Smooth Reflection Classes

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The relation $\models$, basic in First Order logic, takes a back seat here, and the main relation $\leq_K$ (a generalization of the elementary submodel relation $\prec$ of first order) now leads the game.

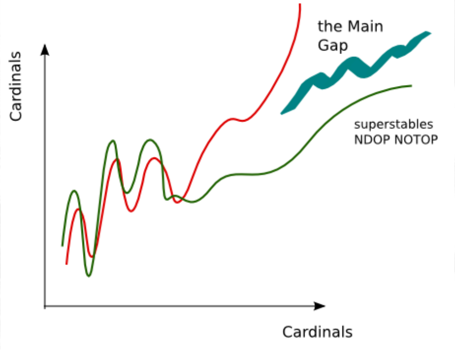
The definition [Abstract Elementary Class]

Fix a language L. A class $\mathcal{K}$ of L-structures, together with a binary relation $\leq_{\mathsf{K}}$ on $\mathcal{K}$ is an abstract elementary class (for short, AEC) if:

1. Both $\mathcal{K}$ and $\leq_{\mathsf{K}}$ are closed under isomorphism. This means two things: first, if $\mathsf{M}' \approx \mathsf{M} \in \mathcal{K}$ then $\mathsf{M}' \in \mathcal{K}$; second, if M', N' are L-structures with $\mathsf{M}' \subset \mathsf{N}'$, $\mathsf{M}' \approx \mathsf{M}$, $\mathsf{N}' \approx \mathsf{N}$ and $\mathsf{M} \leq_{\mathsf{K}} \mathsf{N}$ then $\mathsf{M}' \leq_{\mathsf{K}} \mathsf{N}'$.
2. If $\mathsf{M}, \mathsf{N} \in \mathcal{K}$, $\mathsf{M} \leq_{\mathsf{K}} \mathsf{N}$ then $\mathsf{M} \subset \mathsf{N}$,
3. $\leq_{\mathsf{K}}$ is a partial order,
4. (**Coherence**) If $\mathsf{M} \subset \mathsf{N} \leq_{\mathsf{K}} \mathsf{N}'$ and $\mathsf{M} \leq_{\mathsf{K}} \mathsf{N}'$ then $\mathsf{M} \leq_{\mathsf{K}} \mathsf{N}$,
5. (**LS**) There is a cardinal (called "the Löwenheim-Skolem number" of the class) $\kappa = \mathsf{LS}(\mathcal{K}) \geq \aleph_0$ such that if $\mathsf{M} \in \mathcal{K}$ and $\mathsf{A} \subset |\mathsf{M}|$, then there is $\mathsf{N} \leq_{\mathsf{K}} \mathsf{M}$ with $\mathsf{A} \subset |\mathsf{N}|$ and $|\mathsf{N}| \leq |\mathsf{A}| + \mathsf{LS}(\mathcal{K})$,
6. (**Unions of $\leq_{\mathsf{K}}$-chains**) If $(\mathsf{M}_i)_{i<\delta}$ is a $\leq_{\mathsf{K}}$-increasing chain of length δ (δ a limit ordinal), then
   - $\bigcup_{i<\delta}(\mathsf{M}_i)_{i<\delta} \in \mathcal{K}$,
   - for each $j < \delta$, $\mathsf{M}_j \leq_{\mathsf{K}} \bigcup_{i<\delta} \mathsf{M}_i$,
   - if for each $i < \delta$, $\mathsf{M}_i \leq_{\mathsf{K}} \mathsf{N} \in \mathcal{K}$ then $\bigcup_{i<\delta} \mathsf{M}_i \leq_{\mathsf{K}} \mathsf{N}$.

Main Conjecture: The Main Gap

The Main Gap Theorem for FO logic



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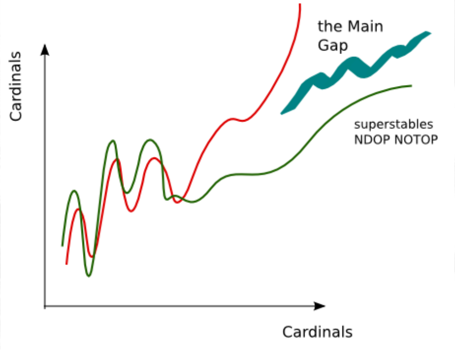

The "gold standard" of mathematical logic, of model theory, in various ways, and the main conjecture in AECs.

At this point, we have the following situation:

- So far, no control on possible axiomatization of the class $\mathcal{K}$. The emphasis is placed on its being closed under the constructions specified in the axioms. However, later (in subsection) we focus on the logical control of these classes. Remember Shelah's "algebraically-minded model theory".
- These are not necessarily amalgamation classes: there is no amalgamation axiom. However, many AECs do satisfy the amalgamation property. Furthermore, the model theory will depend on the kind of amalgamation possible in the class.

How to deal with these AECs?

Theorem (Presentation Theorem, Shelah)

Let $(\mathcal{K}, \leq_K)$ *be an AEC in a language* L*. Then there exist*

- *A language* $L' \supset L$*, with size* $LS(\mathcal{K})$*,*
- *A (first order) theory* T' *in* L' *and*
- *A set of* T'*-types,* Γ'*, such that*

$$\mathcal{K} = PC(L, T', \Gamma') := \{M' \restriction L \mid M' \models T', M' \text{ omits } \Gamma'\}.$$

Moreover, if $M', N' \models T'$*, they both omit* Γ'*,* $M = M' \restriction L$ *and* $N = N' \restriction L$*,*

$$M' \subset N' \Leftrightarrow M \leq_K N.$$

Corollary (“Hanf” number of an AEC)

If an AEC $\mathcal{K}$ has a model of cardinality $\geq \beth_{(2^{\mathrm{LS}(\mathcal{K})})^+}$ then it has arbitrarily large models.

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Proof: Use the Hanf number for PC classes (and the undefinability of well-ordering). □

Theorem (Shelah)

Let $(\mathcal{K}, \leq_{\mathsf{K}})$ be an AEC with amalgamation and arbitrarily large models. If $\mathcal{K}$ is categorical in $\lambda > \mathsf{LS}(\mathcal{K})$ then it is μ-galois-stable for each cardinal $\mu \in [\mathsf{LS}(\mathcal{K}), \lambda)$.

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And many other results on Stability Theory for a.e.c.’s really hinge on this “syntactic embedding” of EM models coming from First Order logic! The proof hinges on Ehrenfeucht-Mostowski models (whose existence in AECs with large enough models is given by the Presentation Theorem).

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The canonical tree of an a.e.c.

This is joint work with Saharon Shelah.
Fix an a.e.c. $\mathcal{K}$ with vocabulary τ and $\mathsf{LS}(\mathcal{K}) = \kappa$.
Let $\lambda = \beth_2(\kappa + |\tau|)^+$.
The **canonical tree** of $\mathcal{K}$:

- $\mathcal{M}_\mathsf{n} := \{\mathsf{M} \in \mathcal{K} \mid$ for some $\bar{\alpha} = \bar{\alpha}_\mathsf{M}$ of length n, M has universe $\left\{\mathsf{a}^*_\alpha \mid \alpha \in \mathsf{S}_{\bar{\alpha}[\mathsf{M}]}\right\}$ and $\mathsf{m} < \mathsf{n} \Rightarrow \mathsf{M} \restriction \mathsf{S}_{\bar{\alpha}\restriction \mathsf{m}[\mathsf{M}]} \prec_\mathcal{K} \mathsf{M}\}$ (and $\mathcal{M}_0 = \left\{\mathsf{M}_\mathsf{empt}\right\}$),
- $\mathcal{M} = \mathcal{M}_\mathcal{K} := \bigcup_\mathsf{n} \mathcal{M}_\mathsf{n}$; this is a tree with ω levels under $\prec_\mathcal{K}$ (equivalenty under $\subseteq$).

Formulas $\varphi_{M,\gamma,n}(\bar{x}_n)$

For M in the canonical tree $\mathcal{M}$ at level n, a formula with $\kappa \cdot n$ free variables, defined by induction on γ.

- $\gamma = 0$: $\varphi_{0,0} = \top$ ("truth"). If $n > 0$,

$$\varphi_{M,0,n} := \bigwedge \mathrm{Diag}^n_\kappa(M),$$

  the atomic diagram of M in $\kappa \cdot n$ variables.

- γ limit: Then

$$\varphi_{M,\gamma,n}(\bar{x}_n) := \bigwedge_{\beta<\gamma} \varphi_{M,\beta,n}(\bar{x}_n).$$

- $\gamma = \beta + 1$: Then $\varphi_{M,\gamma,n}(\bar{x}_n)$ is the $L_{\lambda^+,\kappa^+}(\tau)$ formula

$$\forall \bar{z}_{[\kappa]} \bigvee_{\substack{N \succ_{\mathcal{K}} M \\ N \in \mathcal{M}_{n+1}}} \exists \bar{x}_{=n} \left[\varphi_{N,\beta,n+1}(\bar{x}_{n+1}) \wedge \bigwedge_{\alpha < \alpha_n[N]} \bigvee_{\delta \in S[N]} z_\alpha = x_\delta \right]$$

So we have sentences $\varphi_{\gamma,0}$, for $\gamma < \lambda^+$, such that $i < j < \lambda^+$ implies $\varphi_j \to \varphi_i$. These sentences are better and better approximations of the aec $\mathcal{K}$; they describe how small models of the class embed into arbitrary ones.
Let us take a closer look at low levels:

The catch (beginnings)

When does $M \models \varphi_{1,0}$?

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This is slightly more complicated to unravel:

$$\forall \bar{z}_{[\kappa]} \bigvee_{N \in \mathcal{M}_1} \exists \bar{x}_{=1} \left[\varphi_{N,1,1}(\bar{x}_1) \wedge \bigwedge_{\alpha < \alpha_0[N]} \bigvee_{\delta \in S[N]} z_\alpha = x_\delta \right]$$

For every subset Z of M of size $\leq \kappa$ some model N in the tree (at level 1) M is such that $M \models \varphi_{N,1,1}$, through some "image of N" covering Z...

for all $Z' \subset M$ of size κ there is some $N' \succ_{\mathcal{K}} N$ in the canonical tree, at level 2, extending N, such that some tuple $\bar{x}_{=2}$ from M covers Z' and is the "image" of N' by an embedding

Theorem

$M \in \mathcal{K}$ *implies* $M \models \varphi_{\gamma,0}$ *for each* $\gamma < \lambda^+$

Theorem

$M \models \varphi_{\beth_2(\kappa)^+ + 2, 0}$ *implies* $M \in \mathcal{K}$

This much harder implication requires understanding the tree of possible embeddings of small models; the partition property due to Komjath and Shelah is the key.

The combinatorics behind

Theorem (Komjath-Shelah (2003))

Let α be an ordinal and μ a cardinal. Set $\lambda = \left(|\alpha|^{\mu^{\aleph_0}}\right)^+$ and let $\mathsf{F}(\mathsf{ds}(\lambda^+)) \to \mu$ be a colouring of the tree of finite descending sequences of ordinals $< \lambda$. Then there are an embedding $\varphi : \mathsf{ds}(\alpha) \to \mathsf{ds}(\lambda)$ and a function $\mathsf{c} : \omega \to \mu$ such that for every $\eta \in \mathsf{ds}(\alpha)$ of length $\mathsf{n}+1$

$$\mathsf{F}(\varphi(\eta)) = \mathsf{c}(\mathsf{n}).$$

We apply it with number of colours μ equal to $\kappa^{|\tau|+\kappa} = 2^\kappa$; therefore $(2^\kappa)^{\aleph_0} = 2^\kappa$. We thus obtain a sequence $(\eta_\mathsf{n})_{\mathsf{n}<\omega}$, $\eta_\mathsf{n} \in \mathsf{ds}(\lambda)$ such that:

$$\mathsf{k} \leq \mathsf{m} \leq \mathsf{n}, \ell \in \{1,2\} \Rightarrow \mathsf{N}^\ell_{\eta_\mathsf{m}\restriction\mathsf{k}} = \mathsf{N}^\ell_{\eta_\mathsf{n}\restriction\mathsf{k}}.$$

This enables us to “reconstruct” M (satisfying $\varphi_{\lambda+2,0}$ as a limit of models of size κ, in the class $\mathcal{K}$.

The End...not quite! Next, a video...

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But before the video, let us remember a magical moment of this week, just as set theorists were gathering in this city: