HACES DE ESTRUCTURAS TOPOLÓGICAS SOBRE EL ORDEN PARCIAL DE SUBESPACIOS DE DIMENSIÓN FINITA DE UN ESPACIO DE HILBERT

SHEAVES OF TOPOLOGICAL STRUCTURES OVER THE PARTIAL ORDER OF FINITE DIMENSIONAL SUBSPACES OF A HILBERT SPACE

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Quiero dedicar este trabajo a Elsa Daza, mi madre y mi amiga.

Introduction

The original motivation for this research is external to mathematics. We introduce new elements in the model theory of sheaves, suggest further work in mathematics, and partially solve the question that motivated this thesis. We next provide some context.

The study of nature has inspired many mathematical concepts and theories. History of mathematics was tied to that of the scientific study of nature for centuries. Today, when they are clearly independent fields of human knowledge, there is still a deep interaction between them. Our work is in the juncture.

The most striking crisis within Theoretical Physics in the last century came from chemical and particle physics problems. It was not possible to describe the idea of a stable atom within the realm of Classical Mechanics and Classical Electrodynamics. Quantum Mechanics was developed as a product of the search for the minimal physical principles describing this non-classical system. Since then, it has been an active branch of modern science, still facing unsolved problems.

In particular, it is not possible to do exact Chemistry and only a few simple systems admit a complete quantum mechanical description. Problems not only arise from the intricate nature of the solutions to the wave equation associated to every physical system. They also arise from the difficulties that the mathematical formalization of different concepts in Chemistry and Physics involves. The huge diversity in experimental results, the complexity in the structure of molecular species and the increasing number of techniques used to understand them, have led to a constant evolution of the principles that describe a given set of chemical substances; these principles do not always have good generalizations.

The concept of chemical bond is an example of this. One can find throughout the specialized literature plenty of different definitions for this concept as understood in specific systems, such as metals, organic molecules and metal oxides, for example. Nevertheless, no general agreement exists about what the general concept of chemical bond is. Of course, one can find in the work of many people different definitions for this and other concepts (see for example [1, 6, 10, 13]), but none of these has reached universal agreement.

More generally, we have that every physical observable was traditionally described in classical mechanics by functions between vector spaces and, after the formalization of quantum mechanics, by operators in Hilbert spaces. What is new about many concepts in chemistry, such as the concept of chemical bond, is that they do not fit into this set of observables. They are better described as predicates defined in a vector or Hilbert space. We are led to think about chemical systems as physical systems with additional predicates for notions such as bond and chemical structure. Mathematical Logic (and more specifically Model Theory) is therefore a natural framework for a formal discussion of chemical concepts and models of chemical systems. Axiomatizations of the properties of these predicates are statements about experimental observations or hypotheses about the structure of the system.

We attempted to study different chemical systems. Pursuing this study, we found other problems in the construction of models of Hilbert spaces associated with physical systems. Those are described in subsequent Chapters. In addition, we are interested in an axiomatization of chemical systems consistent with the axioms of quantum mechanics. One can achieve this by associating chemical concepts with the geometric properties of smooth manifolds of physical relevance, as for example, the energy. One may not need to say anything about the differential structure of a manifold, but some expressive power about topological properties of manifolds is required. On the other hand, Model Theory provides nowadays systems of approximations of “limit structures”. Sheaves of structures in a more appropriate language for topological spaces seemed to be a good point to begin. Investigations in this direction lead to the work in Chapter 1 and the Topological Generic Model Theorem (Theorem 1.4). Later, problems around the Cauchy completeness of our models, described in section 1.4, motivates the study of sheaves of metric structures in the context of continuous logic. Chapter 2 presents results in this direction introducing Theorem 2.3, the corresponding metric version of the Generic Model Theorem. We recover the spirit of the main goal in Chapter 3 where we apply what we have learned in previous chapters in a construction of Projective Hilbert spaces.

Before we get into the discussion of the mathematical results of this work, I want to thank Professor Andrés Villaveces for his advising, his constant support in this project and for teaching me the most I could learn about mathematical logic and model theory. In addition, I want to thank Professor

Alexander Berenstein, who supported me in some aspects of the present work and who was always open to talk about it. Finally, I have to say that any mistake that might be found in this document is my whole responsability.

Contents

Chapter 1

Sheaves of Topological Structures

In this chapter we introduce an extended version of the theory of sheaves of first order structures and a topological version of the Generic Model Theorem. In addition, we study the Cauchy completion of the generic model space when this is provided with a metric. The structures of our sheaves are not first order structures but the topological structures defined by Ziegler and Flum [7]. Thus, we begin reviewing some properties of these structures in the frame of the topological model theory of Ziegler and Flum.

1.1 A brief survey of topological model theory

Let us define some basic terminology. More details can be found in Part I of [7].

Definition 1.1. *A* <u>*weak structure*</u> *is a pair* $(\mathfrak{A}, \tau)$ *where* $\mathfrak{A}$ *is a first order structure and* $\tau \subset \mathcal{P}(A)$.

Definition 1.2. *A* <u>*topological structure*</u> *is a pair* $(\mathfrak{A}, \tau)$ *where* $\mathfrak{A}$ *is a first order structure and* τ *is a topology for* $\mathfrak{A}$.

Definition 1.3. $\tilde{\tau} = \{\cup_i s_i | s_i \in \tau\}$.

Therefore, $\tilde{\tau}$ is the closure of τ under arbitrary unions of its elements. In our context, the following notion of invariance will become relevant, since it is located at the roots of the development of topological model theory and allow us to move from topological structures to weak structures and vice versa.

Definition 1.4. *Let ϕ be a second order sentence.*

- *ϕ is invariant if for all $(\mathfrak{A}, \tau)$:*

$$(\mathfrak{A}, \tau) \models \phi \iff (\mathfrak{A}, \tilde{\tau}) \models \phi$$

- *ϕ is topologically invariant if for all $(\mathfrak{A}, \tau)$ such that $\tilde{\tau}$ is a topology in A,*

$$(\mathfrak{A}, \tau) \models \phi \iff (\mathfrak{A}, \tilde{\tau}) \models \phi$$

  holds.

We now proceed to construct the elements of the language L_t for topological structures.

Definition 1.5. *L_t is a set of second order formulas constructed inductively which can be obtained from the classical rules for the first order language and the following additional rules:*

- *If t is a term and $\mathcal{X}$ is a set variable, then $t \in \mathcal{X}$ is an atomic formula.*
- *If t is a term and $\phi(x, \mathcal{X})$ is a positive formula in $\mathcal{X}$, then $\forall \mathcal{X}(t \in \mathcal{X} \to \phi(x, \mathcal{X}))$ is a formula.*
- *If t is a term and $\phi(x, \mathcal{X})$ is a negative formula in $\mathcal{X}$, then $\exists \mathcal{X}(t \in \mathcal{X} \wedge \phi(x, \mathcal{X}))$ is a formula.*

The "Invariance theorem" to be introduced now is relevant in the development of the basic theorems of topological model theory. These theorems are named after their classical counterparts: Compactness, completeness and Löwenheim-Skolem theorem.

Theorem 1.1 (Invariance theorem)**.** *L_t-sentences are invariant.*

Lemma 1.1. *Let $\Gamma \cup \{\phi\} \subset L_t$,*

$$\begin{aligned}\phi_{bas1} =& \forall x \exists \mathcal{X}(x \in \mathcal{X}) \ , \\ \phi_{bas2} =& \forall x \forall \mathcal{X}\Big(x \in \mathcal{X} \to \forall \mathcal{Y} \\ & \quad \big(x \in \mathcal{Y} \to (\exists \mathcal{Z}(x \in \mathcal{Z} \wedge (\forall y \ (y \in \mathcal{Z} \to y \in \mathcal{Y} \wedge y \in \mathcal{X}) \ )))\big)\end{aligned}$$

and $\phi_{bas} = \phi_{bas1} \wedge \phi_{bas2}$ be a sentence that states that τ is a basis for a topology on A.

1. *Γ has a topological model if and only if $\Gamma \cup \phi_{bas}$ has a weak model.*

2. *$\Gamma \models_t \phi$ if and only if $\Gamma \cup \phi_{bas} \models \phi$*

The subscript in the satisfaction relation, means that every model of Γ is in fact a topological structure. Topological model theory has expressive power to describe topological spaces and topological vector spaces to some extent.

1.2 Sheaves of topological structures

The theory of sheaves of first order structures has been developed by various authors including Caicedo in his article [2] and subsequent papers. We use the definitions and ideas in [2] as our starting point to introduce the following more general definitions[1].

Definition 1.6. *A* **topological sheaf over** *X is a pair (E, p), where E is a topological space and p is a local homeomorphism from E into X.*

Definition 1.7. *A* **section** *σ is a function from an open set U of X to E such that $p \circ \sigma = Id_U$. We say that the section is global if $U = X$.*

Definition 1.8 (Sheaf of topological structures $\mathfrak{A}$)**.** *Let X be a topological space and τ its topology. A sheaf of topological structures $\mathfrak{A}$ on X consists of:*

1. *A topological sheaf (E, p) over X.*

2. *For every x in X, a τ-structure*

$$(\mathfrak{A}_x, \tau_x) = \left(E_x, \{R_i^{(n_i)}\}_x, \{f_j^{(m_j)}\}_x, \{c_k\}_x, \{\mathcal{C}_l\}_x\right)$$

where E_x is the fiber $p^{-1}(x)$ over x, and the following conditions hold:

   (a) For all i, $R_i^{\mathfrak{A}} = \bigcup_x R_{i,x}^{(n_i)}$ is open in $\bigcup_x E_x^{n_i}$ as a subspace of the product space E^{n_i}.

   (b) For all j, $f_j^{\mathfrak{A}} = \bigcup_x f_{j,x} : \bigcup_x E_{j,x}^{m_j} \to \bigcup_x E_{j,x}$ is continuous.

   (c) For all k, the function $c_k^{\mathfrak{A}} : X \to E$ such that $c_k^{\mathfrak{A}}(x) = c_{k,x}$ is a continuous global section.

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[1] If $\tau = \emptyset$, our structure is classic.

(d) *For all l, $\mathcal{C}_l^{\mathfrak{A}} = \bigcup_{x \in U} \mathcal{C}_l^{\mathfrak{A}_x}$ is open in $\bigcup_{x \in U} E_x^{n_l}$ as a subspace of the product space $E^{n_l} \restriction U$, where U is open in X.*

The space $\bigcup_x E_x^n$ has as open sets the image of sections given by $\langle \sigma_1, \ldots, \sigma_n \rangle = (\sigma_1, \ldots, \sigma_n) \cap \bigcup_x E_x^n$. These are the sections of a sheaf over X with local homeomorphism p^* defined by $p^* \langle \sigma_1(x), \ldots, \sigma_n(x) \rangle = x$. We are going to drop the symbol * from our notation when talking about this local homeomorphism but it must be clear that it differs from the function p used in the definition of the topological sheaf.

In the following, $\mathcal{X}$ and $\mathcal{Y}$ are going to be used as set variables and $\mathcal{C}, U, V, W$ are going to denote specific open sets. Whenever a sentence ϕ has $\mathcal{X}$ as a variable, it will be explicitly specified, independent from the first order variables (e.g $\phi(x, \mathcal{X}; a, \mathcal{C}) = \ x \in \mathcal{X} \wedge a \in \mathcal{C}$). In addition, σ and ν are going to be used for sections, the former being a variable and the latter being the interpretation of σ at a specific point in X, say, $\nu = \sigma(x)$.

The next lemma is an extension of the *Truth continuity lemma* from the model theory of sheaves of first order structures (see [2] Lemma 2.2).

Lemma 1.2. *Let $\phi(\sigma, \mathcal{X})$ be an L_t formula, obtained inductively from L_t-atomic formulas, and the logical connectives different from the negation symbol, and without the first order universal quantifier or any second order quantifier. If $\mathfrak{A}_x \models_t \phi[\sigma(x), \mathcal{X}]$ then, there exists an open neighborhood U of x such that for all y in U $\mathfrak{A}_y \models_t \phi[\sigma(y), \mathcal{X}]$.*

Proof. By induction on the complexity of the formulas.

- For first order logical connectives and existential quantifier the proof is the same as for sheaves of first order structures (see [2] Lemma 2.2).
- If $\phi(v, \mathcal{X}) = v \in \mathcal{X}$ and $\mathfrak{A}_x \models \sigma(x) \in \mathcal{C}_i$, ($\mathcal{C}_i = \mathcal{X}^{\mathfrak{A}_x}$), then $V = Im(\sigma) \cap \mathcal{C}_i^{\mathfrak{A}}$ is open in E. Therefore $p(V) = U$ is open in X and for every $y \in U$, $\mathfrak{A}_y \models \phi[\sigma(y), \mathcal{X}]$.

□

The above lemma fails when a universal quantifier or negation symbol is present in a sentence. The fact that the lemma is not valid for second order existential quantifiers is due to its definition, since these quantifiers are restricted to formulas that are negative in the set variable.

The following extension of the notion of point forcing on sheaves of first order structures gives us the desired continuity for every logical connective and quantifier. The reader must find that this definition extends its classical counterpart.

Definition 1.9 (point forcing)**.** *Let $t_1, \ldots, t_k$ be first order terms*

- *(atomic first order)*

$$\mathfrak{A} \Vdash_x (t_1(\sigma_1, \ldots, \sigma_n) = t_2(\sigma_1, \ldots, \sigma_n)) \iff t_1^{\mathfrak{A}_x}(\sigma_1(x), \ldots, \sigma_n(x)) = t_2^{\mathfrak{A}_x}(\sigma_1(x), \ldots, \sigma_n(x))$$

$$\mathfrak{A} \Vdash_x R(t_1(\sigma_1, \ldots, \sigma_n), \ldots, (t_k(\sigma_1, \ldots, \sigma_n)) \iff \big(\, t_1^{\mathfrak{A}_x}(\sigma_1(x), \ldots, \sigma_n(x)), \ldots, t_k^{\mathfrak{A}_x}(\sigma_1(x), \ldots, \sigma_n(x) \,\big) \in R_x$$

- *(atomic second order)*

$$\mathfrak{A} \Vdash_x t \in \mathcal{X} \iff t^{\mathfrak{A}_x} \in \mathcal{X}^{\mathfrak{A}_x}$$

- *Logical connectives*

$\mathfrak{A} \Vdash_x \phi(\sigma) \wedge \psi(\sigma) \iff \mathfrak{A} \Vdash_x \phi(\sigma)$ *and* $\mathfrak{A} \Vdash_x \psi(\sigma)$

$\mathfrak{A} \Vdash_x \phi(\sigma) \vee \psi(\sigma) \iff \mathfrak{A} \Vdash_x \phi(\sigma)$ *or* $\mathfrak{A} \Vdash_x \psi(\sigma)$

$\mathfrak{A} \Vdash_x \neg\phi(\sigma) \iff$ *There exists an open set U such that $x \in U$ and for all $y \in U$: $\mathfrak{A} \nVdash_y \phi(\sigma)$*

$\mathfrak{A} \Vdash_x \phi(\sigma) \to \psi(\sigma) \iff$ *There exists an open set U such that $x \in U$ and for all $y \in U$ if $\mathfrak{A} \Vdash_y \phi(\sigma)$ then $\mathfrak{A} \Vdash_y \psi(\sigma)$*

- *First order quantifiers*

$\mathfrak{A} \Vdash_x \exists\mu\phi(\mu) \iff$ *There exists a section σ defined in x such that $\mathfrak{A} \Vdash_x \phi(\sigma)$*

$\mathfrak{A} \Vdash_x \forall\mu\phi(\mu) \iff$ *There exists an open set U such that $x \in U$ and for all $y \in U$ and for every section σ defined in y $\mathfrak{A} \Vdash_y \phi(\sigma)$*

- *(Monadic second order existential quantifier)*
  If $\phi(v, \mathcal{X})$ is a negative L_t-formula in $\mathcal{X}$ in its conjunctive normal form.

$\mathfrak{A} \Vdash_x \exists\mathcal{X}(t \in \mathcal{X} \wedge \phi(\sigma(x), \mathcal{X})) \iff$
there exists an open neighborhood U in x such that $\forall y \in U$ there exist $\mathcal{Y} \in \tau_y$ $\mathfrak{A} \Vdash_y t \in \mathcal{Y}$ and $\mathfrak{A} \Vdash_y \phi(\sigma(y), \mathcal{Y})$.

- *(Monadic second-order universal quantifier)*
  If $\phi(v, \mathcal{X})$ is a positive L_t-formula in $\mathcal{X}$

$$\begin{aligned}\mathfrak{A} \Vdash_x \forall \mathcal{X}(t \in \mathcal{X} \to \phi(\sigma(x), \mathcal{X})) \iff& \\ &\textit{there exists an open neighborhood } U \textit{ of } x \textit{ such that } \forall y \in U \\ &\quad \textit{and for all } \mathcal{Y} \in \tau_y, \textit{ if } \mathfrak{A} \Vdash_y t \in \mathcal{Y} \textit{ then } \mathfrak{A} \Vdash_y \phi(\sigma(y), \mathcal{Y}).\end{aligned}$$

and as a consequence we have the

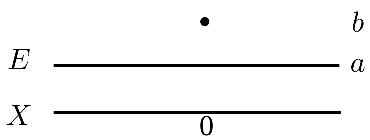
Lemma 1.3 (Truth continuity lemma in L_t)**.** *$\mathfrak{A} \Vdash_x \phi(\sigma(x), \mathcal{X})$ if and only if there exists U open neighborhood of x such that $\mathfrak{A} \Vdash_y \phi(\sigma(y), \mathcal{Y})$ for all y in U.*

Proof. Follows immediately from lemma 1.2 and the above definition. We write the proof only for the case of second order existential quantifier.

Let t be a term, $\phi(v, \mathcal{X})$ negative in $\mathcal{X}$ and $\mathfrak{A}_x \Vdash_x \exists \mathcal{X}(t \in \mathcal{X} \wedge \phi(\sigma(x), \mathcal{X}))$. In that case there exists $\mathcal{C}_x$ such that $\mathfrak{A}_x \Vdash_x (t \in \mathcal{C}_x \wedge \phi(\sigma(x), \mathcal{C}_x))$. Therefore, $\mathfrak{A}_x \Vdash_x t \in \mathcal{C}_x$ and $\mathfrak{A}_x \Vdash_x \phi(\sigma(x), \mathcal{C}_x)$ and by the induction hypothesis there exists $W = U \cap V$ an open neighborhood of x such that for all y in W $\mathfrak{A}_y \Vdash_y t \in \mathcal{C}_y \wedge \phi(\sigma(y), \mathcal{C}_y)$. □

Example 1.1. The following example was previously introduced by Caicedo in [2] in the context of sheaves of first order structures. We define a topology in every fiber so that the sheaf becomes a sheaf of topological structures.

Consider the sheaf on the set of real numbers all whose fibers have only one point a with the trivial topology, except for the fiber at 0 which is a two point set $\{a, b\}$ with open sets $\{\emptyset, \{a\}, \{a, b\}\}$. This sheaf on the set of real numbers can be thought of as a copy of themselves with an additional point b at 0. Sections are the open intervals $U \in \mathbb{R}$ and $(U \setminus \{a\}) \cup \{b\}$.



To avoid confusion, we use T_0^{top} to refer to the topological separability property T_0. This property can be expressed in $\mathcal{L}_t$:

$$T_0^{top} = \forall x \forall y \left(\exists \mathcal{X}(x \in \mathcal{X} \wedge \neg y \notin \mathcal{X}) \vee \exists \mathcal{Y}(y \in \mathcal{Y} \wedge \neg x \notin \mathcal{Y}) \right).$$

For this construction we have $\mathfrak{A}_0 \models T_0^{top}$. However $\mathfrak{A} \nVdash_0 T_0^{top}$. Observe that the image of the section $\sigma_a(x) := a$ is the open set $\{a\}^{\mathfrak{A}}$. If $\mathfrak{A} \Vdash_0 T_0^{top}$, by the truth continuity lemma T_0^{top} would be forced in a neighborhood of 0. However, every section μ such that $\mu(0) = b$, $\mu(x \neq 0) = \sigma_a(x)$. The previous arguments show that $\mathfrak{A} \not\Vdash_0 \neg\mu \in \{a\}^{\mathfrak{A}}$.

The next example shows that the topology of the sheaf may have more open sets than those in the topology of every fiber.

Example 1.2. Let $X = \{0, 1\}$ and $\tau_X = \{\emptyset, \{1\}, X\}$. Fibers in this sheaf are the topological semigroups $E_0 = (X, \tau_X, \circ)$ and $E_1 = (X, \tau_X, *)$ with

$\circ$	0	1
0	0	1
1	0	1

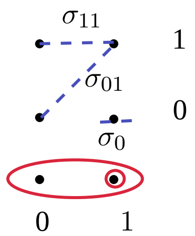
and

$*$	0	1
0	0	1
1	1	1

Notice that every fiber has the same topology as the base space. The topology of the sheaf is given by global sections

$$\sigma_{11}(x) = \begin{cases} 1 & x = 0 \\ 1 & x = 1 \end{cases} \qquad \sigma_{01}(x) = \begin{cases} 0 & x = 0 \\ 1 & x = 1 \end{cases}$$

and the local section defined in $\{1\}$ whose unique image is 0 (name it σ_0).



Let $\dagger = \circ \sqcup *$ be a function defined on $E_1^2 \sqcup E_2^2$. Observe that $\dagger$ induces a function

$$\dagger^{\mathfrak{A}(X)} : \mathfrak{A}(X) \times \mathfrak{A}(X) \to \mathfrak{A}(X),$$

where $\mathfrak{A}(X)$ is the set of sections defined in X, with the following multiplication table

$\dagger^{\mathfrak{A}(X)}$	σ_{01}	σ_{11}
σ_{01}	σ_{01}	σ_{11}
σ_{11}	σ_{01}	σ_{11}

We observe that $\dagger^{\mathfrak{A}(X)}$ is continuous by considering preimages:

$$\left(\dagger^{\mathfrak{A}(X)}\right)^{-1}(\sigma_{11}) = \langle \sigma_{01}, \sigma_{11} \rangle \cup \langle \sigma_{11}, \sigma_{11} \rangle$$
$$\left(\dagger^{\mathfrak{A}(X)}\right)^{-1}(\sigma_{01}) = \langle \sigma_{01}, \sigma_{01} \rangle \cup \langle \sigma_{11}, \sigma_{01} \rangle$$

where

$$\langle \sigma_{01}, \sigma_{11} \rangle = (\sigma_{01}, \sigma_{11}) \cap \left(E_1^2 \sqcup E_2^2 \right)$$

is an open set in $E_1^2 \sqcup E_2^2$. In the same fashion $\langle \sigma_{11}, \sigma_{11} \rangle$, $\langle \sigma_{01}, \sigma_{01} \rangle$ and $\langle \sigma_{11}, \sigma_{01} \rangle$ are open sets in $E_1^2 \sqcup E_2^2$. Also one can see that

$$\left(\dagger^{\mathfrak{A}(\{1\})} \right)^{-1} (\sigma_0) = \langle \sigma_0, \sigma_0 \rangle$$

By writing enough statements, we observe that our sheaf forces that $(\mathfrak{A}(X), \dagger^{\mathfrak{A}(X)})$ is a semigroup. For example, the statement $(\sigma_{11} \dagger \sigma_{01}) \dagger \sigma_{01} = \sigma_{11} \dagger (\sigma_{01} \dagger \sigma_{01})$ is forced in 0 and in every element of X.

The sheaf just constructed also forces that $(\mathfrak{A}, \tau_{\mathfrak{A}}, \dagger^{\mathfrak{A}})$ is a topological semigroup with the topology $\tau_{\mathfrak{A}} = \{\emptyset, \{1\} \times \{1\}, \{1\} \times X, X \times X\} \sqcup \tau_{\mathfrak{A}_1}$. We can write in $\mathcal{L}_t$ that a function is continuous (see below). The above arguments let us conclude that the continuity of $\dagger$ is forced globally: For all $x \in X$

$$\begin{aligned} \mathfrak{A} \Vdash_x \forall \sigma\, \forall \mu\, \forall V (\sigma \dagger \mu \in V \rightarrow & \\ & (\exists U_\sigma (\sigma \in U_\sigma \wedge \exists U_\mu (\mu \in U_\mu \wedge \\ & \quad \forall \eta_1 \forall \eta_2 (\eta_1 \in U_\sigma \wedge \eta_2 \in U_\mu \rightarrow \eta_1 \dagger \eta_2 \in V))))) \end{aligned}$$

Topological properties of interesting dynamical systems can be forced in certain sheaves.

Example 1.3. Let $X = \mathbb{Z}$ be a topological space with a topology τ_X induced by the basis $\{n\mathbb{Z} | n \in \mathbb{Z}\}$. For all $q \in \mathbb{Z}$, let $E_q = S^1$ with the induced topology as a subset of $\mathbb{R}^2$. Elements of S^1 can be written as $x = e^{2\pi\theta i}$ with $\theta \in \mathbb{R}$. We define sections in such a way that they describe periodic functions. For every $x \in S^1$ and every $n \in \mathbb{Z}^+$, let

$$\sigma_x^n : \mathbb{Z} \longrightarrow \bigsqcup_{q \in \mathbb{Z}} E_q$$

$$\sigma_x^n(q) = (q, x e^{2\pi \frac{q}{n} i})$$

The image of each σ_x^n induces a section in $E = \bigsqcup_{q \in \mathbb{Z}} E_q$. Abusing notation, we adopt the name σ_x^n for the section $Im(\sigma_x^n)$.

Every fiber is a topological group under complex multiplication. Observe that for every x, σ_x^1 is a constant function. Observe that[2]

$$\begin{array}{lll} \forall r \in n\mathbb{Z} & \mathfrak{A} \Vdash_r \sigma_x^n = x & \\ \forall r \in q\mathbb{Z} & \mathfrak{A} \Vdash_r \sigma_x^m = \sigma_x^n \text{ with} & q = \mathrm{LCM}(m, n). \end{array}$$

[2]Soon, we will introduce the notion of local forcing. This will allow us to write $\mathfrak{A} \Vdash_{n\mathbb{Z}} \sigma_x^n = x$ and $\mathfrak{A} \Vdash_{q\mathbb{Z}} \sigma_x^m = \sigma_x^n$ with $q = \mathrm{LCM}(m, n)$.

Every fiber is a copy of E_0 as a topological space. The map $\phi_q : E_0 \to E_q$ defined by $\phi_q(x) = x$ is a homeomorphism. Under ϕ_q we can find a 1-1 correspondence between the open sets of E_0 and those of E_q. We interpret our symbols $\mathcal{X}$ for open sets in $\mathfrak{A}$ as the disjoint union of the sets in every fiber that are in correspondence with the open set $\mathcal{X}^{\mathfrak{A}_0}$, through a map ϕ_q. Observe that these sets are also open according to the topology induced by sections: They can be written as the union of the sections σ_x^1 for all $x \in \mathcal{X}^{\mathfrak{A}_0}$.

With this topology in the sheaf and this interpretation for open sets in our language, the dynamics of the circle under complex multiplication for periodic trajectories can be studied. The base space of the sheaf can be thought of as the discrete time and sections as periodic orbits. Let $V = \mathcal{X}^{\mathfrak{A}_0}$ be an open set in E_0. If the sheaf forces that two periodic trajectories are "close" at 0 (corresponding to sections σ_x^n and σ_y^m), then we know that for an open set in $\mathbb{Z}$ they are still close:

$$\begin{aligned}&\text{If } \mathfrak{A} \Vdash_0 \forall\sigma_x^n \forall\sigma_y^m (\sigma_x^n \in V \wedge \sigma_y^m \in V) \text{ then}\\ &\qquad \forall r \in q\mathbb{Z} \quad \mathfrak{A} \Vdash_r \forall\sigma_x^n \forall\sigma_y^m (\sigma_x^n \in V \wedge \sigma_y^m \in V) \quad \text{for } q = LCM(n,m)\end{aligned}$$

The following example describes the continuous cyclic flow in a torus.

Example 1.4. Let $X = S^1$, $E = S^1 \times S^1$ and $p = \pi_1$, be the projection function into the first component. Therefore $E_q = S^1$. Given a set of local coordinates x_i in S_i and a smooth vector field V on E, such that

$$V = V_1 \frac{\partial}{\partial x_1} + V_2 \frac{\partial}{\partial x_2} \qquad\qquad V_1(p) \neq 0 \ \forall p \in S^1,$$

we can take as the set of sections the family of integrable curves of V. Observe that

$$\mathfrak{A} \Vdash_p \sigma \neq \mu \iff \mathfrak{A} \Vdash_X \sigma \neq \mu,$$

for some $p \in S^1$. The open sets of the sheaf can be described as local streams through E. Complex multiplication in every fiber is continuously extended to function between integral curves. Every section can be extended to a global section.

Moving in the same direction as in the theory of Caicedo [2], we now introduce the idea of local forcing in open sets under the equivalence

$$\mathfrak{A} \Vdash_U \phi(\sigma(U), \mathcal{X}^{\mathfrak{A}(U)}) \iff \forall x \in U \, \mathfrak{A} \Vdash_x \phi(\sigma(x), \mathcal{X}^{\mathfrak{A}_x}),$$

where U is an open set of X, and $\mathfrak{A}(U)$ is the sheaf defined on U, which can be seen as a subsheaf of the original one. The concepts already presented allow us to infer the statements for logical connectives that we use in the definition of the local forcing. As expected, our definition of local forcing is an extension of the first order case, hence we have

Definition 1.10 (Local Forcing)**.** *Let U be open in X and $\overline{\sigma} = \sigma_1, \ldots, \sigma_n$. The relation $\mathfrak{A} \Vdash_U \phi(\sigma(U), \mathcal{X}^{\mathfrak{A}(U)})$ is defined by the following statements:*

- *(Atomic first order)*

$$\begin{aligned} \mathfrak{A} \Vdash_U \sigma_1 = \sigma_2 &\iff \sigma_1 \restriction U = \sigma_2 \restriction U \\ \mathfrak{A} \Vdash_U R(\overline{\sigma}) &\iff \langle \sigma_1, \ldots, \sigma_n \rangle(U) \in R^{\mathfrak{A}} \end{aligned}$$

- *(Atomic second order) If $\phi = t \in \mathcal{X}$ then*

$$\mathfrak{A} \Vdash_U t \in \mathcal{X} \iff t^{\mathfrak{A}(U)} \subset \mathcal{X}^{\mathfrak{A}(U)}$$

- *(Logical connectives)*

$$\begin{aligned} \mathfrak{A} \Vdash_U \phi(\overline{\sigma}) \wedge \psi(\overline{\sigma}) &\iff \mathfrak{A} \Vdash_U \phi(\overline{\sigma}) \textit{ and } \mathfrak{A} \Vdash_U \psi(\overline{\sigma}) \\ \mathfrak{A} \Vdash_U \phi(\overline{\sigma}) \vee \psi(\overline{\sigma}) &\iff \textit{There exist open sets } V, W \textit{ such that} \\ & \qquad V \cup W = U \textit{ and } \mathfrak{A} \Vdash_V \phi(\overline{\sigma}) \textit{ and } \mathfrak{A} \Vdash_W \psi(\overline{\sigma}) \\ \mathfrak{A} \Vdash_U \neg\phi(\overline{\sigma}) &\iff \textit{For all } W \subset U, \textit{ with } W \neq \emptyset \;\; \mathfrak{A} \nVdash_W \phi(\overline{\sigma}) \\ \mathfrak{A} \Vdash_U \phi(\overline{\sigma}) \to \psi(\overline{\sigma}) &\iff \textit{For all } W \subset U \\ & \qquad \mathfrak{A} \Vdash_W \phi(\overline{\sigma}) \textit{ then } \mathfrak{A} \Vdash_w \psi(\overline{\sigma}) \end{aligned}$$

- *(First order quantifiers)*

$$\begin{aligned} \mathfrak{A} \Vdash_U \exists\eta\phi(\eta, \overline{\sigma}) &\iff \textit{There exists an open covering } \{U_i\} \textit{ of } U \textit{ and sections} \\ & \qquad \mu_i \in \mathfrak{A}(U_i) \textit{ such that } \mathfrak{A} \Vdash_{U_i} \phi(\mu_i, \overline{\sigma}) \textit{ for all } i. \\ \mathfrak{A} \Vdash_U \forall\eta\phi(\eta, \overline{\sigma}) &\iff \textit{For all } W \subset U \textit{ and every } \mu \textit{ defined in } W \\ & \qquad \mathfrak{A} \Vdash_W \phi(\mu, \overline{\sigma}) \end{aligned}$$

- *(Monadic second order existential quantifier)*
  $\phi(v, \mathcal{C}, \mathcal{X})$ is a negative L_t-formula in $\mathcal{X}$.

$$\begin{aligned} & \mathfrak{A} \Vdash_U \exists\mathcal{X}(t \in \mathcal{X} \wedge \phi(v, \mathcal{C}, \mathcal{X})) \iff \\ & \qquad \textit{there exists a covering } U_i \textit{ of } U \textit{ and open sets } \mathcal{X}_i \subset \mathfrak{A}(U_i) \\ & \qquad\qquad \textit{such that} \\ & \qquad\qquad\qquad \mathfrak{A} \Vdash_{U_i} t \in \mathcal{X}_i \wedge \phi(v, \mathcal{C}, \mathcal{X}_i) \end{aligned}$$

- *(Monadic second order universal quantifier)*

  $\phi(v, \mathcal{C}, \mathcal{X})$ *is a positive* L_t*-formula in* $\mathcal{X}$

$$\begin{aligned} \mathfrak{A} \Vdash_U \forall \mathcal{X}(t \in \mathcal{X} \to \phi(v, \mathcal{C}, \mathcal{X})) \iff \\ \forall W \subset U,\, W \neq \emptyset\, \forall \mathcal{X} \\ \mathfrak{A} \Vdash_W t^{\mathfrak{A}(W)} \in \mathcal{X}^{\mathfrak{A}(W)} \to \phi(v, \mathcal{C}, \mathcal{X}^{\mathfrak{A}(W)}). \end{aligned}$$

Again, we point out that this definition extends the classical notion of local forcing in [2].

Example 1.5. Consider again example 1.3. In this construction is forced that there is a section approaching "arbitrarily close" to a constant function. To show this we take σ_1^0, as a constant in our language. From the definition of local forcing we have

$$\begin{aligned} &\mathfrak{A} \Vdash_{\mathbb{Z}} \forall \mathcal{X} \left(\sigma_1^0 \in \mathcal{X} \to \exists \sigma(\sigma \in \mathcal{X} \wedge \sigma \neq \sigma_1^0)\right) \\ &\iff \text{For all } n \text{ and all } \mathcal{C} \text{ defined in } \mathbb{Z}n \\ &\qquad \mathfrak{A} \Vdash_{\mathbb{Z}n} \left(\sigma_1^0 \in \mathcal{C} \to \exists \sigma(\sigma \in \mathcal{C} \wedge \sigma \neq \sigma_1^0)\right) \\ &\iff \text{If } \mathfrak{A} \Vdash_{\mathbb{Z}n} \sigma_1^0 \in \mathcal{C} \text{ then } \mathfrak{A} \Vdash_{\mathbb{Z}n} \exists \sigma(\sigma \in \mathcal{C} \wedge \sigma \neq \sigma_1^0) \end{aligned}$$

Now, let

$$\tilde{U}_n = \left\{ p \in S^1 \,\middle|\, p = e^{2\pi\theta} \wedge -\frac{1}{n} < \theta < \frac{1}{n} \wedge \theta \neq 0 \right\}$$

for $n \geq 2$, $U_1 = S^1$ and

$$U_n = \sqcup_{\mathbb{Z}} \tilde{U}_n \; .$$

U_n is open in E. For every n there exists m such that

$$\begin{aligned} &\mathfrak{A} \Vdash_{\mathbb{Z}n} \sigma_1^0 \in \mathcal{C} \to \forall \sigma(\sigma \in U_m \to \sigma \in \mathcal{C}) \\ &\mathfrak{A} \Vdash_{\mathbb{Z}n} \sigma_1^0 \in \mathcal{C} \to \forall \sigma(\sigma \in U_m \wedge \sigma \neq \sigma_1^0), \end{aligned}$$

and therefore

$$\mathfrak{A} \Vdash_{\mathbb{Z}n} \sigma_1^0 \in \mathcal{C} \to \exists \sigma(\sigma \in \mathcal{C} \wedge \sigma \neq \sigma_1^0).$$

1.3 The generic model for a sheaf of topological structures

Among the most useful elements of the Model Theory of Sheaves of First Order structures is the Generic Model construction and Generic Model Theorem. In this section, we develop these concepts for our sheaves as natural extensions of the classical statements. First, we present the Maximum principle, which is fundamental in the proof of subsequent theorems.

Theorem 1.2 (Maximum Principle). *If $\mathfrak{A} \Vdash_U \exists \mathcal{X}(t \in \mathcal{X} \wedge \phi(\sigma, \mathcal{X}))$ for given t and σ, then there exists $\mathcal{C}$ defined in an open set W dense in U such that $\mathfrak{A} \Vdash_W t \in \mathcal{C} \wedge \phi(\sigma, \mathcal{C})$.*

Proof. Suppose $\mathfrak{A} \Vdash_U \exists \mathcal{X}(t \in \mathcal{X} \wedge \phi(\sigma, \mathcal{X}))$. This is equivalent to the statement that for each $x \in U$, $\mathfrak{A} \Vdash_x \exists \mathcal{X}(t \in \mathcal{X} \wedge \phi(\sigma, \mathcal{X}))$. Let V_x be a neighborhood of x where is verified that

$$\forall y \in V_x \, \mathfrak{A} \Vdash_y \exists \mathcal{X}(t \in X \wedge \phi(\sigma(y), \mathcal{X}))$$

Then the family $\{V_y | y \in U\}$ is a covering of U and is associated to the family of witnesses $\mathcal{T} = \{\mathcal{Y}^{\mathfrak{A}(V_y)}\}$, such that $\mathfrak{A} \Vdash_{V_y} t \in Y^{\mathfrak{A}(V_y)}$ and $\mathfrak{A} \Vdash_{V_y} \phi(\sigma, \mathcal{Y}^{\mathfrak{A}(V_y)}))$. $\mathcal{T}$ is partially ordered by inclusion. If $\{\mathcal{Y}_i | i \in I\}$ is a chain in $\mathcal{T}$, its union is also in $\mathcal{T}$[3], and by Zorn's lemma there exists a C maximal. We assert that its domain is dense in U. If it was not the case, there would exist an open set $V \subset U$ such that $V \cap dom(C) = \emptyset$. But for some $z \in U$, $V_z \cap V$ is not empty. Therefore, we would be able to define $C^* = C \cup \mathcal{Y}_z^{\mathfrak{A}(V_f \cup V)}$ in contradiction to the maximality of C. □

The next definition will provide us with the generic filter in the topology of $\mathfrak{A}$. Its existence is given by Theorem 1.3. This structure is essential in the construction of the generic model.

Definition 1.11 (Generic Filter). *Let $\mathfrak{A}$ be a sheaf of topological structures on X. If $\mathbb{F}$ is a filter on X such that*

1. *For all $\phi(\sigma, \mathcal{X})$ and for all σ defined on $U \in \mathbb{F}$ holds that there exists $W \in \mathbb{F}$ such that*
   $\mathfrak{A} \Vdash_W \phi(\sigma, \mathcal{X})$ or $\mathfrak{A} \Vdash_W \neg\phi(\sigma, \mathcal{X})$.

[3] this is a consequence of the fact that if $\mathfrak{A} \Vdash_{W_i} \phi(\sigma, Y^{\mathfrak{A}(W_i)})$ for every $i \in I$, then $\mathfrak{A} \Vdash_{\cup W_i} \phi(\sigma, Y^{\mathfrak{A}(\cup W_i)})$.

2. *For all σ defined on $U \in \mathbb{F}$ and for all $\phi(u, \sigma, \mathcal{X})$ if $\mathfrak{A} \Vdash_U \exists u\phi(u, \sigma, \mathcal{X})$, then there exists $W \in \mathbb{F}$ and μ defined in W such that $\mathfrak{A} \Vdash_W \phi(\mu, \sigma, \mathcal{X})$.*

3. *For all $\mathcal{X}$ defined on $U \in \mathbb{F}$, and for all negative $\phi(\sigma, \mathcal{X}, \mathcal{Y})$ and t a term in L_t, if $\mathfrak{A} \Vdash_U \exists \mathcal{Y}(t \in \mathcal{Y} \wedge \phi(\sigma, \mathcal{X}, \mathcal{Y}))$, then there exists $W \in \mathbb{F}$ and $\mathcal{X}'$ defined in W such that $\mathfrak{A} \Vdash_W \phi(t \in \mathcal{X}' \wedge \phi(\sigma, \mathcal{X}, \mathcal{X}'))$.*

then we say that the filter is generic.

Theorem 1.3 (Existence of generic filters). *A maximal filter of open sets on $\mathcal{X}$ is generic for every sheaf of topological structures.*

Proof.
- The first and second properties are the classical conditions and there is nothing new in their proof (details can be found in [2] theorem 5.1).
- (Property 3) By the Maximum Principle, there is an open set W, dense in U, and an open set $\mathcal{C}^{\mathfrak{A}(W)}$ such that $\mathfrak{A} \Vdash_W \phi(t \in \mathcal{C} \wedge \phi(\sigma, \mathcal{X}, \mathcal{C}))$. To proof that $W \in \mathbb{F}$, consider $V \in \mathbb{F}$ and also $V \cap U$. By density $W \cap V \cap U \neq \emptyset$ and therefore $W \cap V \neq \emptyset$. As a consequence there exists filter $\mathbb{F}' \supset \mathbb{F}$ in which $W \in \mathbb{F}'$. By maximality $\mathbb{F} = \mathbb{F}'$.

□

Now, we introduce the idea of the generic model for a sheaf of topological structures.

Definition 1.12 (Generic model). *Let $\mathbb{F}$ be a generic filter on X and $\mathfrak{A}(U) = \{\sigma | dom(\sigma) = U)\}$. We define the* **generic model** *$\mathfrak{A}[\mathbb{F}]$ by*

$$\mathfrak{A}[\mathbb{F}] = \bigsqcup \{\mathfrak{A}(U)/ \sim_{\mathbb{F}} | U \in \mathbb{F}\}$$

wherein $\sigma \sim_{\mathbb{F}} \mu \iff \exists W \in \mathbb{F}$ such that $\sigma \restriction W = \mu \restriction W$. In the same fashion,

$$([\sigma_1], \cdots, [\sigma_n]) \in R^{\mathfrak{A}[\mathbb{F}]} \iff \exists U \in \mathbb{F}\, (\sigma_1, \cdots, \sigma_n) \subset R^{\mathfrak{A}(U)}$$

$$f^{\mathfrak{A}[\mathbb{F}]}([\sigma_1], \cdots, [\sigma_n]) = [f^{\mathfrak{A}(U)}(\sigma_1, \cdots, \sigma_n)]$$

$$[t]^{\mathfrak{A}[\mathbb{F}]} \in \mathcal{X}^{\mathfrak{A}[\mathbb{F}]} \iff \exists U \in \mathbb{F}\, t \in \mathcal{X}^{\mathfrak{A}(U)}.$$

The following theorem provides a connection between what is true in the generic model and what is forced over sections on the sheaf from which it is constructed. As expected, it is a generalization of the theorem of same name in the classical theory.

Theorem 1.4 (Generic Model Theorem)**.** *Let $\mathbb{F}$ be a generic filter on X and $\mathfrak{A}$ a sheaf of topological structures on X. Then*

$$\begin{aligned}\mathfrak{A}[\mathbb{F}] \models \phi([\sigma], \mathcal{X}) &\iff \{x \in X | \mathfrak{A} \Vdash_x \phi^G(\sigma(x), \mathcal{X})\} \in \mathbb{F}. \\ &\iff \exists U \in \mathbb{F} \textit{ such that } \mathfrak{A} \Vdash_U \phi^G(\sigma, \mathcal{X}).\end{aligned}$$

where $\phi^G(\sigma, \mathcal{X})$ is the Gödel translation of ϕ.

Proof. By induction on the complexity of the formulas.

- The proof for first order formulas is the same as that for the classical theorem. Thus, for this part we refer the reader to Theorem 5.2 on [2].

- Second order atomic formula.

$$\begin{aligned}\mathfrak{A}[\mathbb{F}] \models [\sigma] \in \mathcal{C} &\iff \exists U \in \mathbb{F} \quad (\sigma(U)) \subset \mathcal{C}^{\mathfrak{A}} \\ &\iff \exists U \in \mathbb{F}\, \mathfrak{A} \Vdash_U \sigma \in \mathcal{C} \\ &\iff \exists U \in \mathbb{F} \text{ such that } \{z \in U | \mathfrak{A} \Vdash_z \sigma(z) \in \mathcal{C}^{\mathfrak{A}_z}\} \\ &\qquad\qquad \text{is dense in } U. \\ &\iff \exists U \in \mathbb{F}\, \forall y \in U\, \forall W (y \in W \wedge W \subset U) \\ &\qquad\qquad \exists z \in W \mathfrak{A} \Vdash_z \sigma(z) \in \mathcal{C} \\ &\iff \exists U \in \mathbb{F}\, \forall y \in U \mathfrak{A} \nVdash_y \neg\sigma \in \mathcal{C} \\ &\iff \exists U \in \mathbb{F}\, \mathfrak{A} \Vdash_U \neg\neg\sigma(z) \in \mathcal{C}\end{aligned}$$

- (Monadic Second-order Existential Quantifier)

$$\begin{aligned}&\mathfrak{A}[\mathbb{F}] \models \exists \mathcal{X}(t \in \mathcal{X} \wedge \phi(\sigma, \mathcal{X}) \\ &\qquad \iff \mathfrak{A}[\mathbb{F}] \models t \in \mathcal{C} \text{ and } \mathfrak{A}[\mathbb{F}] \models \phi(\sigma, \mathcal{C}) \text{ for some open} \\ &\qquad\qquad \text{set } \mathcal{C} \text{ in } \mathfrak{A}[\mathbb{F}] \\ &\qquad \iff \exists U \in \mathbb{F}\, \mathfrak{A} \Vdash_U \neg\neg t \in \mathcal{C} \text{ and } \exists V \in \mathbb{F}\, \mathfrak{A} \Vdash_V \phi^G(\sigma, \mathcal{C}) \\ &\qquad \iff \exists W \in \mathbb{F}\, \mathfrak{A} \Vdash_W \neg\neg t \in \mathcal{C} \wedge \phi^G(\sigma, \mathcal{C}) \\ &\qquad \iff \exists W \in \mathbb{F}\, \mathfrak{A} \Vdash_W \exists \mathcal{X} \left((t \in \mathcal{X})^G \wedge \phi^G(\sigma, \mathcal{X})\right) \\ &\qquad \iff \exists W \in \mathbb{F}\, \mathfrak{A} \Vdash_W (\exists \mathcal{X} t \in \mathcal{X} \wedge \phi(\sigma, \mathcal{X}))^G\end{aligned}$$

  Note that in the fourth step we have used the third property from our definition of generic filter. Also, we can show from the definition of local forcing that $\mathfrak{A} \Vdash_U \exists \mathcal{X} \phi(\mathcal{X}) \iff \mathfrak{A} \Vdash_U \neg\forall \mathcal{X} \neg\phi(\mathcal{X})$.

- (Monadic Second-order Universal Quantifier)

$$\begin{aligned}
\mathfrak{A}[\mathbb{F}] \models \forall \mathcal{X}(t \in \mathcal{X} \to & \phi(\sigma, \mathcal{X})) \\
& \iff \mathfrak{A} \models (t \in \mathcal{C} \to \phi(\sigma, \mathcal{C})) \\
& \qquad \text{for all open set } \mathcal{C} \text{ in } \mathfrak{A}[\mathbb{F}] \\
& \iff \exists U \in \mathbb{F}\ \mathfrak{A} \Vdash_U (t \in \mathcal{C} \to \phi(\sigma, \mathcal{C}))^G \\
& \qquad \text{for all open set } \mathcal{C} \text{ in } \mathfrak{A}(U) \\
& \iff \exists U \in \mathbb{F}\ \mathfrak{A} \Vdash_U \forall \mathcal{X}(t \in \mathcal{X} \to \phi(\sigma, \mathcal{X}))^G \\
& \iff \exists U \in \mathbb{F}\ \mathfrak{A} \Vdash_U (\forall \mathcal{X}\,[t \in \mathcal{X} \to \phi(\sigma, \mathcal{X})])^G
\end{aligned}$$

□

Example 1.6. Consider again the sheaf in example 1.3. In this case the maximal filter is the whole topology without the empty set. In this case $Th(\mathfrak{A}_0) = Th(\mathfrak{A}[\mathbb{F}])$. To see this observe that $0 \in \mathbb{Z}n$ for all n and therefore $\mathfrak{A}[\mathbb{F}] \models \phi \iff \mathfrak{A} \Vdash_0 \phi^G \iff \mathfrak{A}_0 \models \phi$ as can be proved by induction.

Example 1.7. Let us study the generic model of the sheaf introduced in example 1.4. First, observe that $\mathfrak{A}[\mathbb{F}]$ is a proper subset of the set of local integrable curves. In fact, every element in $\mathfrak{A}[\mathbb{F}]$ is a global section in E: For any element $[\sigma] \in \mathfrak{A}[\mathbb{F}]$, $U = dom(\sigma) \in \mathbb{F}$ and there exists a global integral curve μ in E such that[4]

$$\mathfrak{A} \Vdash_U \sigma = \mu \qquad \text{therefore} \qquad \mathfrak{A}[\mathbb{F}] \models [\sigma] = [\mu]$$

This result leads to the conclusion that every maximal filter of open sets in S^1 generates the same universe for $\mathfrak{A}[\mathbb{F}]$. Complex multiplication can be extended continuously to a binary function in the set of sections that is well defined in $\mathfrak{A}[\mathbb{F}]$. We can show that this binary function, that we denote by juxtaposition of sections, is left continuous in $\mathfrak{A}[\mathbb{F}]$: Let μ be a global section and $[\mu]$ its corresponding equivalence class. Then

$$\begin{aligned}
\mathfrak{A}[\mathbb{F}] \models & \phi_\mu \\
& \models \forall \sigma \forall \mathcal{X}\big([\sigma][\mu] \in \mathcal{X} \to \exists \mathcal{Y}_\sigma([\sigma] \in \mathcal{Y}_\sigma \wedge \forall [\eta]([\eta] \in \mathcal{Y}_\sigma \to [\eta][\mu] \in \mathcal{X}))\big) \\
\iff & \exists U \in \mathbb{F} \text{ such that}
\end{aligned}$$

$$\mathfrak{A} \Vdash_U \phi_\mu^G$$

[4]Observe that if ϕ is an L_t sentence without the universal quantifier $\mathfrak{A} \Vdash \phi^G \iff \mathfrak{A} \Vdash \phi$.

Now, given $\mathcal{C}$ and σ if there exists U such that $\mathfrak{A} \Vdash_U \sigma\mu \in \mathcal{C}$ we can take $U \in \mathbb{F}$ since μ is a global integral curve. Since the vector field is smooth, it is true that $\mathcal{Y}_\sigma = \mathcal{C}\mu^{-1}$ is open in $\mathfrak{A}$, and,

$$\begin{aligned}
&\mathfrak{A} \Vdash_U \eta \in \mathcal{Y}_\sigma \to \eta\mu \in \mathcal{C} \\
&\mathfrak{A} \Vdash_U (\eta \in \mathcal{Y}_\sigma \to \eta\mu \in \mathcal{C})^G \\
&\mathfrak{A} \Vdash_U \forall\eta(\eta \in \mathcal{Y}_\sigma \to \eta\mu \in \mathcal{C})^G \\
&\mathfrak{A} \Vdash_U \big(\sigma \in \mathcal{Y}_\sigma\big)^G \wedge \big(\forall\eta(\eta \in \mathcal{Y}_\sigma \to \eta\mu \in \mathcal{C})\big)^G \\
&\mathfrak{A} \Vdash_U \big(\exists\mathcal{Y}(\sigma \in \mathcal{Y}_\sigma \wedge \forall\eta(\eta \in \mathcal{Y}_\sigma \to \eta\mu \in \mathcal{C}))\big)^G
\end{aligned}$$

from the initial hypothesis

$$\mathfrak{A} \Vdash_U \big(\sigma\mu \in \mathcal{C}\big)^G \to \big(\exists\mathcal{Y}(\sigma \in \mathcal{Y}_\sigma \wedge \forall\eta(\eta \in \mathcal{Y}_\sigma \to \eta\mu \in \mathcal{C}))\big)^G$$

Since $\mathcal{C}$ and σ are arbitrary

$$\begin{aligned}
&\mathfrak{A} \Vdash_U \forall\sigma\forall\mathcal{X}\big(\sigma\mu \in \mathcal{X} \to \exists\mathcal{Y}(\sigma \in \mathcal{Y}_\sigma \wedge \forall\eta(\eta \in \mathcal{Y}_\sigma \to \eta\mu \in \mathcal{X}))\big)^G \\
&\mathfrak{A} \Vdash_U \phi_\mu^G
\end{aligned}$$

Many topological properties (such as being disconnected or Hausdorff), are easily stated in $\mathcal{L}_t$ but not all of them (e.g. being a regular space). Then, we can construct topological models with certain topological properties from sheaves whose Gödel translation is forced only in some open set of X. In that way, the generic model theorem for topological structures "transforms" local properties of the sheaf into global properties of the generic model.

1.4 The completion of the generic model

Some topological spaces are metrizable, and metric spaces whose metric is defined in terms of an inner product are well understood structures in analysis. A Hilbert space is a good example of the last. By definition, it is a vector space with a positive definite inner product $\langle\,,\rangle$, which is complete under the corresponding norm. We are interested in doing some model theory from Cauchy sequences in a Hilbert space. The underlying idea is that if we know something about the sentences that the elements of a sequence satisfy, we must be able to infer something about the type of the limit element when the sequence converges. The model theory just developed is in place to give us some answers about this problem. We will need to define the open sets in terms of an inner product.

Most of what is needed to define open sets from an inner product has been previously stated and is part of the basic literature in the field. Then we can review these ideas. Since, Hilbert spaces associate vector spaces with an algebraic field ($\mathbb{R}$ or $\mathbb{C}$ mainly) by means of the inner product, we propose the following vocabulary intended to be interpreted in a two-sorted structure:

$$\mathcal{L}_H = \mathcal{L}_{vec} \cup \mathcal{L}_{ring} \cup \{\bar{\cdot}\} \cup \{\langle\, , \rangle\} = \\ \{+_H, 0_H, h_v, +_{\mathbb{C}}, -_{\mathbb{C}}, \cdot_{\mathbb{C}}, 0_{\mathbb{C}}, 1_{\mathbb{C}}, \bar{\cdot}, \langle\, , \rangle | v \in \mathbb{C}\}. \quad (1.1)$$

In the present work we are interested in the set of complex numbers as the field of our structures. Symbols identified with the subscript H are going to be defined on a sort that supports a vector space. Symbols identified with the subscript $\mathbb{C}$ are defined in a sort that must satisfy the axioms of the algebraic closed field of complex numbers. In addition, since we cannot define the complex conjugate of a complex number v (it will not be invariant under automorphisms) we have introduced a symbol function $\bar{\cdot}$ to provide complex conjugate numbers (see below). Finally, the symbol $\langle\, , \rangle$ is a binary function to be defined with arguments on the vector space sort and with images on the algebraic field sort. We expect to interpret the symbol $\langle\, , \rangle$ as the inner product of a two-sorted Hilbert space structure. In fact, the following sentences must be satisfied by this structure

- $\langle\, , \rangle$ is a linear transformation
  - **AXI 1** $\forall x \forall y \forall z(\langle x +_H y, z\rangle = \langle x, y\rangle +_{\mathbb{C}} \langle x, z\rangle)$
  - **AXI 2** $\forall x \forall y \forall z(\langle x, y +_H z\rangle = \langle x, y\rangle +_{\mathbb{C}} \langle x, z\rangle)$
  - **AXI 3** $\forall x \forall y(\langle h_v x, y\rangle = v \cdot_{\mathbb{C}} \langle x, y\rangle)$
  - **AXI 4** $\forall x \forall y(\langle x, h_v y\rangle = \overline{v} \cdot_{\mathbb{C}} \langle x, y\rangle)$
- $\langle\, , \rangle$ is an hermitian form
  - **AXI 5** $\forall x \forall y(\langle x, y\rangle = \overline{\langle y, x\rangle})$

In the following, x, y or z are used to represent variables defined on the vector space sort, while c, r and s will be used for the case of the algebraic field sort (e.g., complex numbers).

Fact 1.1. The real numbers are definable in the structure $\{\mathbb{C}, +_\mathbb{C}, -_\mathbb{C}, \cdot_\mathbb{C}, 0_\mathbb{C}, 1_\mathbb{C}, \bar{\cdot} | v \in \mathbb{C}\}$

Proof. take $\phi_\mathbb{R}(c) = (c -_\mathbb{C} \bar{c} = 0_\mathbb{C})$ □

Fact 1.2. The relation of strict order $<$ is definable in the set of real numbers.

In fact, we can define the set of non-negative reals by considering the formula $\phi_+(c) = \exists r(r \cdot_\mathbb{C} r = c)$ and $\phi_\mathbb{R}(c)$. Also, the formula

$$\phi_{inv}(r, c) = \wedge(r +_\mathbb{C} c = 0_\mathbb{C})$$

for c real, defines its additive inverse. Thus, we can extend our language to include the real set as a definable set and the strict order relation $<$, as a definable binary relation given by $a < b \iff \phi_\mathbb{R}(a) \wedge \phi_\mathbb{R}(b) \wedge \phi_+(b-a) \wedge \neg b = a$. Then, we can rewrite our axioms of order in L_t restricted to real numbers, as follows

- $<$ is a dense linear ordering without end points in $\mathbb{R}$
- $\langle\, ,\rangle$ is an hermitian positive definite form
  - **AXI 6** $\langle 0_H, 0_H \rangle = 0_\mathbb{C}$
  - **AXI 7** $\forall x((\neg x = 0_H \to 0_\mathbb{C} < \langle x, x\rangle)$

We stress the fact that the set of reals numbers is invariant under the two possible automorphisms of the set complex numbers when a function which assigns to every complex number its conjugate is included in our language[18]. Indeed, real numbers are pointwise invariant and then we can define the square root of a real positive number c by the formula

$$\phi_{sqrt}(r; c) = \phi_+(r) \wedge r \cdot_\mathbb{C} r = c.$$

Since this is unique, we can define a function $\sqrt{\ldots}$ given by

$$\forall r \forall c(((\phi_+(c) \wedge \phi_{sqrt}(r; c)) \to \sqrt{c} = r) \vee (\neg\phi_+(c) \wedge \sqrt{c} = 0_\mathbb{C}),$$

where r and c are elements of $\mathbb{C}$. We include this function in our language.

Further, we need to include enough axioms to define the induced norm $||\cdot||$ and the induced metric $d(\, ,\,)$. We define $||x|| = \sqrt{\langle x, x\rangle}$ as usual.

Fact 1.3. The following properties are logical consequences of our definitions and set of axioms,

- $\forall x(||x|| \geq 0_{\mathbb{C}})$
- $\forall x(||x|| = 0_{\mathbb{C}} \leftrightarrow x = 0_H)$
- $\forall x(||h_v x|| = |v|\,||x||)$
- $\forall x \forall y(||x +_H y|| \leq ||x|| +_{\mathbb{C}} ||y||)$

At this point we must state our main drawbacks in the axiomatization of Hilbert Spaces in $\mathcal{L}_t$:

1. Given a structure with the complex numbers as the algebraic field, any superstructure of it is going to have a nonstandard set of complex numbers. Since the classical theory is based on the properties of standard complex and real numbers, these superstructures might introduce elements that are not part of the theory. It is also possible, that the generic model has a nonstandard model of $\mathbb{C}$, with properties strongly dependent on the ultrafilter and on the way sections are defined.

2. Completeness under Cauchy sequences seems to be a property only expressible in a language as big as $L_{\omega_1\omega_1}$.

The problem about nonstandard copies of $\mathbb{C}$ is a problem that may not be solved without strong conditions in our sheaves. *This problem is our main motivation to study sheaves of metric structures* in the model theory of Ben Yaacov, Berenstein, Henson, and Usvyatsov [9] in following chapters. Still, the present approach may give rise to interesting nonstandard analysis.

We may try to get rid of the second problem by constructing from the generic model its completion under Cauchy sequences. We must follow the standard procedure to generate the factor space of Cauchy sequences modulo the space of null sequences. Let

$$\{[\sigma]_i | i \in \omega\} \tag{1.2}$$

be a sequence of elements in the generic model. We will say that it is a null sequence if and only if for all $\epsilon > 0$ there exists $N > 0$ such that

$$n > N \rightarrow d([0], [\sigma_n]) < \epsilon, \tag{1.3}$$

where $d(,)$ is the metric induced by the norm of the space. Let

$$S(A[\mathbb{F}]) = S \tag{1.4}$$

be the set of all Cauchy sequences of elements of $A[\mathbb{F}]$. Now we define the equivalence relation $\equiv$ between the elements of S by stating that

$$\{[\mu]_i | i \in \omega\} \equiv \{[\gamma]_i | i \in \omega\}$$
if and only if there exists $\{[\sigma]_i | i \in \omega\}$ null sequence such that
$$[\mu]_i = [\gamma]_i + [\sigma]_i, \text{ for every } i.$$

Consider now the set of all equivalence classes from S, and denote this by $\overline{\mathfrak{A}[\mathbb{F}]}$. For the sake of clarity, we introduce the following notation for the equivalence classes

$$\overline{\mu} = \{[\mu]_i | i \in \omega\}/_{\equiv}. \tag{1.5}$$

We can prove that $\overline{\mathfrak{A}[\mathbb{F}]}$ is an inner product space. For the special case of the inner product, just take as a definition

$$\langle \overline{\mu}, \overline{\gamma} \rangle^{\overline{\mathfrak{A}[\mathbb{F}]}} = \{\langle [\mu]_i, [\gamma]_i \rangle | i \in \omega\}/_{\equiv}. \tag{1.6}$$

That the set of complex numbers is closed under Cauchy sequences guarantees that our definition of the inner product is well defined.

Now we can prove the following proposition which is strongly related the information from the generic model and its completion.

Proposition 1.1. *Let $\mathfrak{A}[\mathbb{F}]$ and $\overline{\mathfrak{A}[\mathbb{F}]}$ be as before. Then the function,*

$$t : \mathfrak{A}[\mathbb{F}] \to \overline{\mathfrak{A}[\mathbb{F}]}$$
$$t[\sigma_x] = \overline{\sigma_x}$$

is an elementary immersion, under the additional conditions that for every relation symbol R in the language , $R^{\overline{\mathfrak{A}[\mathbb{F}]}}$ is open in the product topology and $t(\mathfrak{A}[\mathbb{F}])$ is open in the topology of $\overline{\mathfrak{A}[\mathbb{F}]}$.

Proof. Note that it is an immersion for linear operators in view of the above definition for the inner product in $\overline{\mathfrak{A}[\mathbb{F}]}$, and the additional conditions for predicates. Then

- (Inner product) Suppose $\langle [\sigma_x], [\sigma_y] \rangle = [c^{xy}]$

$$\begin{aligned}
(\langle t[\sigma_x], t[\sigma_y] \rangle) &= (\langle \overline{\sigma_x}, \overline{\sigma_y} \rangle) \\
&= \{\langle [\sigma_x], [\sigma_y] \rangle | i \in \omega\}/_{\equiv} \\
&= \{\langle [c^{xy}] \rangle | i \in \omega\}/_{\equiv} \\
&= \overline{c^{xy}} \\
&= t[c^{xy}]
\end{aligned}$$

- For a linear operator H, suppose $H[\sigma_x] = [\sigma_y]$.

$$\begin{aligned} H(t[\sigma_x]) &= H\overline{\sigma_x} \\ &= \{H[\sigma_x] | i \in \omega\}/_{\equiv} \\ &= \{[\sigma_y] | i \in \omega\}/_{\equiv} \\ &= \overline{\sigma_y} = t[\sigma_y] \end{aligned}$$

We now show that $t(\mathfrak{A}[\mathbb{F}])$ is elementary substructure of $\overline{\mathfrak{A}[\mathbb{F}]}$, by means of the Tarski-Vaught test.

- First, for any relation symbol R we have,

$$\begin{aligned} \overline{\mathfrak{A}[\mathbb{F}]} \models \exists x R(x) &\iff \text{there exists } a^{\overline{\mathfrak{A}[\mathbb{F}]}} \in R^{\overline{\mathfrak{A}[\mathbb{F}]}} \\ &\quad \text{since } t(\mathfrak{A}[\mathbb{F}]) \text{ is a dense set on } \overline{\mathfrak{A}[\mathbb{F}]} \\ &\iff \text{there exists } b^{\overline{\mathfrak{A}[\mathbb{F}]}} \in R^{\overline{\mathfrak{A}[\mathbb{F}]}} \text{ and } b^{\overline{\mathfrak{A}[\mathbb{F}]}} \in t(\mathfrak{A}[\mathbb{F}]) \\ &\iff \overline{\mathfrak{A}[\mathbb{F}]} \models b^{\overline{\mathfrak{A}[\mathbb{F}]}} \in R^{\overline{\mathfrak{A}[\mathbb{F}]}}. \end{aligned}$$

- For the atomic formula $a \in \mathcal{C}$, where $a \in t(\mathfrak{A}[\mathbb{F}])$.

$$\overline{\mathfrak{A}[\mathbb{F}]} \models a \in \mathcal{C} \iff \overline{\mathfrak{A}[\mathbb{F}]} \models a \in \mathcal{C} \cap t(\mathfrak{A}[\mathbb{F}]).$$

since $\mathcal{C} \cap t(\mathfrak{A}[\mathbb{F}])$ is open in $t(\mathfrak{A}[\mathbb{F}])$ and $\overline{\mathfrak{A}[\mathbb{F}]}$. Then, let $\phi(x, \mathcal{X})$ be a L_t-formula where $x \in t(\mathfrak{A}[\mathbb{F}])$ is given.

$$\begin{aligned} \overline{\mathfrak{A}[\mathbb{F}]} \models \exists \mathcal{X}(x \in \mathcal{X} \wedge \phi(x, \mathcal{X})) &\iff \text{there exists } \mathcal{C} \text{ open such that} \\ &\quad \overline{\mathfrak{A}[\mathbb{F}]} \models x \in \mathcal{C} \text{ and } \overline{\mathfrak{A}[\mathbb{F}]} \models \phi(x, \mathcal{C}) \\ &\iff \text{there exists open set } \mathcal{C}' \text{ in } t(\mathfrak{A}[\mathbb{F}]) \\ &\quad \overline{\mathfrak{A}[\mathbb{F}]} \models x \in \mathcal{C}' \text{ and } \overline{\mathfrak{A}[\mathbb{F}]} \models \phi(x, \mathcal{C}') \\ &\iff \overline{\mathfrak{A}[\mathbb{F}]} \models (x \in \mathcal{C}' \wedge \phi(x, \mathcal{C}')). \end{aligned}$$

□

We realize that the additional conditions used in the above proposition are strong. Weakening them does not seem to be an easy task. First, every open set of a substructure is also an open subset of the superstructure if and only if $t(\mathfrak{A}[\mathbb{F}])$ is open. Second, for relations which are not open it might not be possible to find a witness in the substructure. On the other hand, the condition of being open for a predicate R should immediately imply that they are clopen, since $\neg R$ is also a predicate. This may lead to some semantic problems in the boundary of $\neg R$ and R.

1.5 Truth extension

The above difficulties are also present when one is trying to make inferences about what is true at the completion from what is true in the generic model. One might want to "extend" the generic model theorem to its completion, and by means of that, state something about the new elements of the completion. The following theorem gives a partial answer to that point.

Theorem 1.5.
- $\overline{\mathfrak{A}[\mathbb{F}]} \models \overline{\gamma} = \overline{\mu} \iff$ *For all* i*, there exists* U_i *in* $\mathbb{F}$ *such that* $\mathfrak{A} \Vdash_{U_i} [\gamma_i = \sigma_i + \mu_i]^G \wedge [d(0, \sigma_i) < i^{-1}]^G$.

- $\overline{\mathfrak{A}[\mathbb{F}]} \models f\overline{\gamma} = \overline{\mu} \iff$ *For all* i*, there exists* U *in* X *such that* $\mathfrak{A} \Vdash_U [f\gamma_i = \sigma_i + \mu_i]^G \wedge [d(0, \sigma_i) < i^{-1}]^G$

- $\overline{\mathfrak{A}[\mathbb{F}]} \models \overline{\gamma} \in R \iff$ *For all* $i, k \in \omega$*, there exists* U_{ik} *open in* $\mathbb{F}$*,* μ_k *and* σ_{ik} *such that*

$$\mathfrak{A} \Vdash_{U_{ik}} (\mu_k + \sigma_{ik} = \gamma_i)^G \wedge (\mu_k \in R)^G \wedge (d(0, \sigma_{ik}) < i^{-1})^G \wedge (d(0, \sigma_{ik}) < k^{-1})^G$$

Proof. The above statements follow from our definition of the completion of $\mathfrak{A}[\mathbb{F}]$ and from the generic model theorem.

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$$\begin{aligned}
\overline{\mathfrak{A}[\mathbb{F}]} \models \overline{\gamma} = \overline{\mu} &\iff \{[\gamma_i] | i \in \omega\}/_{\equiv} = \{[\mu_i] | i \in \omega\}/_{\equiv} \\
&\iff \text{there exists } [\sigma_i] \text{ null sequence such that} \\
&\quad \mathfrak{A}[\mathbb{F}] \models [\gamma_i] = [\mu_i] + [\gamma_i] \text{ for all } i \in \omega. \\
&\iff \exists U_i \in \mathbb{F}\, \mathfrak{A} \Vdash_{U_i} [\gamma_i = \sigma_i + \mu_i]^G \wedge [d(0, \sigma_i) < i^{-1}]^G \\
&\quad \text{for all } i \in \omega
\end{aligned}$$

The proof is similar for $\mathfrak{A} \Vdash_U [f\gamma_i = \sigma_i + \mu_i]^G$.

- $$\begin{aligned}
\overline{\mathfrak{A}[\mathbb{F}]} \models \overline{\gamma} \in R \iff & \text{There exist } \overline{\mu}_k \text{ and } \overline{\eta_k} \text{ such that} \\
& \overline{\mu}_k \in t(\mathfrak{A}[\mathbb{F}]) \\
& \overline{\mu}_k \in R^{\overline{\mathfrak{A}[\mathbb{F}]}} \\
& \overline{\mu}_k + \overline{\sigma}_k = \overline{\gamma} \text{ for all } k \\
& \overline{\sigma}_k \text{ is a null sequence} \\
\iff & \text{There exist } [\mu]_k, [\eta_k] \text{ and } [\nu_i]_k \text{ such that in } \mathfrak{A}[\mathbb{F}] \text{ satisfies} \\
& [\mu]_k \in R^{\mathfrak{A}[\mathbb{F}]} \\
& [\mu]_k + [\eta]_k + [\nu_i]_k = [\gamma_i] \text{ for all } i \\
& [\eta]_k \text{ and } [\nu_i]_k \text{ are a null sequences} \\
\iff & \exists U_{ik} \in \mathbb{F}\, \exists \mu_k \text{ and } \sigma_{ik} \text{ such that } \mathfrak{A} \text{ forces in } U_{ik} \\
& (\mu_k + \sigma_{ik} = \gamma_i)^G \\
& (\mu_k \in R)^G \\
& (d(0, \sigma_{ik}) < i^{-1})^G \wedge (d(0, \sigma_{ik}) < k^{-1})^G \\
& \text{ for all } i, k \in \omega.
\end{aligned}$$

□

Note that in the last part we have used the fact that R is clopen in $\overline{\mathfrak{A}[\mathbb{F}]}$ and that the proof make use of the generic model theorem.

Chapter 2

Sheaves of Metric Structures

In the previous chapter we described the theory of the completion of the generic topological model. We ultimately introduced conditions on the properties of the predicate symbols. Even under those conditions the connection between what is forced in the sheaf and what is true in the completion was established only for atomic formulas and could not be extended inductively to all formulas in the language. In this chapter we give up some of the basic properties of topological model theory and we construct our sheaves over a model theory more appropriate for the description of metric structures, developed by Ben Yaacov, Berenstein, Henson and Usvyatsov [9].

2.1 Basic Ideas in Model Theory of Metric Structures

This section does not intend to be self contained. It only introduces notation and describes basic features of the model theory for metric structures in a rather informal way. No proof is given of any statement. A complete treatment of these ideas can be found in [9].

In what follows M is a complete bounded metric space, with metric function $d : M \to [0, 1]$. Then M^n is also a metric space with the uniform metric.

Definition 2.1. (Language for metric structures).

1. *An n-ary predicate symbols R is a uniformly continuous function from M^n to $[0, 1]$.*

2. *An n-ary function symbol f is a uniformly continuous function from M^n to M.*

A modulus of uniform continuity is a function $\Delta : (0,1] \to [0,1]$. The uniform continuity of f and R is witnessed by modulus of uniform continuity Δ_f and Δ_R such that

$$d(x,y) < \Delta_f(\varepsilon) \Rightarrow d(f(x), f(y)) \leq \varepsilon \text{ and}$$
$$d(x,y) < \Delta_R(\varepsilon) \Rightarrow |R(x) - R(y)| \leq \varepsilon,$$

for any $\varepsilon \in (0,1]$.

Definition 2.2. *A metric structure $\mathcal{M}$ is a tuple*

$$(M, R_i, f_j, c_k, \Delta_{R_i}, \Delta_{f_j} | i \in I, j \in J, k \in K)$$

with c_k a constant symbol and I, J and K index sets.

Definition 2.3 ($\mathcal{L}$-terms)**.** .

- *Each variable and constant symbol is an $\mathcal{L}$-term.*
- *If f is an n-ary function symbols and $t_1, \ldots, t_n$ are $\mathcal{L}$-terms, then $f(t_1, \ldots, t_n)$ is an $\mathcal{L}$-term.*

Definition 2.4 ($\mathcal{L}$- Formulas)**.** *If $t_1, \ldots, t_n$ are $\mathcal{L}$-terms, then*

- *$d(t_1, t_2)$ and $R(t_1, \ldots, t_n)$ are (atomic) formulas.*
- *If $u : [0,1]^n \to [0,1]$ is a continuous function and $\phi_1, \ldots, \phi_n$ are $\mathcal{L}$-formulas, then $u(\phi_1, \ldots, \phi_n)$ is an $\mathcal{L}$-formula.*
- *If ϕ is an $\mathcal{L}$-formula and x is a variable, then $\sup_x \phi$ and $\inf_x \phi$ are $\mathcal{L}$-formulas.*

Thus, "logical connectives" are continuous functions from $[0,1]^n$ to $[0,1]$ and the supremum and infimum play the role of quantifiers.

Terms are interpreted exactly as in first-order logic. Besides, every sentence has associated with it a real number (its value) in the interval $[0,1]$. An inductive definition of the value of a sentence can be given (see [9] page 16). We now have two important definitions

Definition 2.5 (Logical equivalence)**.** *Let $\phi(x)$ and $\psi(x)$ be $\mathcal{L}$-formulas. They are logically equivalent if $\phi^{\mathcal{M}}(a) = \psi^{\mathcal{M}}(a)$ for every $\mathcal{M}$ and every $a \in M$.*

Definition 2.6 (Logical distance)**.** *Let $\phi(x)$ and $\psi(x)$ be $\mathcal{L}$-formulas. The logical distance between ϕ and ψ is $sup_{a \in M} |\phi^{\mathcal{M}}(a) - \psi^{\mathcal{M}}(a)|$.*

Interesting topological properties in the space of types arise from the latter definition.

Semantics in metric structures differs from that in classical structures by the fact that the satisfaction relation is defined on $\mathcal{L}$-conditions rather than on $\mathcal{L}$-formulas, as shown below.

Definition 2.7 ($\mathcal{L}$- Conditions)**.** *Let $\phi(x)$ and $\psi(y)$ be $\mathcal{L}$-formulas. Expressions of the form $\phi(x) \leq \psi(y)$, $\phi(x) < \psi(y)$, $\phi(x) \geq \psi(y)$, $\phi(x) > \psi(y)$ are $\mathcal{L}$-conditions. If ϕ and ψ are sentences then we say that the condition is closed.*

If we let $E(x, y)$ be the condition $\phi(x) < \psi(y)$, then we write $\mathcal{M} \models E(a, b)$ if $\phi^{\mathcal{M}}(a) < \psi(b)^{\mathcal{M}}$ for $a, b \in M$, and we say E is true of a and b in $\mathcal{M}$. Similar definitions apply for the other conditions.

Thus, every basic classical model theoretic concept (e.g. $\mathcal{L}$-theory, elementary equivalence, ...) is defined in this model theory almost rephrasing its classical definition in terms of conditions. In addition, classical theorems (Tarski-Vaught Test, Löwenheim-Skolem, compactness, ...) have analogs in this setting. However, no further mention of these elements will be made in the rest of this chapter. Still, it may be worth studying the ultraproduct construction of metric models to see an alternative approach to the problem faced in this chapter (see [9] pages 22-29).

One might argue that the set of connectives is too big. However, the set $\mathcal{F} = \{0, 1, x/2, \dot{-}\}$, where 0 and 1 are taken as constant functions and the truncated subtraction $\dot{-}$ is given by Definition 2.8 below; is uniformly dense in the set of all connectives. By that we mean that for every ε and any connective $f(t_1, \ldots, f_n)$ there is a connective $g(t_1, \ldots, t_n)$ in $\overline{\mathcal{F}}$(the closure under composition of F) such that

$$|f(t_1, \ldots, t_n) - g(t_1, \ldots, t_n)| \leq \varepsilon.$$

Thus, we can limit the set of connectives that we use in building formulas to the set $\mathcal{F}$. We close this section defining the special connective $\dot{-}$ mentioned above and some other important connectives.

Definition 2.8. *Let x and y be real numbers in the closed interval $[0, 1]$. We define the functions*

1. $\dot{-}$:$[0, 1]^2 \to [0, 1]$ *by;*

$$x \dot{-} y = \begin{cases} (x - y) & \text{if } x > y \\ 0 & \text{otherwise} \end{cases}$$

2. min:$[0,1]^2 \to [0,1]$ *by*

$$\min(x,y) = x \dot{-} (x \dot{-} y)$$

3. max:$[0,1]^2 \to [0,1]$ *by*

$$\max(x,y) = 1 \dot{-} (\min(1 \dot{-} x, 1 \dot{-} y))$$

2.2 The metric sheaf and forcing relations on conditions

We now investigate the conditions under which a sheaf of metric structures over a topological space is well defined, with a forcing relation and semantic properties similar to those found in classical and topological sheaves. These properties are the tools required to construct new metric structures whose theories are controlled by the forcing relation and the topology of the base space. In what follows we assume that a metric language $\mathcal{L}$ is given and we omit the prefix $\mathcal{L}$ when talking about $\mathcal{L}$-formulas, $\mathcal{L}$-conditions and others.

Definition 2.9 (Sheaf of metric structures)**.** *Let $\mathcal{X}$ be a topological space. A sheaf of metric structures $\mathfrak{A}$ on $\mathcal{X}$ consists of:*

1. *A sheaf (E,p) over X.*

2. *For all x in X we associate a metric structure* $(\mathfrak{A}_x, d) = \Big(E_x, \{R_i^{(n_i)}\}_x, \{f_j^{(m_j)}\}_x, \{c_k\}_x, \Delta_{R_{i,x}}, \Delta_{f_{i,x}}, d, [0,1]\Big)$, *where E_x is the fiber $p^{-1}(x)$ over x, and the following conditions hold:*

   (a) *(E_x, d_x) is a complete, bounded metric space of diameter* 1*.*

   (b) *For all i, $R_i^{\mathfrak{A}} = \bigcup_{x\in X} R_i^{\mathfrak{A}_x}$ is a continuous function according to the topology of $\bigcup_{x\in X} E_x^{n_j}$.*

   (c) *For all j, the function $f_j^{\mathfrak{A}} = \bigcup_x f_j^{\mathfrak{A}_x} : \bigcup_x E_x^{m_j} \to \bigcup_x E_x$ is a continuous function according to the topology of $\bigcup_{x\in X} E_x^{m_j}$.*

   (d) *For all k, the function $c_k^{\mathfrak{A}} : X \to E$ such that $c_k^{\mathfrak{A}}(x) = c_k^{\mathfrak{A}_x}$ is a continuous global section.*

   (e) *We define the premetric function $d^{\mathfrak{A}}$ by $d^{\mathfrak{A}} = \bigcup_{x\in X} d_x : \bigcup_{x\in X} E_x^2 \to [0,1]$, where $d^{\mathfrak{A}}$ is a continuous function according to the topology of $\bigcup_{x\in X} E_x^2$.*

(f) *For all* i, $\Delta^{\mathfrak{A}}_{R_i} = \inf_{x\in X}(\Delta^{\mathfrak{A}_x}_{R_i})$ *with the condition that* $\inf_{x\in X} \Delta^{\mathfrak{A}_x}_{R_i}(\varepsilon) > 0$ *for all* $\varepsilon > 0$.

(g) *For all* j, $\Delta^{\mathfrak{A}}_{f_j} = \inf_{x\in X}(\Delta^{\mathfrak{A}_x}_{f_i})$ *with the condition that* $\inf_{x\in X} \Delta^{\mathfrak{A}_x}_{f_i}(\varepsilon) > 0$ *for all* $\varepsilon > 0$.

(h) *The set* $[0,1]$ *is a second sort and is provided with the usual metric.*

The induced function $d^{\mathfrak{A}}$ is not necessarily a metric nor a pseudometric. Thus, we cannot expect the sheaf defined in Definition 2.9 to be a metric structure, in the sense of continuous logic. Indeed, we want to build the local semantics on the sheaf so that the truth continuity lemma is still valid, i.e, we want to be able to state that for a given sentence ϕ, if ϕ is true at some $x \in X$, then there must exist a neighborhood U of x such that for every y in U, ϕ is also true. In order to accomplish this task, first we have to consider that semantics in continuous logic is not defined on formulas but on conditions. Since the truth of the condition "$\phi < \varepsilon$" for ε small can be thought as a good approximation to the notion of ϕ being true in a first order model, one might choose this as the condition to be forced in our metric sheaf. Therefore, for a given real number $\varepsilon \in (0,1)$, we consider conditions of the form $\phi < \varepsilon$ and $\phi > \varepsilon$. Our first result comes from investigating to what extend the truth in a fiber "spreads" into the sheaf.

Lemma 2.1.
- *Let* ε *be a real number,* $x \in X$, ϕ *an* $\mathcal{L}$*-formula composed only of the logical connectives* $\max, \min, \dot{-}$ *and the quantifier* $\inf$. *If* $\mathfrak{A}_x \models \phi(\sigma(x)) < \varepsilon$ *then there exists an open neighborhood* U *of* x*, such that for every* y *in* U, $\mathfrak{A}_y \models \phi(\sigma(y)) < \varepsilon$.
- *Let* ε *be a real number,* $x \in X$, ϕ *an* L*-formula composed only of the logical connectives* $\max, \min, \dot{-}$ *and the quantifier* $\sup$. *If* $\mathfrak{A}_x \models \phi(\sigma(x)) > \varepsilon$ *then there exists an open neighborhood* U *of* x*, such that for every* y *in* U, $\mathfrak{A}_y \models \phi(\sigma(y)) > \varepsilon$.

Proof.
- Atomic formulas
  - If $\mathfrak{A}_x \models d_x(\sigma_1(x), \sigma_2(x)) < \varepsilon$ and $d_x(\sigma_1(x), \sigma_2(x)) = r$, let $V = (r-\delta, r+\delta)$ where $r + \delta < \varepsilon$. Since $d^{\mathfrak{A}}$ is a continuous function, $(d^{\mathfrak{A}})^{-1}(V)$ is open in $\bigcup_{x\in X} E^2_x$, and then $p(\langle \sigma_1, \sigma_2 \rangle \cap (d^{\mathfrak{A}})^{-1}(V))$ is the desired set.
  - For an n-ary predicate R, if $\mathfrak{A}_x \models R(\sigma_1(x), \dots, \sigma_n(x)) < \varepsilon$ and $R(\sigma_1(x), \dots, \sigma_n(x)) = r$, let $V = (r-\delta, r+\delta)$ where $r+\delta < \varepsilon$. Since R is continuous function on E^n, then the set $W = R^{-1}(V)$ must be open in $\bigcup_{x\in X} E^n_x$. Consider $p(\langle \sigma_1, \dots, \sigma_n \rangle \cap W)$.

  - Similar arguments show the statement for the opposite inequalities.

- Logical connectives

  - Consider the max function and let ϕ and ψ be formulas. Suppose $\mathfrak{A}_x \models \max(\phi, \psi) < \varepsilon$, then $\mathfrak{A}_x \models \phi < \varepsilon$, $\mathfrak{A}_x \models \psi < \varepsilon$ and by induction hypothesis, there exist open sets V and W such that for every $y \in V$ $\mathfrak{A}_y \models \phi < \varepsilon$ and for every $z \in W$ $\mathfrak{A}_z \models \psi < \varepsilon$. Then, for every element in the intersection set, it must be true that $\max(\phi, \psi) < \varepsilon$.
  - If $\mathfrak{A}_x \models \max(\phi, \psi) > \varepsilon$ then either $\mathfrak{A}_x \models \phi > \varepsilon$ or $\mathfrak{A}_x \models \psi > \varepsilon$. Without loss of generality, we may assume $\mathfrak{A}_x \models \phi > \varepsilon$ holds. By induction hypothesis there exists a $U \ni x$ such that for all y in U, $\mathfrak{A}_y \models \phi > \varepsilon$ and then $\mathfrak{A}_y \models \max(\phi, \psi) > \varepsilon$.
  - Consider the min function and let ϕ and ψ be formulas. Suppose $\mathfrak{A}_x \models \min(\phi, \psi) < \varepsilon$, then $\mathfrak{A}_x \models \phi < \varepsilon$ or $\mathfrak{A}_x \models \psi < \varepsilon$. Without loss of generality, we may assume $\mathfrak{A}_x \models \phi < \varepsilon$ holds. By induction hypothesis, there exists V neighborhood of x such that $y \in V$ $\mathfrak{A}_y \models \phi < \varepsilon$. Hence, $\min(\phi, \psi) < \varepsilon$ must hold in for all $y \in V$.
  - If $\mathfrak{A}_x \models \min(\phi, \psi) > \varepsilon$ then $\mathfrak{A}_x \models \phi > \varepsilon$ and $\mathfrak{A}_x \models \psi > \varepsilon$. By induction hypothesis there exists open neighborhoods U and V of x such that for all $y \in U$ $\mathfrak{A}_Y \models \phi > \varepsilon$ and for all $z \in V$ $\mathfrak{A}_z \models \psi > \varepsilon$. Take the intersection set.
  - Consider the formula $1 \dot{-} \phi$. $\mathfrak{A}_x$ models $1 \dot{-} \phi < \varepsilon$ iff $\mathfrak{A}_x$ models $\phi > 1 - \varepsilon$. By induction hypothesis there exists an open set W such that for all y in W, $\mathfrak{A}_y$ models $\phi > 1 - \varepsilon$ and then models $1 \dot{-} \phi < \varepsilon$.
  - If $\mathfrak{A}_x \models 1 \dot{-} \phi > \varepsilon$ then $\mathfrak{A}_x \models \phi < 1 - \varepsilon$ and by induction hypothesis the open set we are looking for exists.
  - Let $\mathfrak{A}_x \models \phi \dot{-} \psi < \varepsilon$. We consider the following cases

    1. If in addition $\mathfrak{A}_x \models \phi < \psi$, let $\phi = s$ and $\psi = r$. Then $\mathfrak{A}_x \models \phi < (r + s)/2$ and $\mathfrak{A}_x \models \psi > (r + s)/2$. By induction hypothesis there exist U and V open in X, such that for all y in U, $\mathfrak{A}_y \models \phi < (r + s)/2$ and for all z in V, $\mathfrak{A}_z \models \psi > (r + s)/2$. For every element in $U \cap V$, $\phi < \psi$ most hold.
    2. If $\mathfrak{A}_x \models \phi = \psi$, then $r = s$. i) If $r \neq 1$ and $r \neq 0$, then $\mathfrak{A}_x \models \phi < r + \varepsilon/2$ and $\mathfrak{A}_x \models \psi > r - \varepsilon/2$. Use the induction

hypothesis. ii) If $r = 1$, consider that $\mathfrak{A}_x \models \phi = 1$ and $\mathfrak{A}_x \models \psi > 1 - \varepsilon$. iii) If $r = 0$, induction hypothesis applied to $\mathfrak{A}_x \models \phi < \varepsilon$ gives the open set we are looking for.

3. In case, $\mathfrak{A}_x \models \phi \dot{-} \psi < \varepsilon$ and $\mathfrak{A}_x \models \phi > \psi$ the proof is given by cases. i) if $s \neq 1$ and $r \neq 0$, then $\mathfrak{A}_x \models \phi < s + \varepsilon/2$ and $\mathfrak{A}_x \models > r - \varepsilon/2$. ii) if $s = 1$, $\mathfrak{A}_x \models 1 \dot{-} \psi < \varepsilon$. iii) If $r = 0$ then $\mathfrak{A}_x \models \phi < \varepsilon$.

– Suppose $\mathfrak{A}_x \models \phi \dot{-} \psi > \varepsilon$. Take $\delta = s - r - \varepsilon$. Then $A_x \models \phi > s + \delta/2$ and $\mathfrak{A}_x \models \psi < r - \delta/2$. By induction hypothesis, there exist U and V such that for all $y \in U$, $\mathfrak{A}_y \models \phi > s + \delta/2$ and for all $z \in V$, $\mathfrak{A}_y \models \psi < r - \delta/2$. Then for all $y \in U \cap V$, A_y models $\phi \dot{-} \psi > \varepsilon$.

- Quantifier

  – If $\mathfrak{A}_x \models \inf_\sigma \phi(\sigma(x)) < \varepsilon$, then there exists a section ν such that $\mathfrak{A}_x \models \phi(\nu(x)) < \varepsilon$. By induction hypothesis, there exists an open set V such that for every $y \in V$ $\mathfrak{A}_y \models \phi(\nu(y)) < \varepsilon$ and consequently $\mathfrak{A}_y \models \inf_\sigma \phi(\sigma(y)) < \varepsilon$.

□

The reader must realize that the statement of the above Lemma for logical connectives is a simple consequence of the fact that every connective is a continuous function. Thus, a formula $\phi(x_1, ..., x_2)$ constructed inductively only from connectives and atomic formulas is a composition of continuous functions and therefore continuous. If $\mathfrak{A}_x \models \phi(\sigma_1(x), \ldots, \sigma_n(x)) < \varepsilon$, then $p\left(\langle \sigma_1, \ldots, \sigma_2 \rangle \cap \phi^{-1}[0, \varepsilon)\right)$ is an open set in X satisfying that for all y in it $\mathfrak{A}_y \models \phi(\sigma_1(y), \ldots, \sigma_n(y)) < \varepsilon$. What is important about the proof of Lemma 2.1 is that this shows us how to define the point forcing relation inductively for the set of connectives considered.

Definition 2.10 (Point Forcing). *Given a metric sheaf $\mathfrak{A}$ defined on some topological space X, and a real number $\varepsilon \in (0, 1)$ we define the relation $\Vdash_x$ on the set of conditions $\phi < \varepsilon$ and $\phi > \varepsilon$ (ϕ an $\mathcal{L}$-sentence) and for an element $x \in X$ by induction as follows*

Atomic formulas

- $\mathfrak{A} \Vdash_x d(\sigma_1, \sigma_2) < \varepsilon \iff d_x(\sigma_1(x), \sigma_2(x)) < \varepsilon$
- $\mathfrak{A} \Vdash_x d(\sigma_1, \sigma_2) > \varepsilon \iff d_x(\sigma_1(x), \sigma_2(x)) > \varepsilon$
- $\mathfrak{A} \Vdash_x R(\sigma_1, \ldots, \sigma_n) < \varepsilon \iff R^{\mathfrak{A}_x}(\sigma_1(x), \ldots, \sigma_n(x)) < \varepsilon$

- $\mathfrak{A} \Vdash_x R(\sigma_1, \ldots, \sigma_n) > \varepsilon \iff R^{\mathfrak{A}_x}(\sigma_1(x), \ldots, \sigma_n(x)) > \varepsilon$

Logical connectives

- $\mathfrak{A} \Vdash_x \max(\phi, \psi) < \varepsilon \iff \mathfrak{A} \Vdash_x \phi < \varepsilon$ *and* $\mathfrak{A} \Vdash_x \psi < \varepsilon$
- $\mathfrak{A} \Vdash_x \max(\phi, \psi) > \varepsilon \iff \mathfrak{A} \Vdash_x \phi > \varepsilon$ *or* $\mathfrak{A} \Vdash_x \psi > \varepsilon$
- $\mathfrak{A} \Vdash_x \min(\phi, \psi) < \varepsilon \iff \mathfrak{A} \Vdash_x \phi < \varepsilon$ *or* $\mathfrak{A} \Vdash_x \psi < \varepsilon$
- $\mathfrak{A} \Vdash_x \min(\phi, \psi) > \varepsilon \iff \mathfrak{A} \Vdash_x \phi > \varepsilon$ *and* $\mathfrak{A} \Vdash_x \psi > \varepsilon$
- $\mathfrak{A} \Vdash_x 1\dot{-}\phi < \varepsilon \iff \mathfrak{A} \Vdash_y \phi > 1\dot{-}\varepsilon$
- $\mathfrak{A} \Vdash_x 1\dot{-}\phi > \varepsilon \iff \mathfrak{A} \Vdash_y \phi < 1\dot{-}\varepsilon$
- $\mathfrak{A} \Vdash_x \phi\dot{-}\psi < \varepsilon \iff$ *One of the following holds*
  i) $\mathfrak{A} \Vdash_x \phi < \psi$
  ii) $\mathfrak{A} \nVdash_x \phi < \psi$ *and* $\mathfrak{A} \nVdash_x \phi > \psi$.
  iii) $\mathfrak{A} \Vdash_x \phi > \psi$ *and* $\mathfrak{A} \Vdash_x \phi < \psi + \varepsilon$.
- $\mathfrak{A} \Vdash_x \phi\dot{-}\psi > \varepsilon \iff \mathfrak{A} \Vdash_x \phi > \psi + \varepsilon$.

Quantifiers

- $\mathfrak{A} \Vdash_x \inf_\sigma \phi(\sigma) < \varepsilon \iff$ *There exists a section* μ *such that* $\mathfrak{A} \Vdash_x \phi(\mu) < \varepsilon$.
- $\mathfrak{A} \Vdash_x \inf_\sigma \phi(\sigma) > \varepsilon \iff$ *There exists an open set* $U \ni x$ *and a real number* $\delta_x > 0$ *such that for every* $y \in U$ *and every section* μ *defined on* y, $\mathfrak{A} \Vdash_y \phi(\mu) > \varepsilon + \delta_x$
- $\mathfrak{A} \Vdash_x \sup_\sigma \phi(\sigma) < \epsilon \iff$ *There exists an open set* $U \ni x$ *and a real number* δ_x *such that for every* $y \in U$ *and every section* μ *defined on* y $\mathfrak{A} \Vdash_y \phi(\mu) < \varepsilon - \delta_x$.
- $\mathfrak{A} \Vdash_x \sup_\sigma \phi(\sigma) > \epsilon \iff$ *There exist a section* μ *defined on* x *such that* $\mathfrak{A} \Vdash_x \phi(\mu) > \varepsilon$

The above definition and the previous lemma lead to the equivalence between $\mathfrak{A} \Vdash_x \inf_\sigma(1\dot{-}\phi) > 1\dot{-}\varepsilon$ and $\mathfrak{A} \Vdash_x \sup_\sigma \phi < \varepsilon$. In addition, we can state the truth continuity lemma for the forcing relation on sections as follows.

Lemma 2.2. *Let* $\phi(\sigma)$ *be a* $\mathcal{F}-$*restricted formula. Then*

1. *if* $\mathfrak{A} \Vdash_x \phi(\sigma) < \varepsilon$ *iff there exist* U *open neighborhood of* x *in* X *such that* $\mathfrak{A} \Vdash_y \phi(\sigma) < \varepsilon$ *for all* $y \in U$.

2. *if* $\mathfrak{A} \Vdash_x \phi(\sigma) > \varepsilon$ *iff there exist* U *open neighborhood of* x *in* X *such that* $\mathfrak{A} \Vdash_y \phi(\sigma) > \varepsilon$ *for all* $y \in U$.

We can also define the point forcing relation for non-strict inequalities by

- $\mathfrak{A} \Vdash_x \phi \leq \varepsilon$ iff $\mathfrak{A} \nVdash_x \phi > \varepsilon$ and
- $\mathfrak{A} \Vdash_x \phi \geq \varepsilon$ iff $\mathfrak{A} \nVdash_x \phi < \varepsilon$,

for $\mathcal{F}-$restricted formulas. This definition let us show the following proposition

Proposition 2.1. *Let* $0 < \varepsilon' < \varepsilon$ *be real numbers. Then*

1. *If* $\mathfrak{A} \Vdash_x \phi(\sigma) \leq \varepsilon'$ *then* $\mathfrak{A} \Vdash_x \phi(\sigma) < \varepsilon$.

2. *If* $\mathfrak{A} \Vdash_x \phi(\sigma) \geq \varepsilon$ *then* $\mathfrak{A} \Vdash_x \phi(\sigma) > \varepsilon'$.

Proof. By induction on the complexity of formulas. □

At this point, we go back to the problem of defining a metric in the set of sections for a metric sheaf. The fact that sections may have different domains brings additional difficulties to the problem of defining such a function with the triangle inequality holding for an arbitrary triple. However, we do not need to consider the whole set of sections of a sheaf but only those whose domain is in a filter of open sets (as will be evident in the construction of the "Metric Generic Model"). One may consider a construction of such a metric by defining the ultraproduct and the ultralimit for an ultrafilter of open sets. However, the ultralimit may not be unique since E is not always a compact set in the topology defined by the set of sections. In fact, it would only be compact if every fiber is finite. Besides, it may not be the case that the ultraproduct is complete. Thus, we proceed in a different way by observing that a pseudometric can be defined for the set of sections with domain in a given filter.

Lemma 2.3. *Let* $\mathbb{F}$ *be a filter of open sets. For all sections* σ *and* μ *with domain in* $\mathbb{F}$*, define the family* $\mathbb{F}_{\sigma\mu} = \{U \cap dom(\sigma) \cap dom(\mu) | U \in \mathbb{F}\}$. *Then the function*

$$\rho_{\mathbb{F}}(\sigma, \mu) = \inf_{U \in \mathbb{F}_{\sigma\mu}} \sup_{x \in U} d_x(\sigma(x), \mu(x))$$

is a pseudometric in the set of sections σ *such that* $dom(\sigma) \in \mathbb{F}$.

Proof. We prove the triangle inequality. Let σ_1, σ_2 and σ_3 be sections with domains in $\mathbb{F}$, and let V be their intersection. Then it is true that

$$\sup_{x\in V} d_x(\sigma_1(x),\sigma_2(x)) \le \sup_{x\in V} d_x(\sigma_1(x),\sigma_3(x)) + \sup_{x\in V} d_x(\sigma_3(x),\sigma_2(x)),$$

and since $\sup_{x\in A} f(x) \le \sup_{x\in B} f(x)$ whenever $A \subset B$,

$$\inf_{W\in\mathbb{F}_{\sigma_1\sigma_2}} \sup_{x\in W} d_x(\sigma_1(x),\sigma_2(x)) \le$$

$$\inf_{W\in\mathbb{F}_{\sigma_1\sigma_2}} \left(\sup_{x\in W} d_x(\sigma_1(x),\sigma_3(x)) + \sup_{x\in W} d_x(\sigma_3(x),\sigma_2(x)) \right).$$

Given an $\varepsilon > 0$, there exist W' and W'' such that

$$\sup_{x\in W'} d_x(\sigma_1(x),\sigma_3(x)) < \inf_{W\in\mathbb{F}_{\sigma_1,\sigma_3}} \sup_{x\in W} d_x(\sigma_1(x),\sigma_3(x)) + \varepsilon/2$$
$$\sup_{x\in W''} d_x(\sigma_3(x),\sigma_2(x)) < \inf_{W\in\mathbb{F}_{\sigma_2,\sigma_3}} \sup_{x\in W} d_x(\sigma_3(x),\sigma_2(x)) + \varepsilon/2.$$

Therefore,

$$\sup_{x\in W'\cap W''} d_x(\sigma_1(x),\sigma_3(x)) + \sup_{x\in W'\cap W''} d_x(\sigma_3(x),\sigma_2(x)) <$$
$$\inf_{W\in\mathbb{F}_{\sigma_1\sigma_2}} \sup_{x\in W} d_x(\sigma_1(x),\sigma_3(x)) + \inf_{W\in\mathbb{F}_{\sigma_2\sigma_3}} \sup_{x\in W} d_x(\sigma_3(x),\sigma_2(x)) + \varepsilon.$$

Since $W'\cap W''$ is in $\mathbb{F}_{\sigma_1\sigma_2}$ and ε was chosen arbitrary, the triangle inequality holds for $\rho_{\mathbb{F}}(\sigma,\mu)$. □

In the following, whenever we talk about a filter $\mathbb{F}$ in X we will be considering a filter of open sets. For any pair of sections σ , μ with domains in a filter we define the binary relation $\sim_{\mathbb{F}}$ by saying that $\sigma \sim_{\mathbb{F}} \mu$ if and only if $\rho_{\mathbb{F}}(\sigma,\mu) = 0$. This is an equivalence relation and the quotient space is therefore a metric space with the metric function $d_{\mathbb{F}}$ defined by

$$d_{\mathbb{F}}([\sigma],[\mu]) = \rho_{\mathbb{F}}(\sigma,\mu).$$

The quotient space provided with the metric $d_{\mathbb{F}}$ is the metric space associated to the filter $\mathbb{F}$. If the filter is a principal filter and the topology of the base space X is given by a metric, then the associated metric space of that filter is complete. In fact completeness would be a trivial consequence of the fact that sections are continuous and bounded in case of a σ-complete filter (X

being a metric space). However, principal filters are not interesting from the semantic point of view and σ-completeness might not hold for filters or even ultrafilters of open sets. The good news is that we can still guarantee completeness in certain kind of ultrafilters.

Theorem 2.1. *Let $\mathfrak{A}$ be a sheaf of metric structures defined over a regular topological space X. Let $\mathbb{F}$ be an ultrafilter of regular open sets. Then, the induced metric structure in the quotient space $\mathfrak{A}[\mathbb{F}]$ is complete under the induced metric.*

In order to prove this theorem we need to state a few useful lemmas.

Lemma 2.4. *Let A and B be two regular open sets (i.e. open sets U such that $\operatorname{int}(\overline{U}) = U$). If $A \setminus B \neq \emptyset$ then $\operatorname{int}(A \setminus B) \neq \emptyset$.*

Proof. If $x \in A \setminus B$ and $int(A \setminus B) = \emptyset$, then $x \in \overline{B}$ and $A \subset \overline{B}$. Therefore $A \subset \operatorname{int}(\overline{B}) = B$ which is in contradiction to the initial hypothesis. □

Lemma 2.5. *Let σ and μ be sections of a given metric sheaf, such that $dom(\sigma) \cap dom(\mu) \neq \emptyset$. If for some x in the intersection of the domains $\sigma(x) \neq \mu(x)$, then $\sigma \restriction V \neq \mu \restriction V$ for some open set $V \ni x$.*

Proof. This is a consequence of $d^{\mathfrak{A}}$ being continuous. If $d_x(\sigma(x), \mu(x)) > \varepsilon$ for some ε, then $\langle \sigma, \mu \rangle \cap (d^{\mathfrak{A}})^{-1}(\varepsilon, 1]$ is open in $\bigcup_{x \in X} E_x^2$. Let V be the projection $p(\langle \sigma, \mu \rangle \cap (d^{\mathfrak{A}})^{-1}(\varepsilon, 1]$. □

Lemma 2.6. *Let $\mathbb{F}$ be a filter and $\{\sigma_n\}$ be a Cauchy sequence of sections according to the pseudometric $\rho_{\mathbb{F}}$ all of them defined in an open set U in $\mathbb{F}$. Then*

1. *There exists a limit function μ_∞ not necessarily continuous defined on U such that $\lim_{n\to\infty} \rho_{\mathbb{F}}(\sigma_n, \mu_\infty) = 0$.*

2. *If X is a regular topological space and $\operatorname{int}(\mu_\infty) \neq \emptyset$, there exists an open set $V \subset U$, such that $\mu_\infty \restriction V$ is continuous.*

Proof.
1. This follows from the fact that $\{\sigma_n(x)\}$ is a Cauchy sequence in the complete metric space (E_x, d_x). Then let μ_∞ be equal to the set $\{\lim_{n\to\infty} \sigma_n(x) | x \in U\}$.

2. Consider the set of points e in μ_∞ such that there exists a section η defined in some open neighborhood of $x \in U$, with $\eta(x) = e$ and $\eta \subset \sigma_\infty$. Let V be the projection set in X of that set of points e. This is an open subset of U and $\mu_\infty \restriction V$ is a section.

□

We can now prove Theorem 2.1.

Proof. Let $\{[\sigma^m] | m \in \omega\}$ be a Cauchy sequence in the associated metric space of an ultrafilter of regular open sets $\mathbb{F}$. If the limit exists, it is unique and the same for every subsequence. Thus, we define the subsequence $\{[\mu^k] | k \in \omega\}$ by making $[\mu^k]$ equal to $[\sigma^m]$ for the minimum m such that for all $n \geq m$, $d_{\mathbb{F}}([\sigma^m], [\sigma^n]) < k^{-1}$. Since $d_{\mathbb{F}}([\mu^k], [\mu^{k+1}]) < k^{-1}$, for every pair $(k, k+1)$, there exists U_k open set, such that

$$\sup_{x \in U_k} d_x(\mu^k(x), \mu^{k+1}(x)) < k^{-1}.$$

Let $W_1 = U_1$, $W_m = \cap_{i=1}^m U_k$ and define a function μ_∞ on W_1 as follows.

- If $x \in W_k \setminus W_{k+1}$ for some k, let $\mu_\infty(x) = \mu^k(x)$.
- Otherwise, if $x \in W_k$ for all k, we can take $\mu_\infty(x) = \lim_{k\to\infty} \mu^k(x)$.

The function μ_∞ might not be a section but, based on the above construction, one can find a suitable restriction σ_∞ that is indeed a section but defined on a smaller domain. We show this by analyzing different cases.

1. If $W_1 = W_k$ for all k, $\bigcap W_k = W_1$ then for all x in W_1, $\sigma_\infty(x) = \lim_{k\to\infty} \mu^k(x)$.

   (a) Suppose $\mathrm{int}(\mu_\infty) = \emptyset$. Let $\tilde{B}_1 = W_1$. For every x in B_1 choose a section η_x, such that $\eta_x(x) = \mu_\infty(x)$ and by the nature of $d^{\mathfrak{A}}$, the set $\tilde{B}_k = p(\langle \eta_x, \mu^k \rangle \cap (d^{\mathfrak{A}})^{-1}[0, k^{-1}))$ for $k \geq 2$ is an open neighborhood of x. Consider $\bigcap_{k\in\omega} \tilde{B}_k$. It is clear that this set is not empty and that $\mathrm{int}(\bigcap_{k\in\omega} \tilde{B}_k) = \emptyset$ as we assumed that $\mathrm{int}(\mu_\infty) = \emptyset$. Since the base space is regular, there exists a local basis on x consisting of regular open sets. We can define a family $\{B_k\}$ of open regular set so that

   - $B_1 := \tilde{B}_1$
   - $B_k \subset \tilde{B}_k$
   - $B_{k+1} \subset B_k$
   - $x \in B_k$

   Let $C_1 := B_1 = W_1$. For all $k \geq 2$, define $C_{k+1} \subset C_k \cap B_{k+1}$ with the condition that C_{k+1} is a regular open set and let $V_k \subset C_k \setminus C_{k+1}$ be some regular open set such that

   $$\overline{V}_k \cap \overline{C_k \setminus C_{k+1}} \setminus \mathrm{int}(C_k \setminus C_{k+1}) = \emptyset,$$

(if $C_{k+1} \subsetneq C_k$, this is possible by Lemma 2.4; if $C_{k+1} = C_k$ let $V_k = \emptyset$) i.e. the closure of V_k does not contain any point in the boundary of $C_k \setminus C_{k+1}$ (Use Lemma 2.4 and the fact that X is regular). Then $\bigcap_{k\in\omega} C_k \supset \{x\}$. Now, if necessary, we renumber the family V_k so that all the empty choices of V_k are removed from this listing. Let $\Gamma = \Gamma_{odd} := \bigcup_{k=1}^{\infty} V_{2k-1}$ and observe that this is an open regular set:

$$\overline{\Gamma} = \overline{\bigcup_{k\in\omega} V_{2k-1}} = \bigcup_{k\in\omega} \overline{V_{2k-1}}$$
$$\operatorname{int}\big(\overline{\Gamma}\big) = \operatorname{int}\Big(\bigcup_{k\in\omega} \overline{V_{2k-1}}\Big) = \bigcup_{k\in\omega} \operatorname{int}(\overline{V_{2k-1}}) = \Gamma.$$

For the first equality observe that if $z \in \overline{\bigcup_{k\in\omega} V_{2k-1}}$ then $z \in \overline{V}_{2l-1}$ for only one l since $\overline{V_n} \cap \overline{V_m} = \emptyset$ for $m \neq n$. In the second line, if $z \in \operatorname{int}\big(\bigcup_{k\in\omega} \overline{V_{2k-1}}\big)$ then every open set containing z is a subset of $\bigcup_{k\in\omega} \overline{V_{2k-1}}$ and again since $\overline{V_n} \cap \overline{V_m} = \emptyset$ for $m \neq n$ they are all subsets of a unique V_{2l-1}.

If Γ is an element of $\mathbb{F}$ then we can define the section σ_∞ in the open regular set Γ by $\sigma_\infty \restriction V_{2k-1} := \mu^{2k-1} \restriction V_{2k-1}$. This is a limit section of the original Cauchy sequence.

Now, consider the family $\mathbb{G}$ of all Γs that can be defined as above for the same element x in W_1 and for the same family $\{C_k\}$. $\mathbb{G}$ is partially ordered by inclusion. Consider a chain $\{\Gamma_i\}$ in $\mathbb{G}$. Observe that $\bigcup_i \Gamma_i$ is an upper bound for this and that $\bigcup_i \Gamma_i$ is regular, since

$$\bigcup_i \Gamma_i \subset \bigcup_i \operatorname{int}\Big(\overline{C_{2i-1} \setminus C_{2i}}\Big)$$
$$\overline{C_{2i-1} \setminus C_{2i}} \,\cap\, \overline{C_{2i+1} \setminus C_{2i+2}} = \emptyset$$

Thus, by Zorn Lemma there is a maximal element Γ_{max}. This intersects every element in the ultrafilter and therefore is an element of the same, otherwise we would be able to construct $\tilde{\Gamma}_{max} \supsetneq \Gamma_{max}$. Suppose that there exists $A \in \mathbb{F}$ such that $\Gamma_{max} \cap A = \emptyset$, then we could repeat the above arguments taking an element x' in $W_1' := A \cap W_1 = A \cap C_1$, finding $\Gamma' \subset W_1'$ with $\Gamma' \cap \Gamma_{max} = \emptyset$. Take $\tilde{\Gamma}_{max} = \Gamma' \cup \Gamma_{max}$.

(b) If $\operatorname{int}(\mu_\infty) \neq \emptyset$, then $U = p(\operatorname{int}(\mu_\infty))$ is an open subset of W_1. Observe that $W_1 \setminus U$ is an open set that contains all possible points of discontinuity of μ_∞. If some regular open set $V \subset U$ is an element of the ultrafilter, then $\mu_\infty \restriction V$ is a limit section. If that is not the case, $V = \mathrm{int}(X \setminus U)$ is in the ultrafilter and $V \cap W_1$ is an open regular set where μ_∞ is discontinuous at every point. Proceed as in case 1a.

2. If there exists N such that for all $m > N$ $W_N = W_m$, we rephrase the arguments used in 1a, this time defining σ_∞ in a subset of W_N. Also, if $\bigcap_{k\in\omega} W_k$ is open and nonempty we follow the same arguments in 1b.

3. If for all k $W_{k+1} \neq W_k$ and $\mathrm{int}(\bigcap_{k\in\omega} W_k) \neq \emptyset$, let $W_1' = \mathrm{int}(\bigcap_{k\in\omega} W_k)$ and use the same ideas than in cases 1a and 1b.

4. If $\mathrm{int}(\bigcap_{k\in\omega} W_k) = \emptyset$, for all k such that $W_k \setminus W_{k+1} \neq \emptyset$ define σ_∞ on some open regular set $\overline{V}_k \subset \mathrm{int}(W_k \setminus W_{k+1})$ so that $\sigma_\infty \restriction V_k = \mu^k \restriction V_k$. Then, σ_∞ is defined in $\bigcup_{k\in\omega} V_k$, and repeat the ideas used in case 1a.

Note that for all μ^k

- If $\sigma_\infty(x) = \mu^{k+n}(x)$, then $d_x(\sigma_\infty(x), \mu^k(x)) < k^{-1}$ for x in the common domain.

- If $\sigma_\infty(x) = \lim_{n\in\omega} \mu^n(x)$, then there exists N such that for $m > N$ $d_x(\sigma_\infty(x), \mu^m(x)) < k^{-1}$ and taking $m > k$, by the triangle inequality $d_x(\sigma_\infty(x), \mu^k(x)) < 2k^{-1}$.

This shows that

$$\sup_{x\in W_k \cap \bigcup V_n} d_x(\sigma_\infty(x), \mu^k(x)) < 2(k-1)^{-1}$$

and then σ_∞ is a limit section. Finally, check that $p \circ \sigma^\infty = Id_{dom(\sigma_\infty)}$. □

We would like to generalize the above result to a bigger set of filters. Looking back to the proof of Theorem 2.1 we can extract from it the additional properties that a filter should obey in order to be complete. This analysis motivates our next definition of generic filter.

Definition 2.11 (Generic Filter for Metric Structures)**.** *Let X be a topological space and $\mathbb{F}$ be a filter of open sets of a given subclass $\mathcal{C}$ in X. We say that $\mathbb{F}$ is generic if*

1. *For any open set U there exist $V \subset U$ such that $V \in \mathcal{C}$.*

2. *Let $\{U_i\}$ be a countable family of open sets in $\mathcal{C}$ such that for all $i \neq j$ $\overline{U_i} \cap \overline{U_j} = \emptyset$, then $\bigcup U_i \in \mathcal{C}$.*

3. *For any open set U in $\mathcal{C}$, either $U \in \mathbb{F}$ or $X \setminus \overline{U} \in \mathbb{F}$*

4. *For any pair U, V in $\mathbb{F}$, if $U \setminus V \neq \emptyset$ then there exists $W \subset U \setminus V$ such that $W \in \mathcal{C}$.*

Note that the first two conditions in the above definition are conditions only on the class $\mathcal{C}$ and that the first one of them implies that there must exist a basis for the topology of X all whose elements are of class $\mathcal{C}$. Finally, observe that an ultrafilter of regular open sets in a regular space is generic.

Before studying the semantics of the quotient space of a generic filter, we define the relation $\Vdash_U$ of local forcing in an open set U for a sheaf of metric structures. The definition is intended to make the following statements about local and point forcing to be valid

$$\mathfrak{A} \Vdash_U \phi(\sigma) < \varepsilon \iff \forall x \in U \; \mathfrak{A} \Vdash_x \phi(\sigma) < \delta \text{ and} \tag{2.1}$$

$$\mathfrak{A} \Vdash_U \phi(\sigma) > \delta \iff \forall x \in U \; \mathfrak{A} \Vdash_x \phi(\sigma) > \varepsilon, \tag{2.2}$$

for some $\delta < \varepsilon$. That this is possible is a consequence of the truth continuity lemma.

Definition 2.12 (Local forcing for Metric Structures). *Let $\mathfrak{A}$ be a Sheaf of metric structures defined in X, ε a positive real number, U and open set in X, and $\sigma_1, \ldots, \sigma_n$ sections defined in U. If ϕ is an $\mathcal{F}$- restricted formula the relations $\mathfrak{A} \Vdash_U \phi(\sigma) < \epsilon$ and $\mathfrak{A} \Vdash_U \phi(\sigma) > \epsilon$ are defined by the following statements*

Atomic formulas

- $\mathfrak{A} \Vdash_U d(\sigma_1, \sigma_2) < \varepsilon \iff \sup_{x \in U} d_x(\sigma_1(x), \sigma_2(x)) < \varepsilon$
- $\mathfrak{A} \Vdash_U d(\sigma_1, \sigma_2) > \varepsilon \iff \inf_{x \in U} d_x(\sigma_1(x), \sigma_2(x)) > \varepsilon$.
- $\mathfrak{A} \Vdash_U R(\sigma_1, \ldots, \sigma_n) < \varepsilon \iff \sup_{x \in U} R^{\mathfrak{A}_x}(\sigma_1(x), \ldots, \sigma_n(x)) < \varepsilon$.
- $\mathfrak{A} \Vdash_U R(\sigma_1, \ldots, \sigma_n) > \varepsilon \iff \inf_{x \in U} R^{\mathfrak{A}_x}(\sigma_1(x), \ldots, \sigma_n(x)) > \varepsilon$.

Logical connectives

- $\mathfrak{A} \Vdash_U \max(\phi, \psi) < \varepsilon \iff \mathfrak{A} \Vdash_V \phi < \varepsilon$ *and* $\mathfrak{A} \Vdash_W \psi < \varepsilon$.
- $\mathfrak{A} \Vdash_U \max(\phi, \psi) > \varepsilon \iff$ *There exist open sets V and W such that $V \cup W = U$ and $\mathfrak{A} \Vdash_V \phi > \varepsilon$ and $\mathfrak{A} \Vdash_W \psi > \varepsilon$.*

- $\mathfrak{A} \Vdash_U \min(\phi,\psi) < \varepsilon \iff$ *There exist open sets V and W such that $V \cup W = U$ and $\mathfrak{A} \Vdash_V \phi < \varepsilon$ and $\mathfrak{A} \Vdash_W \psi < \varepsilon$.*

- $\mathfrak{A} \Vdash_U \min(\phi,\psi) < \varepsilon \iff \mathfrak{A} \Vdash_U \phi < \varepsilon$ *and* $\mathfrak{A} \Vdash_U \psi < \varepsilon$.

- $\mathfrak{A} \Vdash_U 1\dot{-}\psi < \varepsilon \iff \mathfrak{A} \Vdash_U \psi > 1\dot{-}\varepsilon$.

- $\mathfrak{A} \Vdash_U 1\dot{-}\psi > \varepsilon \iff \mathfrak{A} \Vdash_U \psi < 1\dot{-}\varepsilon$.

- $\mathfrak{A} \Vdash_U \phi\dot{-}\psi < \varepsilon \iff$ *One of the following holds*
  i) $\mathfrak{A} \Vdash_U \phi < \psi$
  ii) $\mathfrak{A} \nVdash_U \phi < \psi$ *and* $\mathfrak{A} \nVdash_U \phi > \psi$.
  iii) $\mathfrak{A} \Vdash_U \phi > \psi$ *and* $\mathfrak{A} \Vdash_U \phi < \psi + \varepsilon$.

- $\mathfrak{A} \Vdash_U \phi\dot{-}\psi > \varepsilon \iff \mathfrak{A} \Vdash_U \phi > \psi + \varepsilon$.

Quantifiers

- $\mathfrak{A} \Vdash_U \inf_\sigma \phi(\sigma) < \varepsilon \iff$ *there exist an open covering $\{U_i\}$ of U and a family of section μ_i each one defined in U_i such that $\mathfrak{A} \Vdash_{U_i} \phi(\mu_i) < \varepsilon$ for all i.*

- $\mathfrak{A} \Vdash_U \inf_\sigma \phi(\sigma) > \epsilon \iff$ *there exist ε' such that $0 < \varepsilon < \varepsilon'$ and an open covering $\{U_i\}$ of U such that for every section μ_i defined in U_i $\mathfrak{A} \Vdash_{U_i} \phi(\mu_i) > \varepsilon'$.*

- $\mathfrak{A} \Vdash_U \sup_\sigma \phi(\sigma) < \varepsilon \iff$ *there exist ε' such that $0 < \varepsilon' < \varepsilon$ and an open covering $\{U_i\}$ of U such that for every section μ_i defined in U_i $\mathfrak{A} \Vdash_{U_i} \phi(\mu_i) < \varepsilon'$.*

- $\mathfrak{A} \Vdash_U \sup_\sigma \phi(\sigma) > \varepsilon \iff$ *there exist an open covering $\{U_i\}$ of U and a family of section μ_i each one defined in U_i such that $\mathfrak{A} \Vdash_{U_i} \phi(\mu_i) > \varepsilon$ for all i.*

Observe that the definition of local forcing leads to the equivalences

$$\mathfrak{A} \Vdash_U \inf_\sigma (1\dot{-}\phi(\sigma)) > 1\dot{-}\varepsilon \iff \mathfrak{A} \Vdash_U \sup_\sigma \phi(\sigma) < \varepsilon,$$
$$\mathfrak{A} \Vdash_U \inf_\sigma (\phi(\sigma)) < \varepsilon \iff \mathfrak{A} \Vdash_U \sup_\sigma (1\dot{-}\phi(\sigma)) > 1\dot{-}\varepsilon.$$

Even more important is the fact that we can obtain a similar statement to the classical Maximum Principle.

Theorem 2.2 (Maximum Principle for Metric structures)**.** *If $\mathfrak{A} \Vdash_U \inf_\sigma \phi(\sigma) < \varepsilon$ then there exists a section μ defined in an open set W dense in U such that $\mathfrak{A} \Vdash_U \phi(\mu) < \varepsilon'$.*

Proof. That $\mathfrak{A} \Vdash_U \inf_\sigma \phi(\sigma) < \varepsilon$ is equivalent to the existence of an open covering $\{U_i\}$ and a family of sections $\{\mu_i\}$ such that $\mathfrak{A} \Vdash_{U_i} \phi(\mu_i) < \varepsilon'$ for some $\varepsilon' < \varepsilon$. Then family of section $\mathcal{S} = \{\mu | \ dom(\mu) \subset U$ and $\mathfrak{A} \Vdash_{dom(mu)} \phi(\mu) < \varepsilon'\}$ is nonempty and is partially ordered by inclusion. Consider the maximal element $\mu*$ of a chain of sections in $\mathcal{S}$. Then $dom(\mu*)$ is dense in U and $\mathfrak{A} \Vdash_{dom(\mu*)} \phi(\mu*) < \varepsilon'$. $\square$

2.3 The Metric Generic Model and its semantic connection with its metric sheaf

The quotient space of the metric sheaf is the universe of a metric structure in the same language as every one of the fibers. Conditions are given in the following definition.

Definition 2.13 (Metric Generic Model). *Let $\mathfrak{A} = (X, p, E)$ be a sheaf of metric structures and $\mathbb{F}$ a generic filter in the topology of X. We define the Metric Generic Model $\mathfrak{A}[\mathbb{F}]$ by*

$$\mathfrak{A}[\mathbb{F}] = \{[\sigma]/_{\sim_\mathbb{F}} | dom(\sigma) \in \mathbb{F}\},$$

provided with the metric $d_\mathbb{F}$ defined above, and with

- $$f^{\mathfrak{A}[\mathbb{F}]}([\sigma_1]/_{\sim_\mathbb{F}}, \ldots, [\sigma_n]/_{\sim_\mathbb{F}}) = [f^{\mathfrak{A}}(\sigma_1, \ldots, \sigma_n)]/_{\sim_\mathbb{F}}$$

  with modulus of uniform continuity $\Delta_f^{\mathfrak{A}[\mathbb{F}]} = \inf_{x \in X} \Delta_f^{\mathfrak{A}_x}$.

- $$R^{\mathfrak{A}[\mathbb{F}]}([\sigma_1]/_{\sim_\mathbb{F}}, \ldots, [\sigma_n]/_{\sim_\mathbb{F}}) = \inf_{U \in \mathbb{F}_{\sigma_1 \cdots \sigma_n}} \sup_{x \in U} R_x(\sigma_1(x), \ldots, \sigma_n(x))$$

  with modulus of uniform continuity $\Delta_R^{\mathfrak{A}[\mathbb{F}]} = \inf_{x \in X} \Delta_R^{\mathfrak{A}_x}$.

- $$c^{\mathfrak{A}[\mathbb{F}]} = [c]/_{\sim_\mathbb{F}}$$

Note that the triangle inequality and that $R^{\mathfrak{A}}$ is continuous make the Metric Generic Model be well defined. Special attention should be paid to the uniform continuity of $R^{\mathfrak{A}[\mathbb{F}]}$:

Proof. It is enough to show this for an unary relation. First, suppose $d_{\mathbb{F}}([\sigma],[\mu]) < \inf_{x\in X} \Delta_R^{\mathfrak{A}_x}(\varepsilon)$, then

$$\inf_{U\in\mathbb{F}_{\sigma\mu}} \sup_{x\in U} d_x(\sigma(x),\mu(x)) < \inf_{x\in X} \Delta_R^{\mathfrak{A}_x}(\varepsilon)$$

which implies that there exists $V \in \mathbb{F}_{\sigma\mu}$ such that

$$\sup_{x\in V} d_x(\sigma(x),\mu(x)) < \inf_{x\in X} \Delta_R^{\mathfrak{A}_x}(\varepsilon),$$

and by the uniform continuity of each $R^{\mathfrak{A}_x}$

$$\sup_{x\in V} |R(\sigma(x)) - R(\mu(x))| \leq \varepsilon.$$

We now state that

$$\left|\inf_{U\in\mathbb{F}_\sigma} \sup_{x\in U} R(\sigma(x)) - \inf_{U\in\mathbb{F}_\mu} \sup_{x\in U} R(\sigma(x))\right| \leq \sup_{x\in V} |R(\sigma(x)) - R(\mu(x))|\,.$$

First consider $R^{\mathfrak{A}[\mathbb{F}]}([\sigma]/_{\sim\mathbb{F}}) \geq R^{\mathfrak{A}[\mathbb{F}]}([\mu]/_{\sim\mathbb{F}})$

$$\begin{aligned} |R^{\mathfrak{A}[\mathbb{F}]}([\sigma]/_{\sim\mathbb{F}}) - R^{\mathfrak{A}[\mathbb{F}]}([\mu]/_{\sim\mathbb{F}})| &= R^{\mathfrak{A}[\mathbb{F}]}([\sigma]/_{\sim\mathbb{F}}) - R^{\mathfrak{A}[\mathbb{F}]}([\mu]/_{\sim\mathbb{F}}) \\ &\leq \sup_{x\in V} R(\sigma(x)) - R^{\mathfrak{A}[\mathbb{F}]}([\mu]/_{\sim\mathbb{F}}) \end{aligned}$$

Now, for all $\delta > 0$ there exists $W \in \mathbb{F}_\mu$ such that

$$\sup_{x\in W} R(\mu(x)) < \inf_{U\in\mathbb{F}_\mu} \sup_{x\in U} R(\mu(x)) + \delta$$

and indeed the same is true for the $V' = V \cap W \in \mathbb{F}_{\sigma\mu}$. Therefore

$$|R^{\mathfrak{A}[\mathbb{F}]}([\sigma]/_{\sim\mathbb{F}}) - R^{\mathfrak{A}[\mathbb{F}]}([\mu]/_{\sim\mathbb{F}})| \leq \sup_{x\in V'} R(\sigma(x)) - \sup_{x\in V'} R(\mu(x)) + \delta,$$

where we have substituted V by V' in the first term since $V' \subset V$, and we can apply the same arguments for it. Also, since δ is arbitrary

$$\begin{aligned} |R^{\mathfrak{A}[\mathbb{F}]}([\sigma]/_{\sim\mathbb{F}}) - R^{\mathfrak{A}[\mathbb{F}]}([\mu]/_{\sim\mathbb{F}})| &\leq \sup_{x\in V'} R(\sigma(x)) - \sup_{x\in V'} R(\mu(x)) \\ &\leq \sup_{x\in V'} \left(R(\sigma(x)) - R(\mu(x))\right) \\ &\leq \left|\sup_{x\in V'} \left(R(\sigma(x)) - R(\mu(x))\right)\right| \\ &\leq \sup_{x\in V'} |R(\sigma(x)) - R(\mu(x))| \leq \varepsilon \end{aligned}$$

In case $R^{\mathfrak{A}[\mathbb{F}]}([\sigma]/_{\sim\mathbb{F}}) \leq R^{\mathfrak{A}[\mathbb{F}]}([\mu]/_{\sim\mathbb{F}})$ similar arguments hold. $\square$

Example 2.1. Consider again example 1.4. Observe that every fiber can be made a metric structure with a metric given by the length of the shortest path joining two points. This, of course, is a Cauchy complete and bounded metric space. Dividing by π the distance function, we may redefine this to make $d(x, y) \leq 1$. Therefore, the manifold studied in example 1.4 is also a metric sheaf. In addition, observe that complex multiplication in S^1 extends to the sheaf as a uniformly continuous function in the set of sections.

We can now present the main result of this Chapter, i.e, the Generic Model Theorem (GMT) for metric structures.

Theorem 2.3 (Metric Generic Model Theorem)**.** *Let $\mathbb{F}$ be a generic filter on X and $\mathfrak{A}$ a sheaf of metric structures on X. Then*

1.

$$\mathfrak{A}[\mathbb{F}] \models \phi([\sigma]/_{\sim\mathbb{F}}) < \varepsilon \iff \exists U \in \mathbb{F} \textit{ such that } \mathfrak{A} \Vdash_U \phi(\sigma) < \varepsilon$$

2.

$$\mathfrak{A}[\mathbb{F}] \models \phi([\sigma]/_{\sim\mathbb{F}}) > \varepsilon \iff \exists U \in \mathbb{F} \textit{ such that } \mathfrak{A} \Vdash_U \phi(\sigma) > \varepsilon$$

Proof. Atomic formulas

- $\mathfrak{A}[\mathbb{F}] \models d_{\mathbb{F}}([\sigma_1]/_{\sim\mathbb{F}}, [\sigma_2]/_{\sim\mathbb{F}}) < \varepsilon$ iff $\inf_{U \in \mathbb{F}_{\sigma_1\sigma_2}} \sup_{x \in U} d_x(\sigma_1(x), \sigma_2(x)) < \varepsilon$. This is equivalent to say that there exist $U \in \mathbb{F}_{\sigma_1\sigma_2}$ such that $\sup_{x \in U} d_x(\sigma_1(x), \sigma_2(x)) < \varepsilon$ and by definition, that is equivalent to $\mathfrak{A} \Vdash_U d(\sigma_1, \sigma_2) < \varepsilon$.
- For $\mathfrak{A}[\mathbb{F}] \models R([\sigma_1]/_{\sim\mathbb{F}}, \ldots, [\sigma_n]/_{\sim\mathbb{F}})$ use similar arguments as before.
- $\mathfrak{A}[\mathbb{F}] \models d_{\mathbb{F}}([\sigma_1]/_{\sim\mathbb{F}}, [\sigma_2]/_{\sim\mathbb{F}}) > \varepsilon$ iff $\inf_{U \in \mathbb{F}_{\sigma_1\sigma_2}} \sup_{x \in U} d_x(\sigma_1(x), \sigma_2(x)) > \varepsilon$.
  - ($\Rightarrow$) Let $\inf_{U \in \mathbb{F}_{\sigma_1\sigma_2}} \sup_{x \in U} d_x(\sigma_1(x), \sigma_2(x)) = r$ and $\varepsilon' = (r + \varepsilon)/2$. Then, the set $V = p(\langle \sigma_1, \sigma_2 \rangle \cap d^{\mathfrak{A}\,-1}(\varepsilon', 0])$ is nonempty and intersects every open set in $\mathbb{F}$. If $V \notin \mathbb{F}$, consider $X \setminus \overline{V}$. That set is not an element of $\mathbb{F}$ since $dom(\sigma_1) \cap dom(\sigma_2) \cap X \setminus \overline{V}$ would also be in $\mathbb{F}$ with $d_x(\sigma_1(x), \sigma_2(x)) \leq \varepsilon'$ for all x in it. Therefore $\text{int}(\overline{V}) \cap dom(\sigma_1) \cap dom(\sigma_2) \in \mathbb{F}$ and for every element in this set $d_x(\sigma_1(x), \sigma_2(x)) \geq \varepsilon'$, which implies that there exist $U' \in \mathbb{F}$ such that $\inf_{x \in U'} d_x(\sigma_1(x), \sigma_2(x)) \geq \varepsilon' > \varepsilon$.

– ($\Leftarrow$) If $\mathfrak{A} \Vdash_V d(\sigma_1, \sigma_2) > \varepsilon$ for some $V \in \mathbb{F}_{\sigma_1\sigma_2}$, then V intersects any open set in the generic filter and for any element in V, $d_x(\sigma_1(x), \sigma_2(x)) \geq r$ where $r = \inf_{x\in V} d_x(\sigma_1(x), \sigma_2(x))$. Thus, for all $U \in \mathbb{F}$ $\sup_{x\in U} d_x(\sigma_1(x), \sigma_2(x)) \geq r$ and then

$$\inf_{U\in\mathbb{F}_{\sigma_1\sigma_2}} \sup_{x\in U} d_x(\sigma_1(x), \sigma_2(x)) \geq r > \varepsilon.$$

- Similar statements to those claimed above, shows the case of $\mathfrak{A}[\mathbb{F}] \models R([\sigma_1]/_{\sim\mathbb{F}}, \ldots, [\sigma_n]/_{\sim\mathbb{F}}) > \varepsilon$.

Logical connectives

- For the connectives $1\dot{-}$, min and max, it follows by simple induction in each case. We only show the proof for one of these connectives.

$$\begin{aligned}
&\mathfrak{A}[\mathbb{F}] \models \min(\phi([\sigma_1]/_{\sim\mathbb{F}}), \psi([\sigma_2]/_{\sim\mathbb{F}})) < \varepsilon \\
&\quad\Longleftrightarrow \mathfrak{A}[\mathbb{F}] \models \phi([\sigma_1]/_{\sim\mathbb{F}}) < \varepsilon \text{ or } \mathfrak{A}[\mathbb{F}] \models \psi([\sigma_2]/_{\sim\mathbb{F}}) \\
&\quad\overset{\text{ind}}{\Longleftrightarrow} \exists U_1 \in \mathbb{F}\ \mathfrak{A} \Vdash_{U_1} \phi(\sigma_1) < \varepsilon \text{ or } \exists U_2 \in \mathbb{F}\ \mathfrak{A} \Vdash_{U_2} \psi(\sigma_2) < \varepsilon \\
&\quad\Longleftrightarrow \exists U \in \mathbb{F} \text{ such that } \mathfrak{A} \Vdash_U \min(\phi(\sigma_1), \psi(\sigma_2)) < \varepsilon.
\end{aligned}$$

- If $\mathfrak{A}[\mathbb{F}] \models \phi([\sigma_1]/_{\sim\mathbb{F}})\dot{-}\psi([\sigma_2]/_{\sim\mathbb{F}}) < \varepsilon$ we analyze this by cases

  – if $\mathfrak{A}[\mathbb{F}] \models \phi([\sigma_1]/_{\sim\mathbb{F}}) < \psi([\sigma_2]/_{\sim\mathbb{F}})$

$$\begin{aligned}
&\Longleftrightarrow \exists r \text{ such that } \mathfrak{A}[\mathbb{F}] \models \phi([\sigma_1]/_{\sim\mathbb{F}}) < r \text{ and } \mathfrak{A}[\mathbb{F}] \models \psi([\sigma_2]/_{\sim\mathbb{F}}) > r \\
&\overset{\text{ind}}{\Longleftrightarrow} \exists r \text{ such that } \exists U_1\ \mathfrak{A} \Vdash_{U_1} \phi(\sigma_1) < r \text{ and } \exists U_2\ \mathfrak{A} \Vdash_{U_2} \psi(\sigma_2) > r \\
&\Longleftrightarrow \exists U\ \mathfrak{A} \Vdash_U \phi(\sigma_1) < \psi(\sigma_2)
\end{aligned}$$

  – If $\mathfrak{A}[\mathbb{F}] \not\models \phi([\sigma_1]/_{\sim\mathbb{F}}) < \psi([\sigma_2]/_{\sim\mathbb{F}})$ and $\mathfrak{A}[\mathbb{F}] \not\models \phi([\sigma_1]/_{\sim\mathbb{F}}) > \psi([\sigma_2]/_{\sim\mathbb{F}})$

$$\Longleftrightarrow \forall U\ \mathfrak{A} \not\Vdash_U \phi(\sigma_1) < \psi(\sigma_2) \text{ and } \forall U\ \mathfrak{A} \not\Vdash_U \phi(\sigma_1) > \psi(\sigma_2)$$

  – $\mathfrak{A} \models \phi([\sigma_1]/_{\sim\mathbb{F}}) > \psi([\sigma_2]/_{\sim\mathbb{F}})$ and $\mathfrak{A} \models \phi([\sigma_1]/_{\sim\mathbb{F}}) < \psi([\sigma_2]/_{\sim\mathbb{F}})+ \varepsilon$.

$$\begin{aligned}
\Longleftrightarrow\ & \exists U_1 \in \mathbb{F}\ \mathfrak{A} \Vdash_{U_1} \phi(\sigma_1) > \psi(\sigma_2) \text{ and} \\
& \qquad \exists U_2 \in \mathbb{F}\ \mathfrak{A} \Vdash_{U_2} \phi([\sigma_1]/_{\sim\mathbb{F}}) < \psi([\sigma_2]/_{\sim\mathbb{F}}) + \varepsilon.
\end{aligned}$$

Quantifiers

- $$\begin{aligned}\mathfrak{A}[\mathbb{F}] \models \inf_{[\sigma_i]/\sim_{\mathbb{F}}} \phi([\sigma_i]/\sim_{\mathbb{F}}) < \varepsilon \\ &\iff \exists[\sigma_1]/\sim_{\mathbb{F}} \text{ such that } \mathfrak{A}[\mathbb{F}] \models \phi([\sigma_1]/\sim_{\mathbb{F}}) < \varepsilon \\ &\iff \exists U_1 \in \mathbb{F}\, \exists \sigma_1 \text{ such that } \mathfrak{A} \Vdash_{U_1} \phi(\sigma_1) < \varepsilon \\ &\Rightarrow \exists U \in \mathbb{F} \text{ such that } \mathfrak{A} \Vdash_U \inf_\sigma \phi(\sigma) < \varepsilon\end{aligned}$$

  For the other direction suppose that there exists $U \in \mathbb{F}$ such that $\mathfrak{A} \Vdash_U \inf_\sigma \phi(\sigma) < \varepsilon$. Then the family $\mathcal{I}_\varepsilon = \{U \in \mathbb{F} | \mathfrak{A} \Vdash_U \inf_\sigma \phi(\sigma) < \varepsilon\}$ is nonempty and can be partially ordered by the binary relation $\prec$ defined by: $U \prec V$ if and only if $U \supset V$. Consider the maximal element U' of a chain defined in $\mathcal{I}_\varepsilon$. Then there exists a covering $\{V_i\}$ of U' all whose elements are basic open sets of class $\mathfrak{C}$, and a family of sections $\{\mu_i\}$, such that $\mathfrak{A} \Vdash_{V_i} \phi(\mu_i) < \varepsilon$. If any $V_i \in \mathbb{F}$ then $V_i = U'$ otherwise it will contradict the maximality of U'. Also, if $\mathrm{int}(X \setminus V_i) \in \mathbb{F}$ then $\mathfrak{A} \Vdash_{\mathrm{int}(X \setminus V_i) \cap U'} \inf_\sigma \phi(\sigma) < \varepsilon$ in contradiction to the maximality of U'. We conclude that there exists μ such that $\mathfrak{A} \Vdash_{U'} \phi(\mu) < \varepsilon$.

- $$\begin{aligned}\mathfrak{A}[\mathbb{F}] \models \sup_{[\sigma_i]/\sim_{\mathbb{F}}} \phi([\sigma_i]/\sim_{\mathbb{F}}) < \varepsilon \\ &\iff \forall[\sigma_i]/\sim_{\mathbb{F}}\, \mathfrak{A}[\mathbb{F}] \models \phi([\sigma_i]/\sim_{\mathbb{F}}) < \varepsilon \\ &\iff \forall \sigma_i \exists U_i \in \mathbb{F} \text{ such that } \mathfrak{A} \Vdash_{U_i} \phi(\sigma_i) < \varepsilon\end{aligned}$$

  ($\Rightarrow$) We prove this by contradiction. Suppose there exists V in $\mathbb{F}$ such that for some σ_i $\mathfrak{A} \Vdash_V \phi(\sigma_i) \geq \varepsilon$, then $V \cap U_i$ is also in $\mathbb{F}$ and in this set $\phi(\sigma_i) < \varepsilon$ and $\phi(\sigma_i) \geq \varepsilon$ are forced simultaneously.

  ($\Leftarrow$) Suppose that there exists $U \in \mathbb{F}$ such that $\mathfrak{A} \Vdash_U \sup_\sigma \phi(\sigma) < \varepsilon$. Then the family $\mathcal{S}_\varepsilon = \{U \in \mathbb{F} | \mathfrak{A} \Vdash_U \sup_\sigma \phi(\sigma) < \varepsilon\}$ is nonempty. The proof follows by similar arguments to those used in the case of $\inf_{[\sigma_i]/\sim_{\mathbb{F}}} \phi([\sigma_i]/\sim_{\mathbb{F}}) < \varepsilon$ above.

□

Example 2.2. Let us study the metric generic model for the sheaf discussed in example 2.1. Similarities with its topological analog in example 1.7 are expected. If the filters coincide for both the topological and the metric sheaf, then their generic models have the same set as universe. However, there is a difference in the way the metric sheaf compares sections. For example, for any element $[\sigma] \in \mathfrak{A}[\mathbb{F}]$, let $U = dom(\sigma) \in \mathbb{F}$ and μ be the global integral curve that extends σ. Thus, for arbitrary $\varepsilon > 0$

$$\mathfrak{A} \Vdash_U d^{\mathfrak{A}}(\sigma, \mu) < \varepsilon \quad \text{ and as a consequence } \quad \mathfrak{A}[\mathbb{F}] \models d^{\mathfrak{A}[\mathbb{F}]}([\sigma], [\mu]) = 0.$$

In addition, the metric generic model knows that the *multiplication* between sections is left continuous. Let η and μ be sections whose domain is an element of the ultrafilter. For any $\varepsilon < 1/2$, if

$$\mathfrak{A} \Vdash_{dom(\eta)\cap dom(\mu)} d(\eta, \mu) < \varepsilon$$

then for any other section σ defined in an element of $\mathbb{F}$, it is true that in $V = dom(\eta) \cap dom(\mu) \cap dom(\sigma)$

$$\mathfrak{A} \Vdash_V d(\eta\sigma, \mu\sigma) < \varepsilon$$

and also

$$\mathfrak{A} \Vdash_V 1 \dot{-} \max(d(\eta, \mu), 1 \dot{-} d(\eta\sigma, \mu\sigma)) < \varepsilon.$$

By the metric GMT, we can conclude that

$$\mathfrak{A}[\mathbb{F}] \models 1 \dot{-} \max(d^{\mathfrak{A}[\mathbb{F}]}([\eta], [\mu]), 1 \dot{-} d([\eta][\sigma], [\mu][\sigma])) < \varepsilon$$

and since σ, η and μ were chosen arbitrarily.

$$\mathfrak{A}[\mathbb{F}] \models \sup_{\sigma} \sup_{\eta} \sup_{\mu} \big[1 \dot{-} \max(d^{\mathfrak{A}[\mathbb{F}]}([\eta], [\mu]), 1 \dot{-} d([\eta][\sigma], [\mu][\sigma]))\big] < \varepsilon.$$

Right continuity, the left and the right invariance of this metric can be expressed and in the same fashion.

Note that the proof of the Metric Generic Model Theorem does not make use of the Maximum Principle for metric structures. The condition that there is a basis for the topology of X of class $\mathcal{C}$ is used instead where the Maximum Principle was invoked to prove the Classical Theorem in the case of the existential quantifier.

To close this chapter, it is worth stressing that the Topological and the Metric Generic Model Theorems have distinct but strong connections with the Classical Theorem. That the Topological GMT is an extension of the CMT was evident from the definition of Generic Filter for topological structures and the definition of point and local forcing. In the case of the Metric GMT, we can observe similarities in the forcing definitions if we make the parallel between the minimum function and the disjunction, the maximum function and the conjunction, the infimum and the existential quantifier. On the other hand, differences are evident if we compare the supremum with the universal quantifier. The reason for this is that in this case the sentence $1\dot{-}(1\dot{-}\phi)$, which is our analog for the double negation in continuous logic, is equal to the sentence ϕ. Note that the point and local forcing definitions are consistent with this fact, i.e.

$$\mathfrak{A} \Vdash_U 1\dot{-}(1\dot{-}\phi) < \varepsilon \iff \mathfrak{A} \Vdash_U \phi < \varepsilon,$$
$$\mathfrak{A} \Vdash_U 1\dot{-}(1\dot{-}\phi) > \varepsilon \iff \mathfrak{A} \Vdash_U \phi > \varepsilon.$$

As another consequence, the metric version of the GMT does not require an analog definition to the Gödel translation.

Chapter 3

Application: Projective Hilbert Spaces

In this chapter we want to sketch some of the possible applications that the logic over sheaves may have in the appropriate description of physical systems. In particular, we are interested in two different problems: The geometrical description of quantum mechanics and the local behavior of wave functions. We anticipate that the results in this chapter are not conclusive and there is more to understand before the connection of these structures with the physical world is clear.

The description of quantum mechanical systems in the last century has led to the idea that all possible states of a system are well represented by the elements of a Complex Hilbert space. In agreement with this point of view, the class of all self-adjoint operators is associated with the set of all magnitudes that can be measured from the physical system. The mathematical properties of these Hilbert spaces became the principal interest of many people in quantum mechanics. Around forty years ago, a different but still related approach to quantum mechanics was postulated in terms of Projective Hilbert spaces. This description eliminates the redundancy in the association of many vectors in a Hilbert space with the same state, i.e., for physical purposes an element x of a Hilbert space and any non-vanishing scalar multiple of it represent the same physical state. However Projective Hilbert spaces do not inherit neither the vector space nor the inner product space structure of the Hilbert spaces making difficult to establish the complete connection between these two descriptions. In this work we also explore how that connection should be addressed.

In regard to the first approach, we must say that only a few physical systems admit a complete description in a finite dimensional Hilbert Space. However, for practical purposes in many cases the study of real systems is restricted to finite dimensional Complex Hilbert subspaces. Very few has been done in order to understand the limitations of this approach and the conditions under which properties of a finite dimensional subspace are the same as those expected from the infinite dimensional one. It is easy to find examples in mathematics where finite structures do not capture essential elements of infinite structures where they may be embedded. We also know that the ultraproduct of a family of finite structures can give rise to interesting infinite dimensional structures whose theory is well described by the Łoś theorem.

Thus, we describe an infinite dimensional projective Hilbert spaces by constructing a sheaf where every fiber is a finite dimensional projective space in an appropriate base space.

3.1 A Topological Sheaf for a Projective Hilbert Space

In 1936, John von Neumann and Garrett Birkhoff introduced a propositional calculus based on the lattice of closed subspaces of a Hilbert space ordered by inclusion [16]. This lattice is not Boolean but orthocomplemented, and therefore it is different to the classical propositional calculus. In particular, the classical distributive law fails. Their hope was that the study of this propositional calculus might reflect the differences between the Classical and Quantum Mechanical picture of nature. Many physical and also philosophical questions have been stated in this context, and different kinds of quantum logics have been proposed looking for an answer. For example, Domenech and Freytes have recently presented a contextual logic [5] to investigate how far one can refer to physical objects in Quantum Mechanics without contradiction. In their work they introduce a sheaf over a topological space associated with Boolean sublattices of the ortholattice of closed subspaces of the Hilbert Space of a physical system. Connections with the Kochen-Specker theorem are addressed.

In this work we are not pursuing the questions studied by Domenech and Freytes. We mention them because our construction in this section resembles their "spectral sheaf". Our goal is to construct models of a projective Hilbert space when the Hilbert Space is infinite dimensional. Finite projective spaces have been studied to a good extent and infinite dimensional analogs are conceived as directed limit of these finite spaces. Here, we present a similar construction in the context of topological model theory.

We use the notation $\langle x, y\rangle$ to represent the inner product of two vectors x, y in a Hilbert space. We construct a sheaf for the lattice of finite subspaces of a Hilbert space as follows.

1. (*Base space X*) Let H be a Hilbert space, A a self-adjoint operator defined on H and $\{x_k | k \in K\}$ be a maximal set of pairwise orthogonal eigenvectors of A. Let $I \subset K$ be a finite index set and define $l_I = \{x_i | i \in I\}$. Also, let $X = \mathcal{L}(H) = \{l_I | I \subset K \text{ and } |I| < \aleph_0\}$ be the lattice of finite subsets of the chosen set of eigenvectors. X is provided with a partial order relation $\prec$ defined by

$$I \subset J \rightarrow l_I \prec l_J.$$

X is a topological space with basic open set $[l) = \{l' \in \mathcal{L}(H) | l \prec l'\}$ and the empty set.

2. (*Fibers of the Sheaf*) Every fiber is a two sorted topological structure

$$E_I = ((PV_I, \tau_{PV_I}), (\mathbb{C}, \tau_{\mathbb{C}}), E_{A_I}, A_I, E_{K_\alpha}, K_\alpha, P)$$

being α an index over an arbitrary index set and where

(a) PV_I is the universe of the sort that is a finite projective space. It is constructed as follows. First, for $l_I \in \mathcal{L}(H)$ define on the vector space V_I spanned by l_I in the complex field the equivalence relation $\sim$ by

$$y \sim x \iff \exists c \in \mathbb{C} \setminus \{0\} \text{ such that } y = cx,$$

for x and y different to 0. Let $[y]$ be the equivalence class with representative element y and

$$PV_I = \{[y] | y \in V_I \setminus \{0\}\}.$$

Thus, PV_I is the complex projective space of V_I. The second sort is the set of complex numbers $\mathbb{C}$ in the language of algebraic fields.

(b) The topology τ_{PV_I} is the quotient topology associated with the quotient map $\pi_I : V_I \setminus \{0\} \rightarrow PV_I$ given by $\pi_I(x) = [x]$, and $\tau_{\mathbb{C}}$ is the standard topology of the set of complex numbers that makes it homeomorphic to $\mathbb{R}^2$.

(c) The complex projective sort has symbols A_I, E_{A_I} and P to be interpreted as follows. For the linear operator $A_I^* = A \restriction V_I$[1] we can define a function in PV_I, the expected value of A_I, as follows

$$E_{A_I}([x]) = \frac{\langle A_I^* x, x\rangle}{\langle x, x\rangle}.$$

This is a function from the projective sort to the complex numbers. A_I is function symbol from the projective sort to itself associated with the operator A_I^* and defined as follows

$$A_I[x] = [A_I^* x]$$

Finally P is a binary function symbol from the projective sort to the complex numbers, interpreted as follows

$$P([x],[y]) = \frac{\langle x, y\rangle\langle y, x\rangle}{\langle x, x\rangle\langle y, y\rangle}.$$

This is the square of the projection of an equivalence class $[x]$ into the equivalence class $[y]$. We have also included into the language additional symbols K_α and E_{K_α} to be interpreted in a similar way than A_I and E_{A_I} for a linear operator K with trivial kernel defined in V_I.

3. (*Sheaf topology*) If a sheaf (classical, topological or metric) is such that every fiber is a multisorted structure, the topology for every sort should be given by sections in such a way that every function from one sort to another is continuous. Thus, we must extend our definition for a sheaf to include the following statement.

   Let $S_{x1}, \ldots, S_{xn}$ be sorts of a model E_x, then the function $f_j^{\mathfrak{A}} = \cup_x f_j^x : \cup_x S_{xn_1} \times \cdots \times S_{xn_k} \to S_{xm}$ must be continuous.

   Thus, consider again our sheaf and let B be a basic open set in $\mathcal{L}(H)$. We define for each $[x]$ in PV_I the function

$$\sigma_x : [l_I) \to \bigsqcup_{I \in J} PV_J \tag{3.1}$$

$$\sigma_x(l_J) = [x] \tag{3.2}$$

[1] Not to be confused with the adjoint operator of A.

and for each c in $\mathbb{C}$

$$\mu_c : [l_I) \to \bigsqcup_{I \in J} \mathbb{C} \tag{3.3}$$

$$\mu_c(l_J) = c \tag{3.4}$$

where $l_J \in [l_I)$, and $c \in \mathbb{C}$. These are the sections of our sheaf.

From the set of sections just defined it is clear that there is no global section in the projective sort. Before showing that the sheaf is well defined, we want to emphasize that we may not be able to define an inner product in PV_I as a function of the inner product in V_I only. However, in the absence of an inner product we may choose P as a geometric descriptor for the projective sort. We say that $[x], [y]$ are orthogonal if $P([x], [y]) = 0$. In this case they are also orthonormal as a consequence of the definition of P, since $P([x], [x]) = 1$ for all $[x]$. This function is not an inner product and as we will see, it can be associated with an angle between two elements in PV_I.

To show that the topological sheaf is well defined observe that $\left(A^{\mathfrak{A}}\right)^{-1}(\sigma_x)$ is a section for every σ_x by analyzing the inverse pointwise. To see that $E_A^{\mathfrak{A}}$ is continuous, for every $c \in \mathbb{C}$ define

$$S_c(I) = \left\{ [x] \;\middle|\; c = \frac{\langle A_I^* x, x\rangle}{\langle x, x\rangle} \right\},$$

observe that $E_A^{\mathfrak{A}\,-1}(\mu_c) = \cup_I S_c(I)$ and that this set is a union of sections. Finally define

$$S'_c(I) = \left\{ ([x], [y]) | c = \frac{\langle x, y\rangle\langle y, x\rangle}{\langle x, x\rangle\langle y, y\rangle} \right\}$$

and observe that $P^{\mathfrak{A}-1}(\mu_c) = \cup_I S'_c(I)$ is open in $\bigcup_{[l_I)} PV_I^2$.

Note that the set of "open sets" $\tau_{PV}^{\mathfrak{A}}$ is bigger than the topology of each fiber. This is a consequence of the fact that we have defined open sets as "local monadic predicates". In this case we call it the limit topology.

We claim that the generic model is an appropriate projective model for the description of many quantum mechanical systems. As expected, the structure of the generic model is unveiled through the application of the Topological Generic Model Theorem. This leads to our next lemma

Lemma 3.1. *Let $\mathfrak{A}[\mathbb{F}]$ be the topological generic model just defined. The following are statements satisfied by this model.*

1. *There is an infinite number of orthonormal elements.*

2. *The function $A^{\mathfrak{A}[\mathbb{F}]}$ has an infinite number of eigenvalues and each eigenvalue has finite multiplicity.*

3. *The generic model is Hausdorff.*

4. *If $A^{\mathfrak{A}[\mathbb{F}]}$ is not a bounded operator, $A^{\mathfrak{A}[\mathbb{F}]}$ is still continuous.*

Remark 3.1. We may define the "dimension" of the projective sort on every fiber as the maximum number of mutually orthogonal elements. With this definition, statement 1 in the above Theorem says that our generic model is infinite dimensional.

Remark 3.2. In fact, that A is not bounded operator and at the same time continuous does not contradict the fact that this two properties of an operator are equivalent in a Hilbert space. We know that neither the fibers nor the generic model are vector spaces.

By induction it can be proved that the Gödel translation is not necessary if ϕ is a sentence without any universal quantifier. This statement was previously noted by Caicedo in [2] and is a consequence of the fact that every open set dense in X is in the ultrafilter.

Proof. We only proof statements 1 and 3 since the other cases are similar. All of them follow from the Topological GMT.

(1) For every n the following is a sentence in $\mathcal{L}_T$:

$$\phi_{dim>n} = \exists^{i\leq n}\sigma_i \bigwedge\nolimits_{i\neq j}^{\leq n} P([\sigma_i],[\sigma_j]) = 0$$

By construction, for any $[l_I)$ with $|I| = n+1$ we have $\mathfrak{A} \Vdash_{[l_I)} \phi_{dim>n}$ and therefore $\mathfrak{A}[\mathbb{F}] \models \phi_{dim>n}$.

(3) Observe that being T_1 is a property whose Gödel translation can be expressed in $\mathcal{L}_t$ by

$$\begin{aligned}\phi_{T_1}^G &= \big(\forall\sigma_x\forall\sigma_y(\sigma_x = \sigma_y \vee \exists\mathcal{X}(\sigma_x \in \mathcal{X} \wedge \neg\sigma_y \in \mathcal{X}))\big)^G \\ &= \forall\sigma_x\forall\sigma_y\,\neg\big((\neg\sigma_x = \sigma_y)^G \wedge \neg(\exists\mathcal{X}(\sigma_x \in \mathcal{X} \wedge \neg\sigma_y \in \mathcal{X}))^G\big).\end{aligned}$$

Given two different sections σ_x and σ_y with domain in $\mathbb{F}$, let $U = \mathrm{dom}(\sigma_x)\cup\mathrm{dom}(\sigma_y)$. In the fiber corresponding to the meet of U there exists $\mathcal{C}$ that make σ_x and σ_y topologically distinguishable. There is an extension of $\mathcal{C}$ to U such that

$$\mathfrak{A} \Vdash_U \sigma_x \in \mathcal{C} \wedge \neg\sigma_y \in \mathcal{C}$$

therefore

$$\mathfrak{A} \Vdash_U \exists \mathcal{X} \sigma_x \in \mathcal{X} \wedge \neg \sigma_y \in \mathcal{X}.$$

The statement do not involve any universal quantifier, therefore

$$\mathfrak{A} \nVdash_U \neg(\exists \mathcal{X} \sigma_x \in \mathcal{X} \wedge \neg \sigma_y \in \mathcal{X})^G.$$

Besides

$$\mathfrak{A} \Vdash_U \neg \sigma_y = \sigma_x \iff \mathfrak{A} \Vdash_U (\neg \sigma_y = \sigma_x)^G.$$

hence we have

$$\mathfrak{A} \Vdash_U \neg\big(\neg(\sigma_x = \sigma_y)^G \wedge \neg(\exists(\mathcal{X} \sigma_x \in \mathcal{X} \wedge \neg \sigma_y \in \mathcal{X}))^G\big)$$
$$\mathfrak{A} \Vdash_U \big(\sigma_x = \sigma_y \vee \exists(\mathcal{X} \sigma_x \in \mathcal{X} \wedge \neg \sigma_y \in \mathcal{X})\big)^G.$$

This statement is true for every pair of sections with domain equal to U and in every $V \subset U$. Therefore

$$\mathfrak{A} \Vdash_U \forall \sigma_x \forall \sigma_y \big(\sigma_x = \sigma_y \vee \exists(\mathcal{X} \sigma_x \in \mathcal{X} \wedge \neg \sigma_y \in \mathcal{X})\big)^G.$$

□

3.2 A Metric Sheaf for a Projective Hilbert space

In this section we consider a different but related problem. We define a metric sheaf for a family of complex projective Hilbert spaces that are constructed as above and whose structure depends upon a given parameter or set of parameters R. This parameter might appear in the definition of the operator A as a variable or a constant. That dependence might come from a physical problem were R is, for example, a force constant or just the time. We assume R is an element of a topological space X. if a solution to the eigenvalue problem of A_R exists at every possible value of R, we want to extend such a solution continuously to a neighborhood in X. To be more precise

1. (*Base space X*) We choose X to be a regular space with a basis of regular open sets.

2. (*Fibers of the Sheaf*) Every fiber is a two sorted topological structure

$$E_R = (PV_R, I_R, A_R, ||A_R||, E_{A_R}, P)$$

where

(a) PV_R is the complex projective space of a Hilbert space V_R. The space V_R is the domain of a self-adjoint and *bounded* finite operator A_R that depends parametrically on R. Every operator has associated with it a real number $||A_R||$ that represents its norm and a closed interval I_R corresponding to its numerical range, i.e., $I_R = \{\langle x, A_R x\rangle : ||x|| = 1\}$. The last is our second sort with the standard metric of the real set. From the operator A_R defined in V_R, we can define an operator in PV_R with the same name by

$$\begin{aligned} A^R : PV_R &\to PV_R \\ A^R[x] &= [A^R x] \end{aligned}$$

Since PV_R is not a vector space, we may not expect A_R to be linear.

(b) We provide PV_R with the Fubini-Study metric. This is a Kähler metric that can be written as

$$d([x],[y]) = \arccos \sqrt{\frac{\langle x, y\rangle\langle y, x\rangle}{\langle x, x\rangle\langle y, y\rangle}} = \arccos \sqrt{P([x],[y])}$$

It is the length of the geodesic in the finite dimensional sphere joining $y/||y||$ and $x/||x||$. The above expression shows that $P([x],[y])^{1/2}$ can be interpreted as the angle between $[x]$ and $[y]$.

(c) E_{A_R} is a function symbol from the projective sort to the interval I_R interpreted as follows

$$E_{A_R}([x]) = \frac{\langle Ax, x\rangle}{\langle x, x\rangle}$$

3. (*Modulus of uniform continuity for E_{A_R}*) The modulus of uniform continuity $\Delta_{E_{A_R}}$ for E_{A_R} can be easily defined based on $||A_R||$. It is required that $\sup_R ||A_R||$ be finite for the generic model to be well defined. Since A_R is bounded, we know that A is uniformly continuous with respect to the Euclidean metric. Observe that,

$$||A_R|| = \sup_{||x||=1\ ||y||=1} |\langle y, A_R x\rangle|$$

Let $x = y + h$, then

$$||A_R y - A_R x|| = ||A_R h|| \leq ||A_R|| ||h||$$

Now we show that the function E_{A_R} is uniformly continuous respect to the euclidean metric of the Hilbert space. It is enough to consider x and y of unit length:

$$\begin{aligned}
||\langle A_R y, y\rangle - \langle A_R x, x\rangle|| &= ||\langle A_R(x+h), x+h\rangle - \langle A_R x, x\rangle|| \\
&= ||\langle A_R h, h\rangle + \langle A_R h, x\rangle + \langle A_R x, h\rangle|| \\
&\leq ||h||^2 \, ||\langle A_R h/||h||, h/||h||\rangle|| + \\
&\quad\; ||h|| \, ||\langle A_R h/||h||, x\rangle|| + ||h|| \, ||\langle A_R x, h/||h||\rangle|| \\
&\leq ||h||(||h|| + 2)||A_R|| = \varepsilon_R
\end{aligned}$$

given that $||y - x|| = ||h||$ we can take as a modulus of uniform continuity

$$\delta(\varepsilon) = \sqrt{1 + \frac{\varepsilon}{||A_R||}} - 1.$$

It can be shown that this also implies that $E_A[x]$ is uniformly continuous as a function from PV_R to I_R. We use the fact that $d_{FS}([x],[y])$ equals in magnitude the angle of the geodesic that joins two points $x/||x||$ and $y/||y||$ in the unit sphere S^n. Thus we can define the modulus of uniform continuity respect to d_{FS} according to

$$\Delta_{E_{A_R}}(\varepsilon_R) = \arccos\left(1 - \frac{\delta^2(\varepsilon_R)}{2}\right)$$

Similar arguments can be used to show that there is a modulus of uniform continuity for $A^{\mathfrak{A}}$ and $P^{\mathfrak{A}}$.

4. (*Sheaf's topology*) For the projective sort, we define sections in such a way that they respect the basis of the initial Hilbert space V_R. Let $\{x_i^R\}$ be a basis of orthonormal eigenvectors of A_R ordered according to

$$x_i^R \leq x_j^R \iff E_{A_R}[x_i^R] \leq E_{A_R}[x_j^R].$$

In case of $E_{A_R}[x_i^R] = E_{A_R}[x_j^R]$, choose any consistent ordering. Given a regular open set U in X, and $c = [c_i] \in \mathbb{CP}^n$ we define a section σ_c

as

$$\sigma_c : U \to E_{PV_R}$$
$$\sigma_c(R) = [\sum c_i x_i^R]$$

For the numerical range sort, we define sections as follows

$$\mu_c : U \to E_{I_R}$$
$$\mu_c(R) = \frac{\sum |c_i|^2 \lambda_i^R}{\sum |c_i|^2}$$

where $\{\lambda_i^R\}$ is the set of real eigenvalues $A_R x_i^R = \lambda_i^R x_i^R$.

It remains to show that $E^{\mathfrak{A}}$, $A^{\mathfrak{A}}$ and $P^{\mathfrak{A}}$ the conditions in the definition of a metric sheaf, i.e., that they are continuous according to the topology of the sheaf. Observe that $E_A^{\mathfrak{A}-1}(\mu_c) = \sigma_c$. If A_R changes continuously as a function of R, then $P^{\mathfrak{A}}$ and $A^{\mathfrak{A}}$ are continuous. We impose this additional condition in the definition of our sheaf.

Now we investigate the theory of the metric generic model. Consider the formula

$$\phi_{norm}^{\mathfrak{A}_R} = \inf_{[x]} \left| E_{A_R}[x] - ||A_R|| \right|$$

We know that $\mathfrak{A} \Vdash_R \phi_{norm} < \varepsilon$ for all $\varepsilon > 0$. There exists σ_{norm}, a global section with $\sigma_{norm}(R) = ||A_R||$ for all $R \in X$. Then $[\sigma_{norm}]/_{\sim F}$ is a constant in $\mathfrak{A}[\mathbb{F}]$ and $\phi_{norm}^{\mathfrak{A}[\mathbb{F}]} < \varepsilon$ is true in $\mathfrak{A}[\mathbb{F}]$ as well. In that case we can also state that this "norm" is unique. However, note that in this case the norm is not a complex number but an equivalent class.

The idea behind this construction is to solve "locally" the eigenvalue equation for A, in such a way that the parametric dependence is removed. Far from accomplishing this task, we can still present results in this direction. For example, the conditions

$$d_{FS}(A_R[\sigma_{1,0,\ldots,0}]/_{\sim\mathbb{F}}, [\sigma_{1,0,\ldots,0}]/_{\sim\mathbb{F}}) < \varepsilon,$$
$$\left| E_A^{\mathfrak{A}[\mathbb{F}]}[\sigma_{1,0,\ldots,0}]/_{\sim\mathbb{F}} - [\mu_{1,0,\ldots,0}]/_{\sim\mathbb{F}} \right| < \varepsilon,$$

are forced in X, for every ε. Observe that in the sheaf the eigenvectors are sections.

We might not have enough tools in our language to let the model know about the dimension of the projective spaces PV_R through local isomorphism to open subsets of $\mathbb{R}^n$. However, the generic model may know about its dimension by means of the projective function P. The analogy comes from a classical model of an inner product space. As an example, consider the first order sentence in the language of an inner product space of "dimension 2":

$$\phi^{classic}_{\dim 2} = \exists x_1 \exists x_2 \Big(x_1 \neq x_2 \wedge \langle x_1, x_2 \rangle = 0 \wedge \\ \forall x_3 (x_1 \neq x_3 \wedge x_1 \neq x_3 \wedge \langle x_1, x_3 \rangle \neq 0 \wedge \langle x_2, x_3 \rangle \neq 0) \Big)$$

we write the analog metric sentence for our model, as follows

$$\phi^{metric}_{\dim 2} = \inf_{\sigma_1} \inf_{\sigma_2} \max \Big(1 \dot{-} d_{FS}(\sigma_1, \sigma_2),\ P(\sigma_1, \sigma_2), \\ \sup_{\sigma_3} \big(\max(1 \dot{-} d_{FS}(\sigma_1, \sigma_3),\ 1 \dot{-} d_{FS}(\sigma_1, \sigma_3),\ 1 \dot{-} P(\sigma_1, \sigma_3),\ 1 \dot{-} P(\sigma_2, \sigma_3)\ )\big) \Big)$$

and the condition $\phi^{metric}_{\dim 2} < \varepsilon$ should be forced in a fiber if $\dim(PV_R) = 2$. As a trivial consequence, the same condition should be satisfied by the generic model if the dimension of V_R is the same for all $R \in X$. A more interesting problem is to find a sheaf whose generic model is infinite dimensional and all whose fibers are finite dimensional. The following lemmas state the properties of the sheaf that are associated with an infinite dimensional metric generic model.

Lemma 3.2. *The generic model for the above sheaf is infinite dimensional if and only if for every $k \in \omega$, $U_k = \{R \in X \mid \mathfrak{A} \Vdash_R \phi_{dim>k} < 2^{-k}\}$ is in $\mathbb{F}$.*

Proof. ($\Rightarrow$) Suppose that $\mathfrak{A}[\mathbb{F}] \models \phi_{dim>n} < \varepsilon$ for any $\varepsilon > 0$. Then it is true that there is a family $\{U_k\ , k \in \omega\}$ such that $\mathfrak{A} \Vdash_{U_k} \phi_{dim>k} < 2^{-k}$.

($\Leftarrow$) Given $\varepsilon > 0$, there exists k such that $2^{-k} < \varepsilon$ and then $\mathfrak{A} \Vdash_{U_k} \phi_{dim>k} < \varepsilon$. Thus $\mathfrak{A}[\mathbb{F}] \models \phi_{dim>n} < \varepsilon$ for any ε. □

Lemma 3.3. *If $\mathfrak{A}[\mathbb{F}]$ is infinite dimensional then $\cap U_k = \emptyset$. As a consequence, the ultrafilter would not σ-complete.*

Proof. Suppose $\cap U_k \neq \emptyset$, then for some fiber $R \in X$ $\mathfrak{A} \Vdash_R \phi_{dim>k} < 2^{-k}$ holds for all k and then PV_R is infinite dimensional. □

Lemma 3.4. *If $\mathfrak{A}[\mathbb{F}]$ is infinite dimensional then X is neither a compact nor a connected set.*

Proof. $\{U_k | \mathfrak{A} \Vdash_{U_k} \phi_{dim>k} < 2^{-k}\}$ is a covering of X. If X is a compact set, then there would exists a finite subcovering $\{U_{k_1}, \ldots, U_{k_n}\}$ of X.

Now, consider $V_k = \{R \in X | \mathfrak{A} \Vdash_R \phi_{dim<k} < 2^{-k}\}$. Then $\{V_k, V_{k+1} \cap U_{k-1}, U_k\}$ are open sets, pairwise disjoint and their union is X. □

Further Directions

In this section we want to point out and briefly discuss some possible ways in which the present work can be improved and expanded. This, by no means, is an exhaustive listing.

1. (*Model Theory of Sheaves of Topological and Metric Structures*)
   In Chapters 1 and 2 we defined the Generic Model in the topological and metric language. However, we did not explore the model theory of such models. From the corresponding GMT many questions about the connection between the topological properties of the sheaf and model theoretical concepts may be stated. For instance, one may wonder about the topological properties of the base space that would generate a model whose theory has quantifier elimination with, perhaps, models in the fibers lacking this. It may not only depend on the properties of the base space, but also in the properties of the set of sections and also in the additional properties of the ultrafilter. Similar and more interesting questions can be formulated around the nature of the type space, completeness of the model, saturation of these models and so forth.

2. (*The Topology of a Sheaf of topological structures*)
   One interesting property from a sheaf of topological structures is that it may have more open sets than those in every fiber, as shown in example 1.2. Most of them "collapse" into a unique open set in the generic model much in the same way sections do. Thus, some of the new topological properties of the sheaf may disappear after the construction of the generic model. This does not mean we may not get new open sets and new topological properties in the generic model, however this tells us that the topology in sheaf may have more interesting topological properties.

3. (*Metric Generic Filter*)

The definition of the generic filter for metric structures may be a little bit restrictive. Its definition came after we showed that an ultrafilter of open regular sets in a regular space leads to Cauchy complete Generic metric model. There is no reason up to now to think that more general ultrafilters, hopefully less restrictive, may have Cauchy complete generic models.

4. (*Noncommuting operators in projective Hilbert spaces*)
   In our discussion about projective Hilbert spaces, we never talk about any operator other than that involved in the construction of such spaces. Noncommuting operators deserve special attention, since the restriction of these to a finite dimensional projective space (or to a finite dimensional Hilbert space) may not be well defined. A common practice in physical sciences is to define an operator in the finite space through a matrix with elements given by $\langle Be_i, e_j \rangle$ where e_i and e_j are vectors in the basis set of the space, and B is the original operator. Through the construction of appropiate sheaves one can construct operators $B^{\mathfrak{A}}$ that may resemble or not the original operator B. Questions of this kind about restrictions of noncommutative operator have a lot of thing to say about the way quantum mechanics is done in finite spaces nowadays.

Bibliography

[1] BADER R., "Atoms in Molecules: A Quantum Theory". *The International series of monomgraphs on Chemistry 22*. Oxford University Press. Great Britain. 1990.

[2] CAICEDO X. "Lógica de los haces de estrutura". Rev. Acad. Colomb. Cienc. **19** (74): 569-586, 1995.

[3] DAUDEL R. "Quantum Chemistry" John Wiley & Sons Ltd. Brisbane. 1983

[4] DEBNATH L. "Introduction to Hilbert spaces with applications" Academic Press Inc.USA. 1990

[5] DOMENECH G., FREYTES H. "Contextual logic for quantum systems" J. Math. Phys. (**46**) 012102-1 (2005)

[6] FLAPAN E. "When Topology Meets Chemistry". Cambridge University Press. United States of America. 1980.

[7] FLUM J., ZIEGLER M. "Topological Model Theory". LNM 769. Springer-Verlag. Berlin. 1980.

[8] FRINK O. J., "Topology in Lattices" Trans. Am. Math. Soc. Vol 51, No 3. (May, 1942) 569-582.

[9] BEN YAACOV I; BERENSTEIN A.; HENSON C. W; USVYATSOV A. "Model theory for metric structures" in *Model Theory with Applications to Algebra and Analysis, Vol II.* Lecture Notes series of the London Mathematical Society, No. 350, Cambridge University Press, 2008, 315-427.

[10] IACHELLO F., LEVINE R. D. "Algebraic Theory of Molecules". Oxford University Press. United States of America. 1995.

[11] LANG S. "Real and Functional Analysis" Springer-Verlag. New York.1993

[12] LANG S. "Fundamentals of Differential Geometry" Springer-Verlag. New York.1999

[13] MEZEY P. G., "Shape in Chemistry: An introduction to Molecular Sahpe and topoloy". VCH Publishers, Inc. New York, 1993.

[14] MONTOYA A., "Contribuciones a la teoría de modelos de haces". *Lecturas Matemáticas* Vol 28, No 1 (2007) 5-37.

[15] MÜLLER V. "Spectral Theory of Linear Operator". Birkhauser Verlag. Alemania. 2003

[16] BIRKHOFF G., Von NEUMANN J., "The logic of Quantum Mechanics". Ann. Math.(**37**) (4) October (1936) 823-843.

[17] TIERSCH M., VIVIESCAS C., BUCHLEITNER A., "Entaglement and Decoherence. Foundations and Modern Trends". LNP 768. Springer-Verlag. Berlin. 2009.

[18] YALE. P "Automorphisms of the Complex Numbers".*Math. Magaz.* Vol 39, No 3(May, 1966), pp. 135-141.