1)

# Some New Infinitary Logics and a Canonical Tree

CUNY GG

① $L^1_\kappa$, $L^1_{c,\kappa}$, ... $\Delta$

② Sentences for AECs ... the canonical tree of models. | A logic for AECs?

$L_{\infty,\omega}$

$\vdots$

$L_{\lambda^+,\omega}$ — $L_{\kappa,\lambda}$

$\vdots$

$L_{\omega_1,\omega}$

$L_{\omega,\omega}$ — $L_{\omega_1,\omega_1}$

Malitz (1971): Craig$(L_{\kappa^+\omega}, L_{(2^\kappa)^+,\kappa^+})$

(Consistency property)

Also, Craig$(L_{\omega\omega})$. and Craig$(L^*)$

Can we find $L^*$ s.t. $L_{\kappa^+\omega} \le L^* \le L_{(2^\kappa)^+,\omega}$

2) Shelah 2012 [797] :

If $\kappa$ is singular strong limit of cf $\omega$ there is a logic $L'_\kappa$ s.t.

$$\bigcup_{\lambda<\kappa} L_{\lambda^+\omega} \leq L'_\kappa \leq \bigcup_{\lambda<\kappa} L_{\lambda^+\lambda^+}$$

Also, if $\kappa = \beth_\kappa$, $L'_\kappa$ has a Lindström-type charact. as the maximal logic with (a strong form of) undef. of well order.

Digression: $L^{chain}_\kappa$ (Karp)

syntax: $L_{\kappa\kappa}$ $\kappa$ sing. strong limit, $cf(\kappa)=\omega$

semantics: chain models

Craig($L^{chain}_\kappa$) Cunningham - 1975

$L'_\kappa \leq L^c_\kappa \leq L_{\kappa\kappa}$ Džamonja-Väänänen

3)

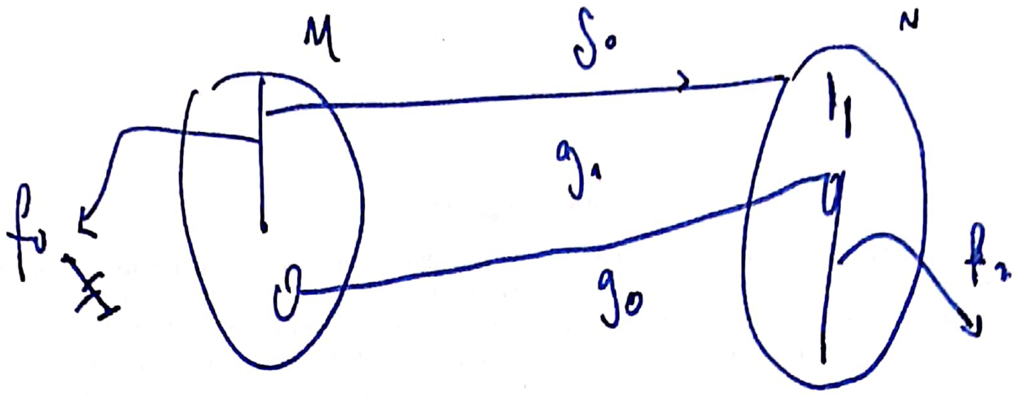
$L^1_\kappa$ : not given by a syntax!

instead: "$M \equiv_{L^1_\kappa} N$"

via a game (EF) $G^\beta_\theta(M,N)$

| $\mathrm{I}$ ($\forall$) | $\mathrm{II}$ ($\exists$) |
| --- | --- |
| pick $\beta_0 < \beta$, $\vec{a}_0$ size $\leq \theta$ | |
| | pick $f_0 : \vec{a}_0 \to \omega$, give $g_0 : A$ , p.i. $\mathrm{dom}(g_0) \supseteq f^{-1}(0)$ |
| $\beta_1 < \beta_0$, $\bar{a}_1$ size $\leq \theta$ | pick $f_1 : \bar{b}_1 \to \omega$, give $g_1 : A \rightharpoonup B$ p.i. $g_1 \supseteq g_0$, $\mathrm{dom}(g_1) \supseteq f_1^{-1}(0) \cup f_0^{-1}(1)$ |

4)




$M \sim^{\beta}_{\theta} N \iff$ $\underline{\mathrm{II}}$ has winning strategy

$M \equiv^{\beta}_{\theta} N$ : tr. cl. of $M \sim^{\beta}_{\theta} N$

A "sentence" of $L^1_\kappa$ : a union of $\leq \beth_{\beta+\omega}(\theta)$ eq. classes of $\equiv^{\beta}_{\theta}$ for some $\theta < \kappa$, $\beta < \theta^+$

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Approaching $L^1_\kappa$ from below, up to $\Delta$
(joint with Väänänen)

$L^{1,c}_{\kappa} \leq L^1_\kappa$ (Cartagena logic)

5) if $\operatorname{len}(\bar{x}) = \theta$

$f : \vec{x} \to \omega$

$\varphi_{f,n}(\bar{x}, \bar{y})$ are flas of $L^{1,c}_{\kappa}$

with vars. $x_\alpha$, $\bar{y}$

s.t. $f(\alpha) = n$ free among $\vec{x}$

then

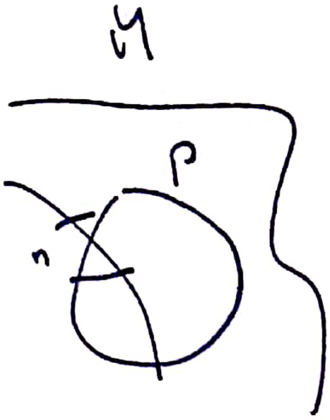
$$\forall \bar{x} \bigvee_f \bigwedge_n \varphi_{f,n}(\bar{x}, \bar{y})$$

is a fla of $L^{1,c}_{\kappa}$.

Ex: (1) $|P| < \theta$

let $\theta < \kappa$, $cf(\theta) > \omega$

$\operatorname{len}(\bar{x}) = \theta$




$$\forall \vec{x} \bigvee_f \bigwedge_n \Big[ \bigwedge_{f(i)=n} P(x_i) \to \bigvee_{i \neq j \in f^{-1}(n)} x_i = x_j \Big]$$

(2) no long chains ( $<$ has no chains of length $\theta$ )

$$\forall \bar{x} \bigvee_f \bigwedge_n \bigwedge_{i \neq j \in f^{-1}(n)} \neg\, x_i < x_j$$

6)

Ex. $len(\bar{x}) = \theta$

$len(\bar{y}) = \omega$

$$\forall \bar{x} \bigvee_{f} \bigwedge_{n} \exists \bar{y} \bigwedge_{g} \bigvee_{m} \bigwedge_{f(i)=n} \bigvee_{g(j)=m} R(y_j, x_i)$$

Says every set of size $\leq \theta$ can be covered by ctbly many of the form $R(a, \cdot)$

Corollary: if $\theta < \kappa$ there is $\psi_\theta \in L^{1,c}_{\kappa}$ with a model of cardinality $\theta$ iff $\theta^\omega = \theta$

anti LS close to $L^1_\kappa$

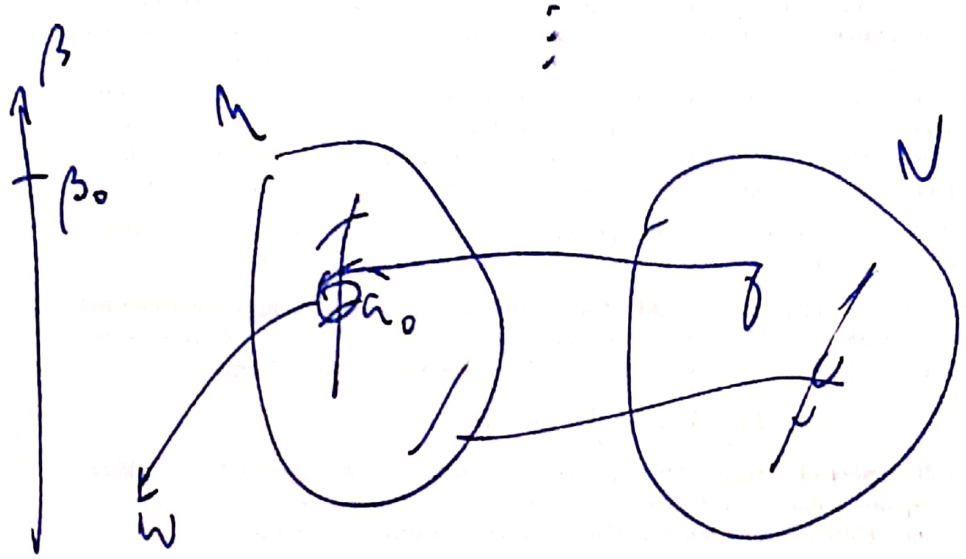
EF game for $L^{1,c}_{\infty,\lambda}$ : $G^{\beta,c}_{\theta}(M, N)$

| | |
|---|---|
| $\beta_0 < \beta$, $\bar{a}_0$ | |
| | $f_0 : \bar{a}_0 \to \omega$ |
| $n_0 < \omega$ | |
| | $g_0 : A \to B$ p.i. $dom(g_0) \supset f_0^{-1}(n_0)$ |

7)

| | |
|---|---|
| ~~A,A~~ $\beta_1 < \beta_0$ <br> $\vec{b}_1$ | |
| | $f_1 : \vec{b}_1 \to \omega$ |
| $n_1 < \omega$ | |
| | $g_1 : A \to B$, $g_1 \supset g_0$ <br> $\mathrm{ran}(g_0) \supset f_1^{-1}(n_1)$ |

⋮




↻ $\underline{\mathrm{II}}$ has winning strat. in $G_\theta^{\beta,c}(M,N)$

$$M \equiv_{L^{1,c}_{\theta^+}} N \text{ up to quant. rank } \in \beta$$

<u>Cor.</u> $L^{1,c}_\kappa \subseteq L^1_\kappa$

2] Thm.

$\kappa = \beth_\kappa \Rightarrow \Delta(L^{1,c}_\kappa) = L^1_\kappa$

$\Delta(L)$: $Mod(\varphi)$, $Str_\tau \setminus Mod(\varphi)$ are $\Sigma(L)$

$K$ is $\Sigma(L)$: $K$ class of rel. reducts of an $L'$-def'ble model class, $L' \supseteq L$

$\Delta(L_{\omega\omega}) = L_{\omega\omega}$

$\Delta(L_{\omega_1,\omega}) = L_{\omega_1,\omega}$

$\Delta(L(Q_0)) = L_{HYP}$

$\Delta\Delta L = \Delta L$

$\Delta$ preserves | compactness, axiomatizability, LS

$$\begin{tikzcd}
 & L_{KR} & \\
L^c_\kappa \arrow[ur] & L^1_\kappa \arrow[l] \arrow[u] \arrow[r] & L^1_{\kappa,a} \\
 & L^{1,c}_\kappa \arrow[u] & \\
 & L^1_{\kappa\omega} \arrow[u] &
\end{tikzcd}$$