One Puzzling Logic, Two Approximations…

…and a Bonus

Andrés Villaveces - *Universidad Nacional de Colombia - Bogotá*

Contents

When is a logic "appropriate" for model theory?

- Of course, logics "similar to" $\mathsf{L}_{\omega,\omega}$, ${}^{\mathsf{cont}}\mathsf{L}_{\omega,\omega}$, …(they have Löwenheim-Skolem, Compactness, Interpolation, etc.)

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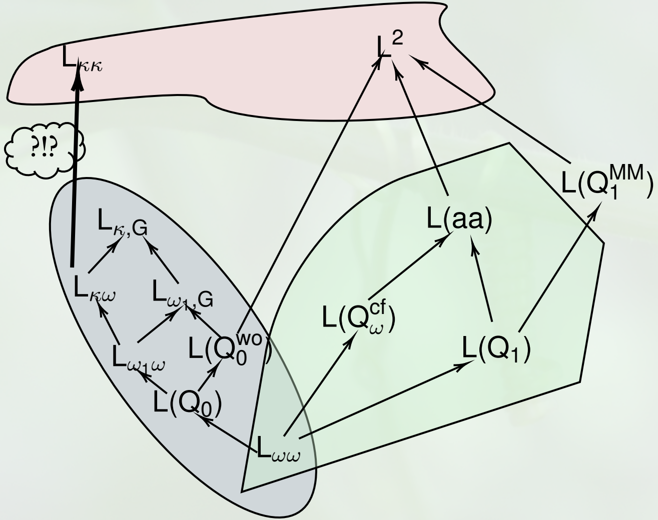
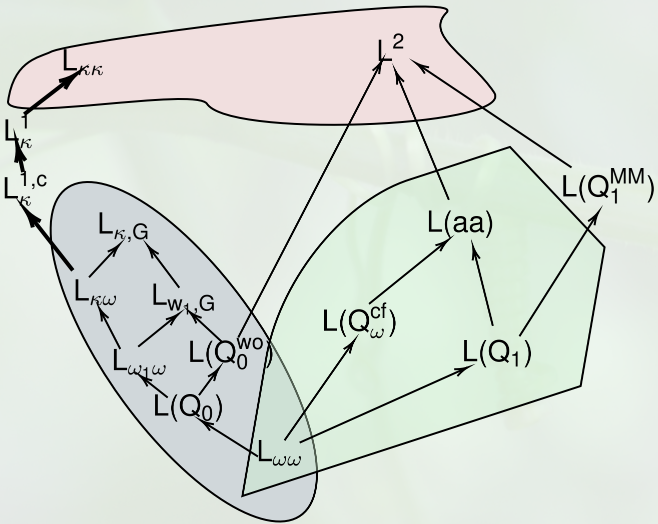
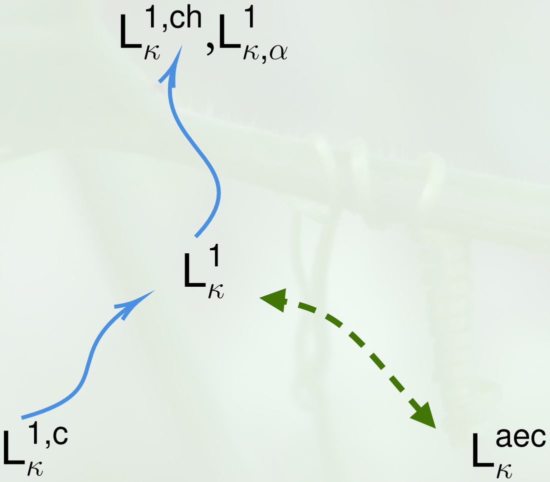
- Väänänen: "infinitary logic may still serve as a 'yardstick' for model theoretic constructs, permits fragments of model theory and is preserved under (reasonable) forcing"...
- More recently, Espíndola (following Makkai and Kueker) has captured λ-categoricity of $L_{\kappa,\kappa}$ in terms of categorical logic (Boolean behaviour of the λ-classifying topos of the logic...)

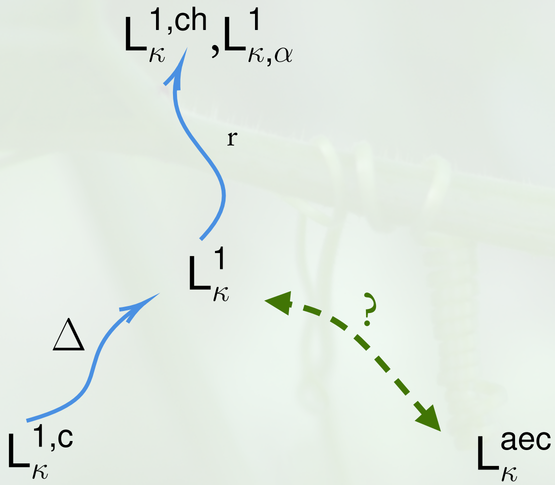
A (Väänänen) map of various infinitary logics



New Logics


$L_{\kappa\kappa}$
L^2
L^1_κ
$L^{1,c}_\kappa$
$L(Q_1^{MM})$
$L_{\kappa,G}$
$L(aa)$
$L_{\kappa\omega}$
$L_{w_1,G}$
$L(Q^{cf}_\omega)$
$L(Q^{wo}_0)$
$L(Q_1)$
$L_{\omega_1\omega}$
$L(Q_0)$
$L_{\omega\omega}$


$L^{1,ch}_\kappa, L^1_{\kappa,\alpha}$
L^1_κ
$L^{1,c}_\kappa$
L^{aec}_κ


$L^{1,ch}_\kappa, L^1_{\kappa,\alpha}$
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Interpolation

- Craig($L_{\kappa^+\omega}, L_{(2^\kappa)^+\kappa^+}$) (Malitz 1971).

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  If $\varphi \vdash \psi$, where φ is a τ_1-sentence and ψ is a τ_2-sentence and both are in $L_{\kappa^+\omega}$ then
  there exists $\chi \in L_{(2^\kappa)^+\kappa^+}(\tau_1 \cap \tau_2)$ such that

$$\varphi \vdash \chi \vdash \psi.$$

- The original argument used "consistency properties". Other proofs have stressed the "Topological Separation" aspect of Interpolation.

So what about "balancing" Interpolation?

- Problem: Find L^* such that

$$L_{\kappa^+\omega} \leq L^* \leq L_{(2^\kappa)^+\kappa^+}$$

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and $\mathsf{Craig}(\mathsf{L}^*)$.

- Shelah, 2012: For singular strong limit κ of cofinality ω there is a logic L^1_κ such that

$$\bigcup_{\lambda<\kappa} \mathsf{L}_{\lambda^+\omega} \leq \mathsf{L}^1_\kappa \leq \bigcup_{\lambda<\kappa} \mathsf{L}_{\lambda^+\lambda^+}$$

and $\mathsf{Craig}(\mathsf{L}^1_\kappa)$.

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$$\bigcup_{\lambda<\kappa} L_{\lambda^+\omega} \leq L^1_\kappa \leq \bigcup_{\lambda<\kappa} L_{\lambda^+\lambda^+}$$

and $\mathrm{Craig}(L^1_\kappa)$.

- Moreover, in the case $\kappa = \beth_\kappa$, the logic L^1_κ also has a Lindström-type characterization as the maximal logic with a peculiar strong form of undefinability of well-order.

A description of Shelah's logic L^1_κ

- Shelah's L^1_κ is not really defined as usual; rather, it is defined by declaring what its elementary equivalence relation is.

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- This elementary equivalence relation is given by an EF-game type equivalence.
- Then…what is the syntax of Shelah's logic?
- We describe two partial answers, one approaching from below (Väänänen-V.), the other one from above (Džamonja, Väänänen).

SHELAH'S GAME $G^\beta_\theta(M, N)$.

ANTI	ISO
$\beta_0 < \beta, \vec{a^0}$	
	$f_0 : \vec{a^0} \to \omega$, $g_0 : M \to N$ a p.i.
$\beta_1 < \beta_0, \vec{b^1}$	$f_1 : \vec{a^1} \to \omega$, $g_1 : M \to N$ a p.i., $g_1 \supseteq g_0$
$\vdots$	$\vdots$

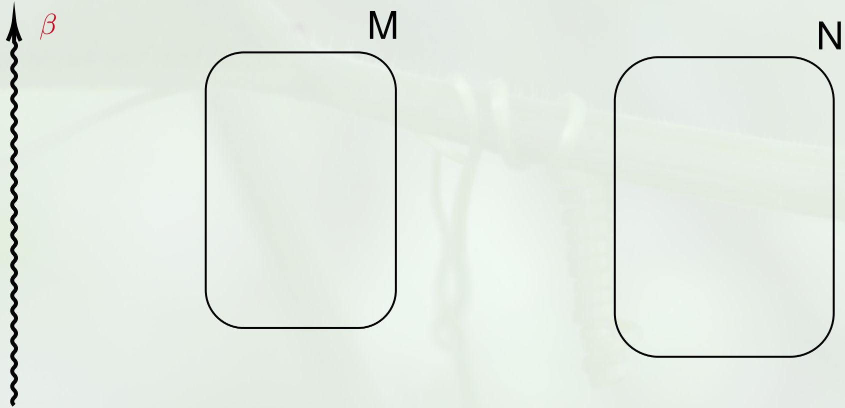
Constraints:

- $\text{len}(\vec{a^n}) \leq \theta$
- $f^{-1}_{2n}(m) \subseteq \text{dom}(g_{2n})$ for $m \leq n$.
- $f^{-1}_{2n+1}(m) \subseteq \text{ran}(g_{2n})$ for $m \leq n$.

ISO wins if she can play all her moves, otherwise ANTI wins.

- $\mathsf{M} \sim^\beta_\theta \mathsf{N}$ iff ISO has a winning strategy in the game.
- $\mathsf{M} \equiv^\beta_\theta \mathsf{N}$ is defined as the transitive closure of $\mathsf{M} \sim^\beta_\theta \mathsf{N}$.
- A union of $\leq \beth_{\beta+1}(\theta)$ equivalence classes of $\equiv^\beta_\theta$ for some $\theta < \kappa$ and $\beta < \theta^+$ is called a sentence of L^1_κ.

Shelah's game $G^\beta_\theta(M, N)$.



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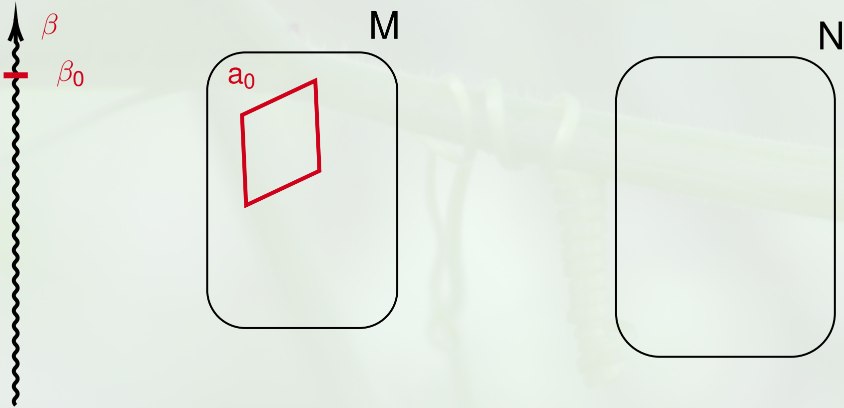

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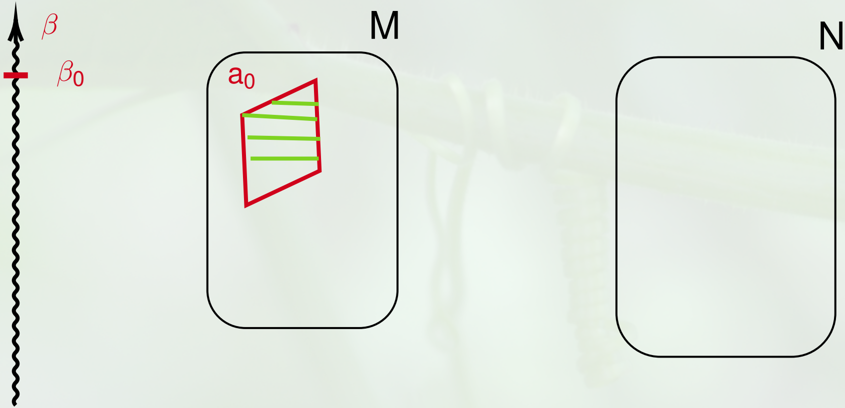

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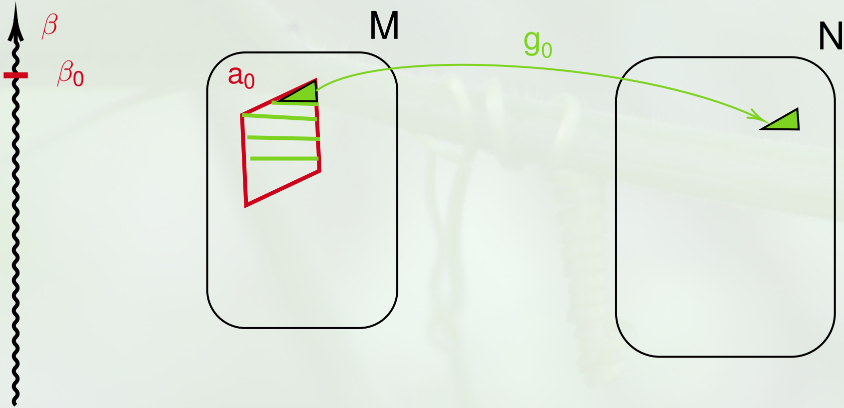

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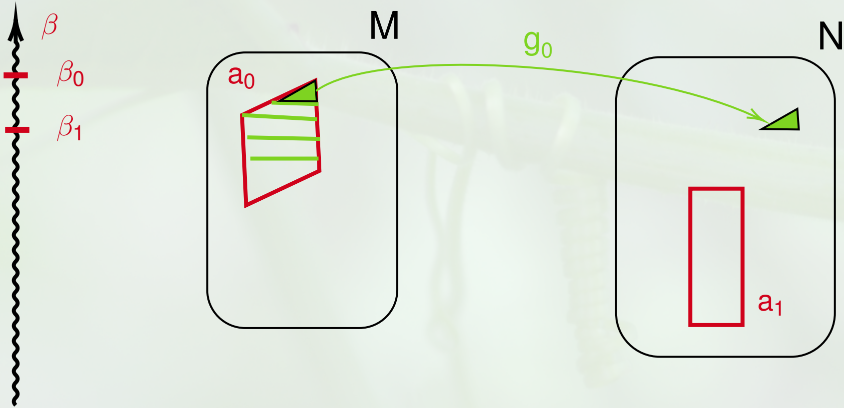

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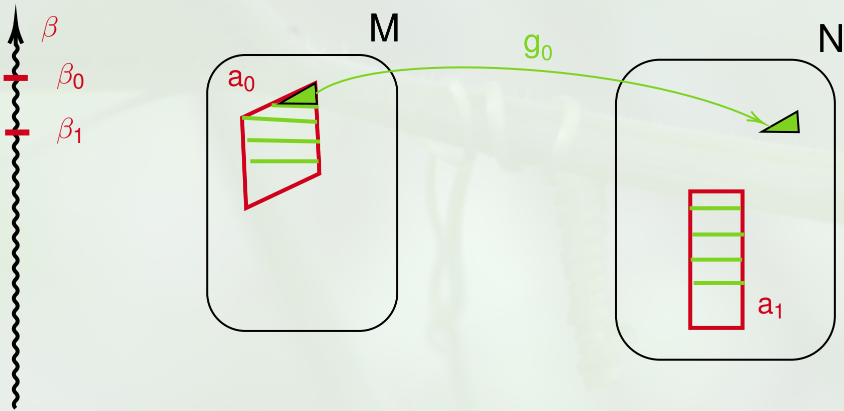

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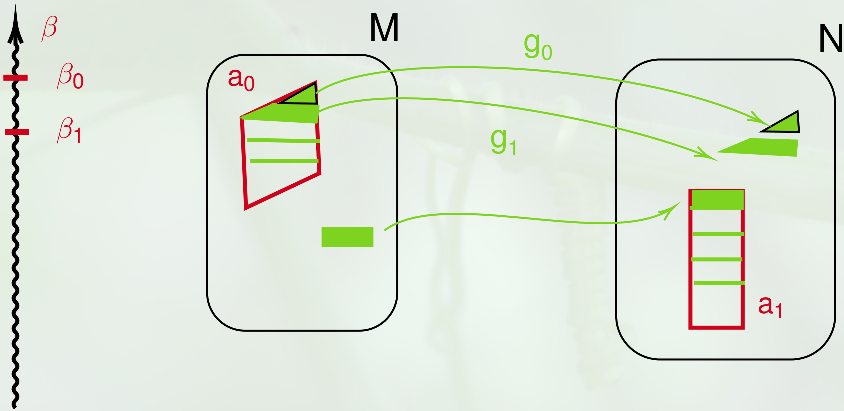

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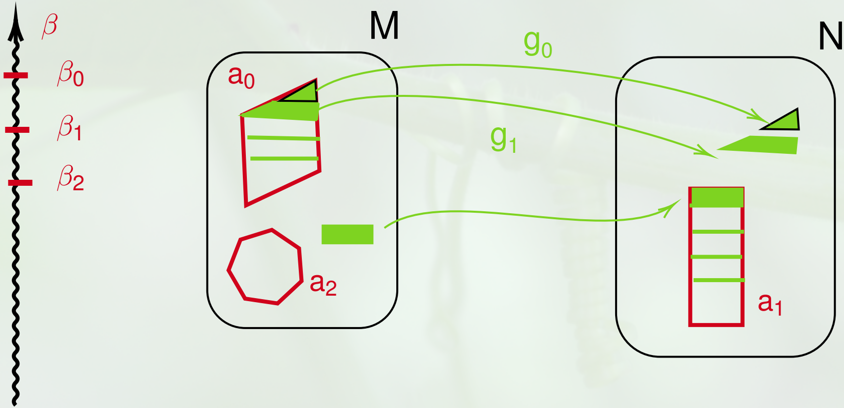


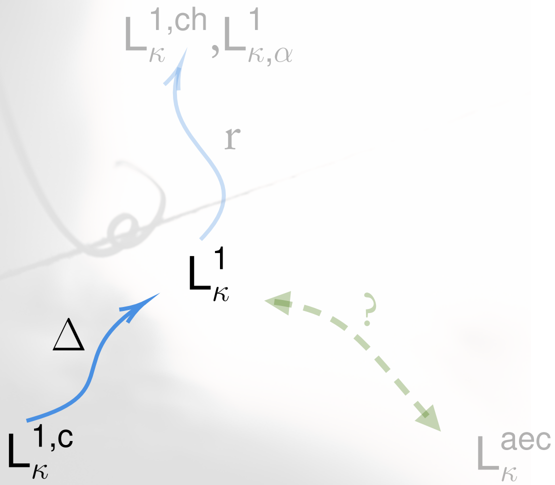
Plan

Shelah's logic L^1_κ

An approximation from below: $\mathsf{L}^{1,c}_\kappa$

Approximations from above: chain logic, …

Bonus: logics to capture aecs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Approaching L^1_κ from below (mod Δ)

- Joint work with **J. Väänänen**

- We define a sublogic $L^{1,c}_\kappa$ of L^1_κ ("Cartagena Logic")
- If $\mathrm{len}(\vec{x}) = \theta$, $f : \vec{x} \to \omega$, $\phi_{f,n}(\vec{x}, \vec{y})$ formulas of $L^{1,c}_\kappa$ with only the variables x_α, $f(\alpha) = n$, free among $\vec{x}$, then the following is a formula of $L^{1,c}_\kappa$:

$$\forall \vec{x} \bigvee_f \bigwedge_n \phi_{f,n}(\vec{x}, \vec{y}).$$

Cardinality quantifiers may be captured: $|P| < \theta$

Example

Let $\theta < \kappa$ such that $\mathrm{cof}(\theta) > \omega$. Let $\mathrm{len}(\vec{x}) = \theta$. The sentence

$$\forall \vec{x} \bigvee_f \bigwedge_n \Big(\bigwedge_{f(i)=n} P(x_i) \rightarrow \bigvee_{i \neq j \in f^{-1}(n)} (x_i = x_j)\Big)$$

says $|P| < \theta$.

An example of expressive power: no long chains

Example

Let $\theta < \kappa$ such that $\mathrm{cof}(\theta) > \omega$. Let $\mathrm{len}(\vec{x}) = \theta$. The sentence

$$\forall \vec{x} \bigvee_f \bigwedge_n \bigwedge_{i \neq j \in f^{-1}(n)} \neg x_i < x_j$$

says $<$ has no chains of length θ.

A covering property: the combinatorial core of L^1_κ!

The combinatorial core of Shelah's L^1_κ is captured by $L^{1,c}_\kappa$...

Example

Let $\theta < \kappa$ such that $\mathrm{cof}(\theta) > \omega$. Let $\mathrm{len}(\vec{x}) = \theta$ and $\mathrm{len}(\vec{y}) = \omega$. The sentence

$$\forall \vec{x} \bigvee_f \bigwedge_n \exists \vec{y} \bigwedge_g \bigvee_m \bigwedge_{f(i)=n} \bigvee_{g(j)=m} R(y_j, x_i)$$

says every set of size $\leq \theta$ can be covered by countably many sets of the form $R(a, \cdot)$.

Corollary

Suppose $\theta < \kappa$. There is a sentence in $L^{1,c}_\kappa$ which has a model of cardinality θ if and only if $\theta^\omega = \theta$.

The EF-game of $L^{1,c}_\kappa$: $G^{\beta,c}_\theta(M,N)$.

$\beta_0 < \beta, \vec{a^0}$	
	$f_0 : \vec{a^0} \to \omega$
$n_0 < \omega$	
	$g_0 : M \to N$ a p.i.
$\beta_1 < \beta_0, \vec{a^1}$	
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Constraints:

- $\text{len}(\vec{a^n}) \leq \theta$
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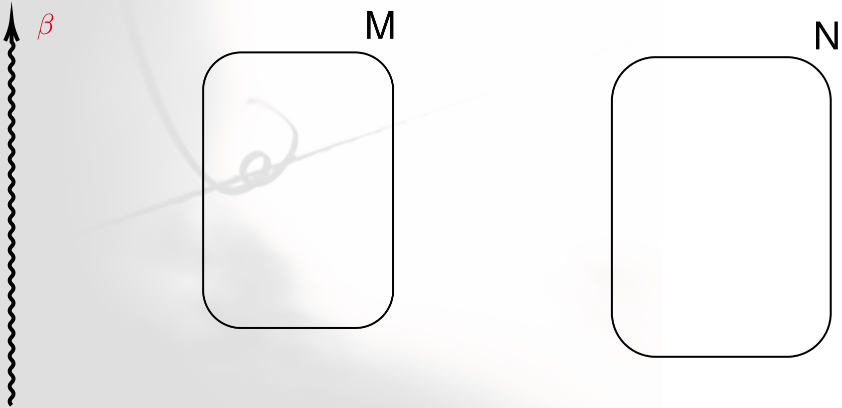
Player II wins if she can play all her moves, otherwise Player I wins.

Our "Cartagena" game $G^{\beta,c}_\theta(M, N)$.



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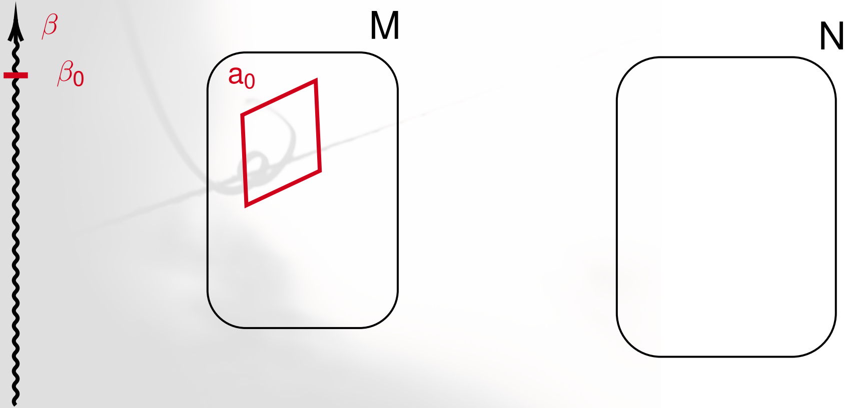

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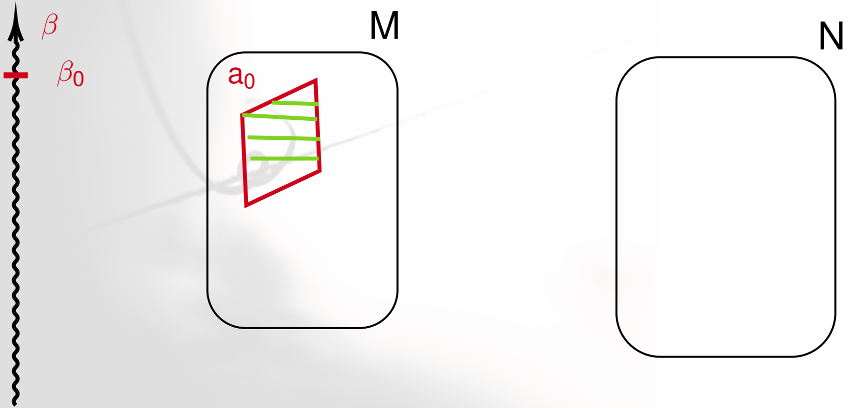

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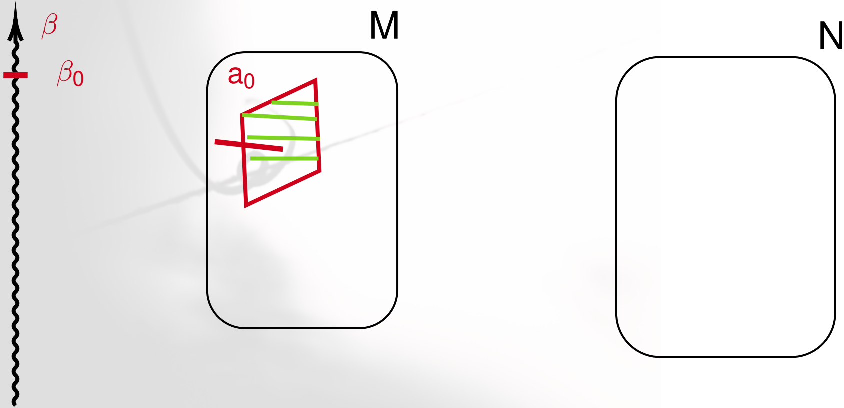

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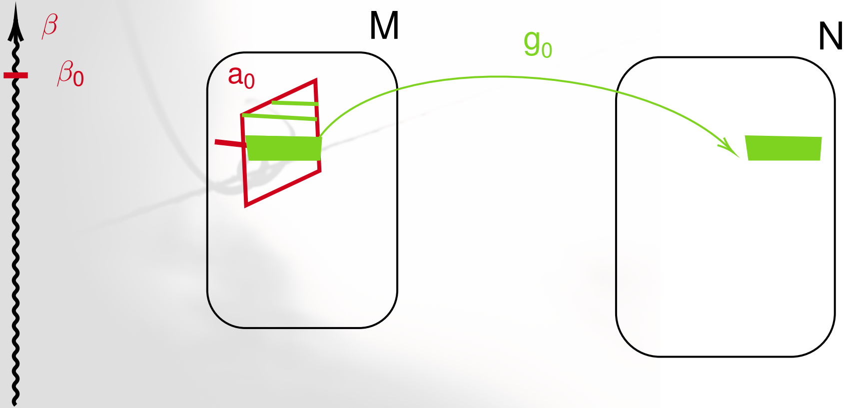

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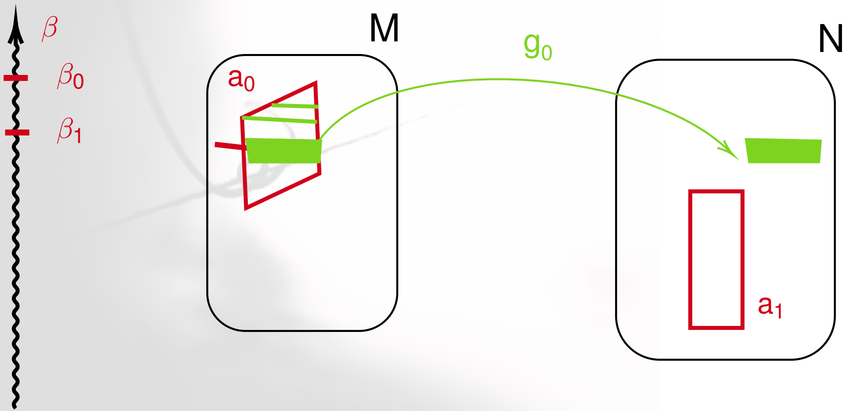

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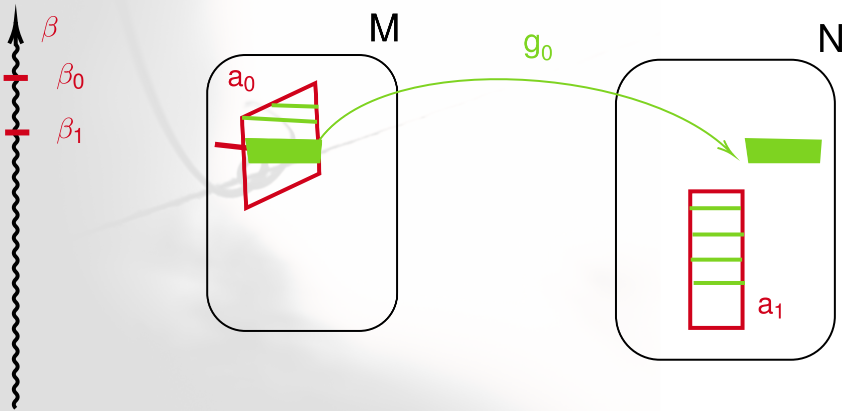

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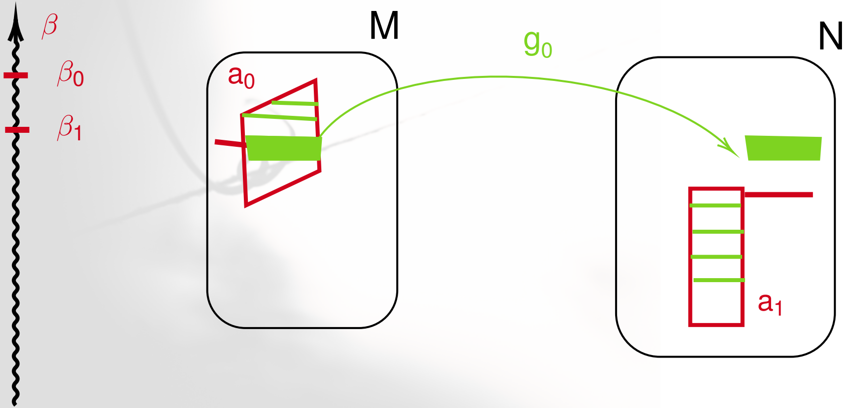

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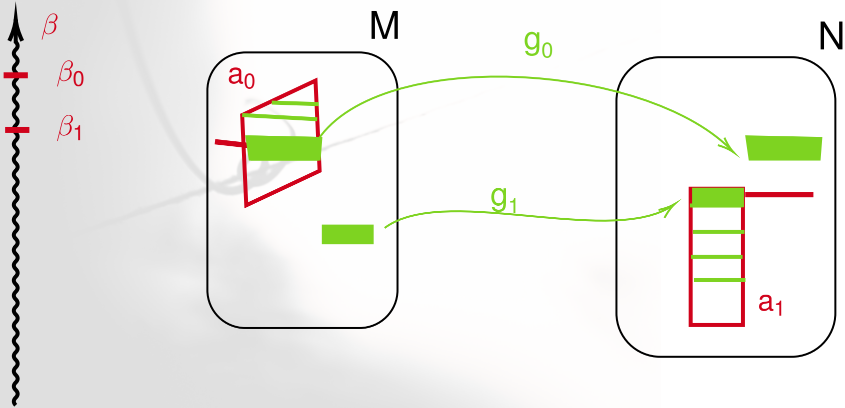

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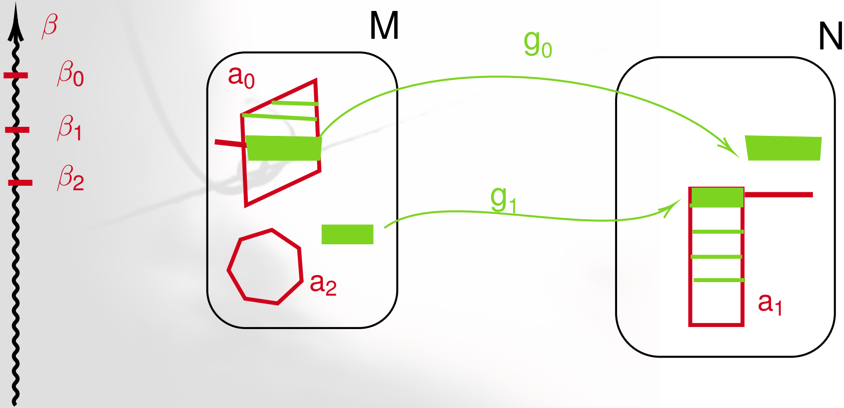

Theorem

The following are equivalent:

1. *Player II has a winning strategy in* $\mathsf{G}^{\beta,c}_\theta(\mathsf{M},\mathsf{N})$.
2. M *and* N *satisfy the same sentences of* $\mathsf{L}^{1,c}_{\theta^+}$ *of quantifier rank* $\leq \beta$.

Corollary

$\mathsf{L}^{1,c}_\kappa \leq \mathsf{L}^1_\kappa$.

Theorem

Assume $\kappa = \beth_\kappa$. *Then* $\Delta(\mathsf{L}^{1,c}_\kappa) = \mathsf{L}^1_\kappa$.

What is $\Delta(\mathsf{L})$?

- A model class $\mathcal{K}$ is $\Sigma(\mathsf{L})$ if it is the class of relativized reducts of an L-definable model class.
- A model class $\mathcal{K}$ is $\Delta(\mathsf{L})$ if both $\mathcal{K}$ and its complement are $\Sigma(\mathsf{L})$.
- $\Delta(\mathsf{L}_{\omega\omega}) = \mathsf{L}_{\omega\omega}$
- $\Delta(\mathsf{L}_{\omega_1\omega}) = \mathsf{L}_{\omega_1\omega}$
- $\Delta(\Delta(\mathsf{L})) = \Delta(\mathsf{L})$
- Δ preserves compactness, axiomatizability, Löwenheim-Skolem properties…

The advantages of $\mathsf{L}^{1,c}_\kappa$

- Simple syntax.
- Can express what L^1_κ does, at least implicitly.
- Its Δ-extension has Craig and Lindström Theorem.
- Undefinability of well-ordering is (also) a consequence of Caicedo's theorem on rigid structures and Uniform Reducibility of Pairs.

Plan

Shelah's logic L^1_κ

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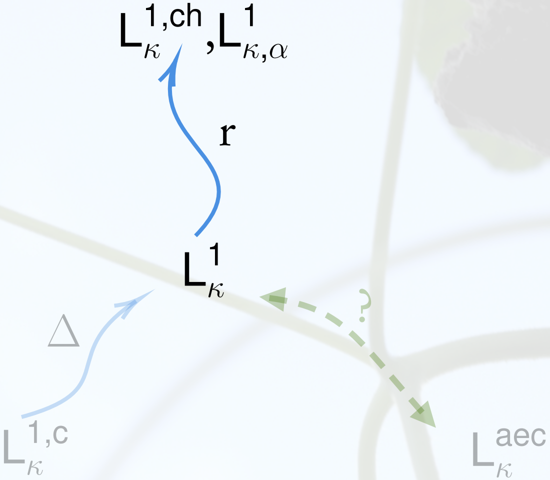
Approximations from above: chain logic, …
- Bonus: logics to capture aecs

Musings on approximation from above



I: Chain logic $L^{1,ch}_\kappa$: Carol Karp

(This is recent work of Džamonja and Väänänen)

- Syntax: $L_{\kappa\kappa}$, κ singular strong limit of cof ω.
- Semantics in chain models ($M_0 \subseteq M_1 \subseteq \ldots$)
- $\exists\vec{x}\phi$ means $\exists\vec{x}((\bigvee_n \bigwedge_j x_j \in M_n) \wedge \phi)$
- Craig($L^{1,ch}_\kappa$) (E. Cunningham, 1975)
- $L_{\kappa\omega} < L^{1,ch}_\kappa < L_{\kappa\kappa}$
- $L^1_\kappa \leq L^{1,c}_\kappa < L_{\kappa\kappa}$
- "Chu-transform" (Chu-spaces) is used as a device to compare logics.

II: From above, a new game (other splittings)

- L^1_κ is robust, but the lack of proper syntax if problematic.
- Väänänen and Velickovic define a deliberately stronger but simpler logic and then show that it is the same as L^1_κ, under conditions on κ.

The modified game $G^{1,\beta}_{\theta,\alpha}(M, N)$.

$\beta_0 < \beta, \vec{a^0}$	
	$f_0 : \vec{a^0} \to \alpha$, $g_0 : M \to N$ a p.i.
$\beta_1 < \beta_0, \vec{b^1}$	$f_1 : \vec{a^0} \cup \vec{b^1} \to \alpha$, $g_1 : M \to N$ a p.i., $g_1 \supseteq g_0$
⋮	⋮

Constraints:

- $\mathrm{len}(\vec{a^n}) \leq \theta$, $\mathrm{len}(\vec{b^n}) \leq \theta$.
- $f_{i+1}(x) < f_i(x)$ if $f_i(x) \neq 0$.
- $f^{-1}_{2n}(0) \subseteq \mathrm{dom}(g_{2n})$ for $m \leq n$.
- $f^{-1}_{2n+1}(0) \subseteq \mathrm{ran}(g_{2n})$ for $m \leq n$.

Player II wins if she can play all her moves, otherwise Player I wins.

FROM ABOVE, THE VÄÄNÄNEN-VELICKOVIC VARIANT OF THE GAME

- $G^{1,\beta}_{\theta,\alpha}(M, N)$ is the EF-game of a logic $L^1_{\theta,\alpha}$ up to the quantifier-rank β.
- If $\omega \leq \alpha \leq \alpha'$ and $\theta \leq \eta$, then $L^1_\theta \leq L^1_{\theta,\alpha} \leq L^1_{\theta,\alpha'} \leq L_{\eta^+\eta^+}$.
- If α is indecomposable, then "Player II has a winning strategy in $G^{1,\beta}_{\theta,\alpha}(M, N)$" is transitive and $L^1_{\kappa,\alpha}$ has a syntax (less clear than that of our $L^{1,c}_\kappa$).

From above, the Väänänen-Velickovic variant of the game

Theorem

If $\kappa = \beth_\kappa$ *and* α *is indecomposable, then* $\mathsf{L}^1_\kappa = \mathsf{L}^1_{\kappa,\alpha}$.

Comparison of the two games:

Trivially: If $\beta' \leq \beta$, $\theta' \leq \theta$ and $\alpha \leq \alpha'$, then

$$\mathrm{II} \uparrow G^{1,\beta}_{\theta,\alpha}(A, B) \Rightarrow \mathrm{II} \uparrow G^{1,\beta'}_{\theta',\alpha'}(A, B).$$

Theorem

For every β there is β^* such that

$$\mathrm{II} \uparrow G^{1,\beta^*}_{2^\theta,\alpha}(A, B) \Rightarrow \mathrm{II} \uparrow G^{1,\beta}_{\theta,\omega}(A, B).$$

Here if $\kappa = \beth_\kappa$ and $\beta < \kappa$, then $\beta^* < \kappa$. The proof uses a lemma by Komjath and Shelah (A partition theorem for scattered order types. Combin. Probab. Comput. 12 (2003), no. 5-6, 621-626.)

For any α let $\mathsf{FS}(\alpha)$ be the tree of all descending sequences of elements of α. We use $\mathsf{len}(\mathsf{s})$ to denote the length of $\mathsf{s} \in \mathsf{FS}(\alpha)$.

Lemma (Komjath-Shelah 2003)

Assume that α *is an ordinal and* I *a set. Set* $\lambda = (|\alpha|^{|\mathsf{I}|^{\aleph_0}})^{++}$. *Suppose* $\mathsf{T} = \mathsf{FS}(\lambda)$ *and* $\mathsf{F} : \mathsf{T} \to \mathsf{I}$. *Then there is a subtree* $\mathsf{T}^* = \{(\delta^{\mathsf{s}}_0, \ldots, \delta^{\mathsf{s}}_{\mathsf{n}}) : \mathsf{s} = (\mathsf{s}_0, \ldots, \mathsf{s}_{\mathsf{n}}) \in \mathsf{FS}(\alpha)\}$ *of* T *and a function* $\mathsf{c} : \omega \to \mathsf{I}$ *such that for all* $\mathsf{s} \in \mathsf{T}^*$ *we have* $\mathsf{F}(\mathsf{s}) = \mathsf{c}(\mathsf{len}(\mathsf{n}))$.


$L^{1,ch}_\kappa, L^1_{\kappa,\alpha}$
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Δ
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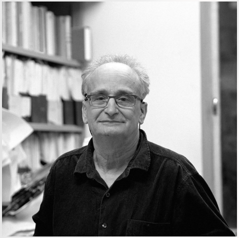
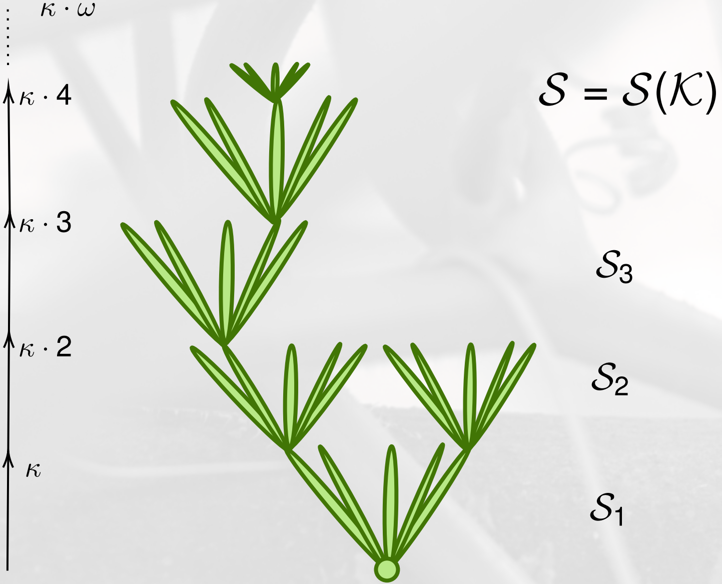
The canonical tree of an a.e.c.



This is joint work with Saharon Shelah.
Fix an a.e.c. $\mathcal{K}$ with vocabulary τ and $\mathsf{LS}(\mathcal{K}) = \kappa$.
Let $\lambda = \beth_2(\kappa + |\tau|)^+$.
The **canonical tree** of $\mathcal{K}$:

- $\mathcal{S}_n := \{M \in \mathcal{K} \mid$ for some $\bar{\alpha} = \bar{\alpha}_M$ of length n, M has universe $\{a^*_\alpha \mid \alpha \in S_{\bar{\alpha}[M]}\}$ and $m < n \Rightarrow M \restriction S_{\bar{\alpha}\restriction m[M]} \prec_{\mathcal{K}} M\}$ (and $\mathcal{S}_0 = \{M_{empt}\}$),
- $\mathcal{S} = \mathcal{S}_{\mathcal{K}} := \bigcup_n \mathcal{S}_n$; this is a tree with ω levels under $\prec_{\mathcal{K}}$ (equivalenty under $\subseteq$).

$\mathcal{S}(\mathcal{K})$



Formulas $\varphi_{M,\gamma,n}(\bar{x}_n)$

For M in the canonical tree $\mathcal{S}$ at level n, a formula with $\kappa \cdot n$ free variables, defined by induction on γ.

- $\gamma = 0$: $\varphi_{0,0} = \top$ ("truth"). If $n > 0$,

$$\varphi_{M,0,n} := \bigwedge \mathrm{Diag}^n_\kappa(M),$$

  the atomic diagram of M in $\kappa \cdot n$ variables.
- γ limit: Then

$$\varphi_{M,\gamma,n}(\bar{x}_n) := \bigwedge_{\beta<\gamma} \varphi_{M,\beta,n}(\bar{x}_n).$$

- $\gamma = \beta + 1$: Then $\varphi_{M,\gamma,n}(\bar{x}_n)$ is the $L_{\lambda^+,\kappa^+}(\tau)$ formula

$$\forall \bar{z}_{[\kappa]} \bigvee_{\substack{N \succ_{\mathcal{K}} M \\ N \in \mathcal{S}_{n+1}}} \exists \bar{x}_{=n} \left[\varphi_{N,\beta,n+1}(\bar{x}_{n+1}) \wedge \bigwedge_{\alpha<\alpha_n[N]} \bigvee_{\delta \in S[N]} z_\alpha = x_\delta \right]$$

TESTING THE CLASS AGAINST THE TREE - DOES $M \in \mathcal{K}$?

So we have sentences $\varphi_{\gamma,0}$, for $\gamma < \lambda^+$, such that $i < j < \lambda^+$ implies $\varphi_j \to \varphi_i$. These sentences are better and better approximations of the aec $\mathcal{K}$; they describe how small models of the class embed into arbitrary ones.

Let us take a closer look at low levels:

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When does $M \models \varphi_{1,0}$?

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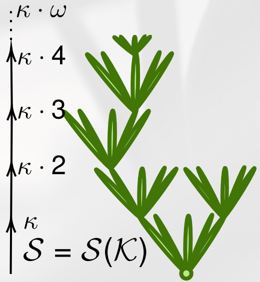
THIS IS SLIGHTLY MORE COMPLICATED TO UNRAVEL:

$\forall \bar{z}_{[\kappa]} \bigvee_{N \in \mathcal{M}_1} \exists \bar{x}_{=1} \left[\varphi_{N,1,1}(\bar{x}_1) \wedge \bigwedge_{\alpha < \alpha_0[N]} \bigvee_{\delta \in S[N]} z_\alpha = x_\delta \right]$

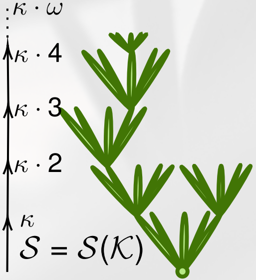
For every subset Z of M of size $\leq \kappa$ some model N in the tree (at level 1) M is such that $M \models \varphi_{N,1,1}$, through some "image of N" covering Z…

for all $Z' \subset M$ of size κ there is some $N' \succ_{\mathcal{K}} N$ in the canonical tree, at level 2, extending N, such that some tuple $\bar{x}_{=2}$ from M covers Z' and is the "image" of N' by an embedding

THE MEZCAL TEST - DOES $\mathsf{M} \in \mathcal{K}$?



The mezcal test - Does $M \in \mathcal{K}$?



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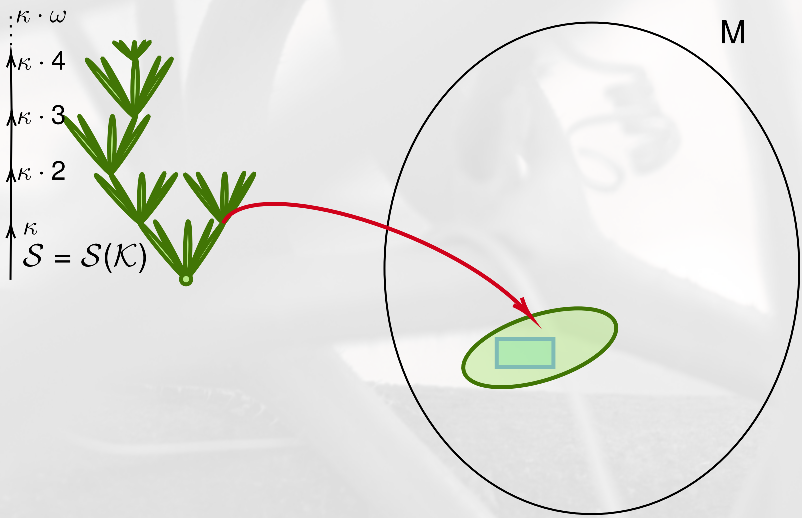

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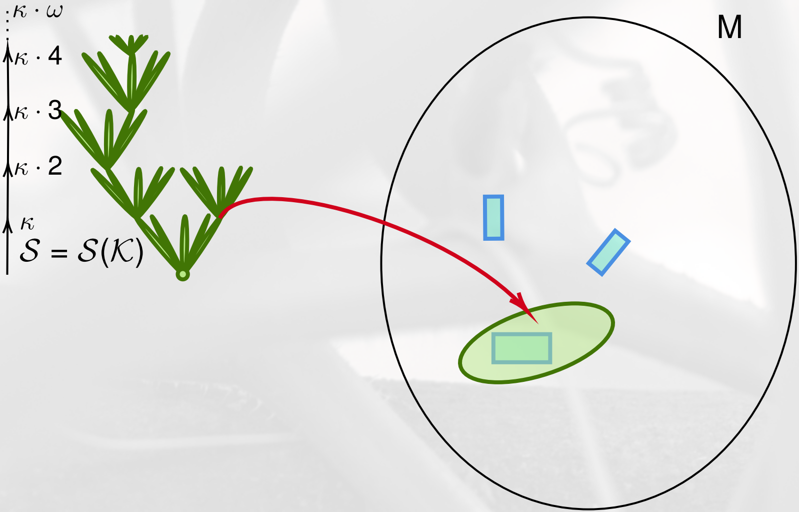

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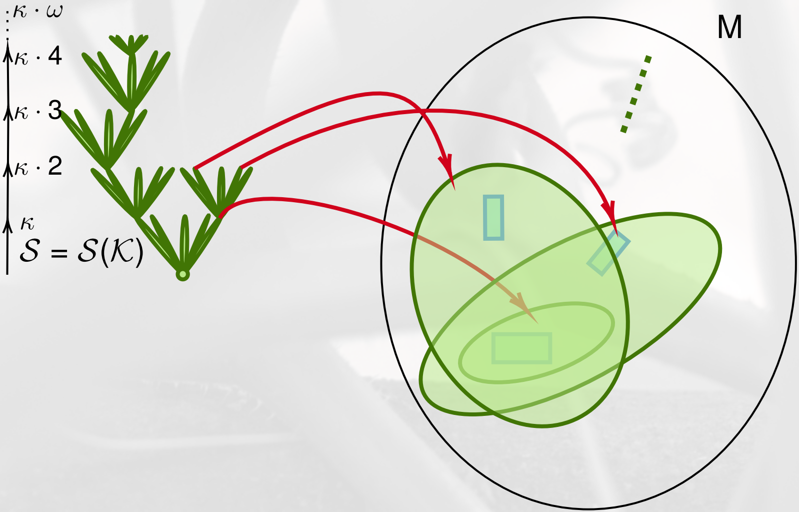

Theorem

$M \in \mathcal{K}$ *implies* $M \models \varphi_{\gamma,0}$ *for each* $\gamma < \lambda^+$

Theorem

$\mathsf{M} \models \varphi_{\beth_2(\kappa)^+ +2,0}$ *implies* $\mathsf{M} \in \mathcal{K}$

This much harder implication requires understanding the tree of possible embeddings of small models; the partition property due to Komjath and Shelah is the key…

The same partition property that worked for Väänänen and Velickovic's reduction of the game!

The tree property enables us to "reconstruct" M (satisfying $\varphi_{\lambda+2,0}$ as a limit of models of size κ, in the class $\mathcal{K}$).
With this we can

- define "quantificational depth" of an aec (variants of Baldwin-Shelah (building on Mekler and Eklöf) give examples of high quantificational depth)…
- get definability of the "strong submodel relation" $\prec_{\mathcal{K}}$ …and genuine variants of a Tarski-Vaught test
- a grip on biinterpretability of AECs…

Kiitos paljon!

From Chía, for the Helsinki Logic Seminar
May 2020