AECs and notions of existential closure

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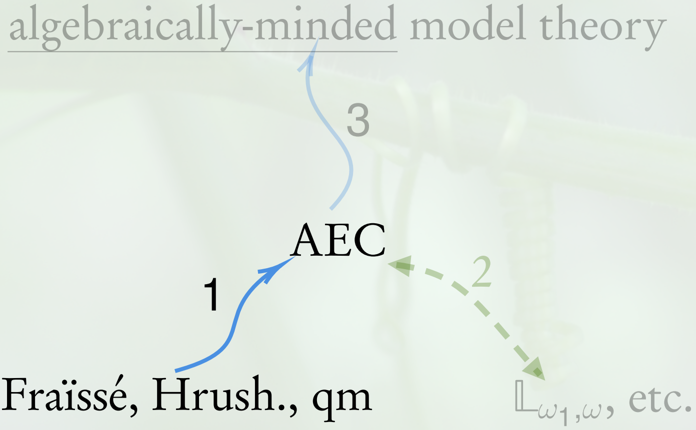
Contents

Various Origins of AECs



A twisted case: quasiminimal classes

In a language $\mathbb{L}$, a quasiminimal pregeometry class $\mathcal{Q}$ is a class of pairs $\langle \mathsf{H}, \mathsf{cl}_\mathsf{H} \rangle$ where H is an $\mathbb{L}$-structure, cl_H is a pregeometry operator on H such that the following conditions hold:

1. Closure under isomorphisms,
2. For each $\langle \mathsf{H}, \mathsf{cl}_\mathsf{H} \rangle \in \mathcal{Q}$, the closure of any finite set is countable,
3. If $\langle \mathsf{H}, \mathsf{cl}_\mathsf{H} \rangle \in \mathcal{Q}$ and $\mathsf{X} \subseteq \mathsf{H}$, then $\langle \mathsf{cl}_\mathsf{H}(\mathsf{X}), \mathsf{cl}_\mathsf{H} \rangle \restriction \mathsf{cl}_\mathsf{H}(\mathsf{X}) \in \mathcal{Q}$,
4. If $\langle \mathsf{H}, \mathsf{cl}_\mathsf{H} \rangle, \langle \mathsf{H}', \mathsf{cl}_{\mathsf{H}'} \rangle \in \mathcal{Q}$, $\mathsf{X} \subseteq \mathsf{H}$, $\mathsf{y} \in \mathsf{H}$ and $\mathsf{f} : \mathsf{H} \to \mathsf{H}'$ is a partial embedding defined on $\mathsf{X} \cup \{\mathsf{y}\}$ then $\mathsf{y} \in \mathsf{cl}_\mathsf{H}(\mathsf{X})$ iff $\mathsf{f}(\mathsf{y}) \in \mathsf{cl}_{\mathsf{H}'}(\mathsf{f}(\mathsf{X}))$,
5. Homogeneity over countable models.

(**Uniqueness** of non-algebraic types…)
Moreover...

On QM classes

(These are joint results with Zaniar Ghadernezhad…)

- If $\mathcal{Q}$ is a quasiminimal pregeometry class, $M \in \mathcal{Q}$ is of size $\aleph_1$, $\mathcal{C} = \{ \mathrm{cl}(A) \mid A \subseteq M, A$ small$\}$, then $\mathcal{C}$ has the free aut-independence amalgamation property.
- $\mathcal{Q}$ quasiminimal pregeom. class $\rightarrow$ for every model M of $\mathcal{Q}$, $\mathrm{Aut}(M)$ has SIP, therefore (e.g.) the so-called Zilber field has the SIP.

Aut-Independence

Another way to get generics is via "aut-independence"

Definition 12. Let $A, B, C \in \mathcal{C}$. Define $A \perp^{a}_{B} C$ if for all $f_1 \in \mathrm{Aut}(A)$ and all $f_2 \in \mathrm{Aut}(C)$ and for all $h_i \in \mathcal{O}_{f_i}$ $(i = 1, 2)$ such that $h_1 \upharpoonright A \cap C = h_2 \upharpoonright A \cap C$ and $h_1 \upharpoonright B = h_2 \upharpoonright B$ then $\mathcal{O}_g \neq \emptyset$ where $g := f_1 \cup f_2 \cup h_1 \upharpoonright B$.

where $\mathcal{O}_f := \left\{ \hat{f} \in \mathrm{Aut}(M) \mid \hat{f} \supset f \right\}$, whenever f is an automorphism of a subset of M.

Definition 13. Let $A, B, C \in \mathcal{C}$. Define $A \perp^{a-s}_{B} C$ if and only if $A' \perp^{a}_{B} C'$ for all $A' \subseteq A$ and $B' \subseteq B$ with $A', B' \in \mathcal{C}$.

Fact 14. *$\perp^{a-s}$ satisfies symmetry, monotonicity and invariance.*

8.4. **Free $\perp^{a-s}_{B}$ $\mathcal{C}$-amalgamation.** The class $\mathcal{C}$ has the *free $\perp^{a-s}$-amalgamation property* if for all $A, B, C \in \mathcal{C}$ with $A \cap B = C$ there exists $B' \in \mathcal{C}$ such that $ga - tp(B'/C) = ga - tp(B/C)$ (or there exists $g \in \mathrm{Aut}_C(M)$ that $g[B] = B'$) and $A \perp^{a-s}_{C} B'$.

Fact 15. *Suppose $\mathcal{C}$ has the free $\perp^{a-s}$-amalgamation property. Then generic automorphisms exist.*

AEC - THE AXIOMS, BRIEFLY

Fix $\mathcal{K}$ be a class of τ-structures, $\prec_{\mathcal{K}}$ a binary relation on $\mathcal{K}$.

Definition

$(\mathcal{K}, \prec_{\mathcal{K}})$ is an abstract elementary class iff

- $\mathcal{K}$, $\prec_{\mathcal{K}}$ are **closed under isomorphism**,
- $M, N \in \mathcal{K}$, $M \prec_{\mathcal{K}} N \Rightarrow M \subset N$,
- $\prec_{\mathcal{K}}$ is a partial order,
- (**TV**) $M \subset N \prec_{\mathcal{K}} \bar{N}, M \prec_{\mathcal{K}} \bar{N} \Rightarrow M \prec_{\mathcal{K}} N$,
- (↘**LS**) There is some $\kappa = LS(\mathcal{K}) \geq \aleph_0$ such that for every $M \in \mathcal{K}$, for every $A \subset |M|$, there is $N \prec_{\mathcal{K}} M$ with $A \subset |N|$ and $\|N\| \leq |A| + LS(\mathcal{K})$,
- (**Unions of $\prec_{\mathcal{K}}$-chains**) A union of an arbitrary $\prec_{\mathcal{K}}$-chain in $\mathcal{K}$ belongs to $\mathcal{K}$, is a $\prec_{\mathcal{K}}$-extension of all models in the chain and is the sup of the chain.

Examples

Natural constructions in Mathematics are examples of AEC (or metric AEC)

1. Complete first order theories
2. Homogeneous Model Theory
3. Excellent, quasiminimal classes
4. Various classes axiomatizable in $L_{\omega_1,\omega}$ or $L_{\kappa\omega}$.
5. Covers of Abelian algebraic groups

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6. Metric AEC - stability theory started by Hirvonen and Hyttinen, Usvyatsov, and continued by Zambrano and V.
7. Hilbert Spaces with Annihilation and Creation operators (Hyttinen, V.)
8. Gelfand triples (Zambrano, V.)

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8. Gelfand triples (Zambrano, V.)
9. AECs of C^*-algebras (Argoty, Berenstein, V.)
10. Zilber analytic classes (pseudoexponentiation)
11. “Hart-Shelah”-like examples (Baldwin, Kolesnikov, V.)
12. Classes of ACVF? - NTP_2 classes? (Not done yet!)

Our "type of types" (Galois), Galois-stability, etc.

The correct notion of types in AECs with amalgamation (and JEP and arbitrarily large models):

1. There is a large homogeneous model - the monster model $\mathbb{C}$ of the class.
2. We may define $\mathrm{ga-tp}(a/M) = \mathrm{ga-tp}(b/M)$ if and only if there exists $f \in \mathrm{Aut}(\mathbb{C}/M)$ such that $f(a) = b$.

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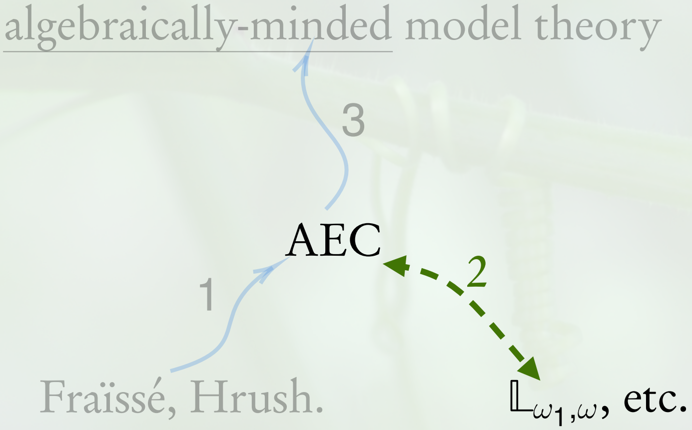
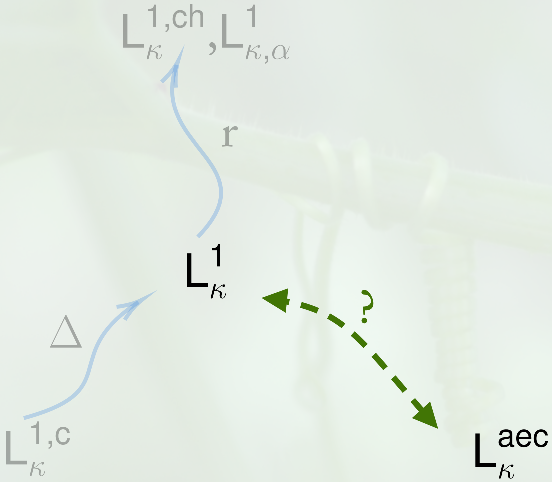
5. By the way, all this generalizes syntactic notions of types.
6. Hyttinen and Kesälä have come up with clever use of "Lascar Galois strong types" in Finitary AEC contexts...
7. Shelah in more recent work has (heavily axiomatized) notions of "regularity" of types in contexts with "no oxygen" (AP)

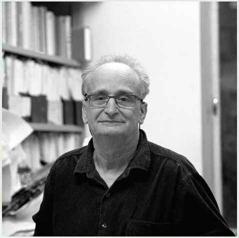
Plan

Various Origins of AECs



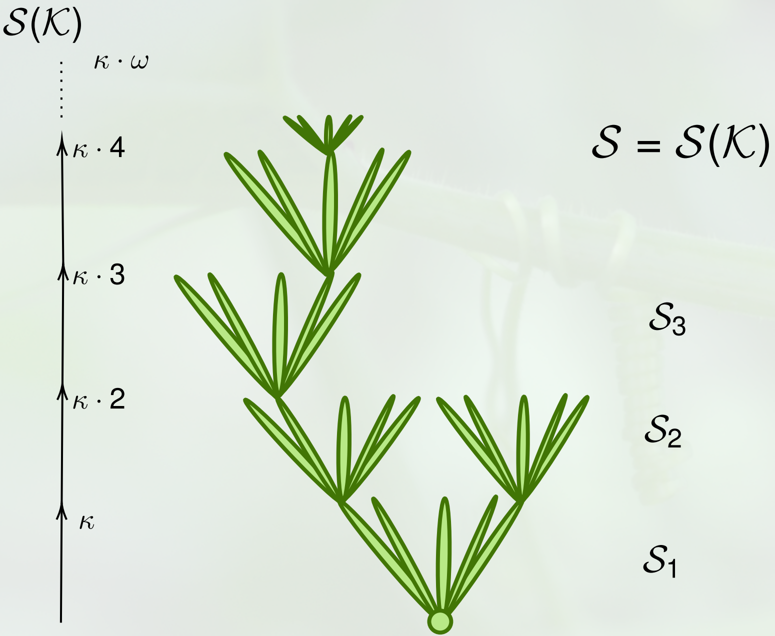

$L_\kappa^{1,ch}$, $L_{\kappa,\alpha}^{1}$
r
L_κ^1
?
$\triangle$
$L_\kappa^{1,c}$
L_κ^{aec}

The canonical tree of an a.e.c.



This is joint work with Saharon Shelah.
Fix an a.e.c. $\mathcal{K}$ with vocabulary τ and $\mathsf{LS}(\mathcal{K}) = \kappa$.
Let $\lambda = \beth_2(\kappa + |\tau|)^+$.
The **canonical tree** of $\mathcal{K}$:

- $\mathcal{S}_n := \{M \in \mathcal{K} \mid$ for some $\bar{\alpha} = \bar{\alpha}_M$ of length n, M has universe $\left\{a^*_\alpha \mid \alpha \in S_{\bar{\alpha}[M]}\right\}$ and $m < n \Rightarrow M \restriction S_{\bar{\alpha}\restriction m[M]} \prec_{\mathcal{K}} M\}$ (and $\mathcal{S}_0 = \left\{M_{empt}\right\}$),
- $\mathcal{S} = \mathcal{S}_{\mathcal{K}} := \bigcup_n \mathcal{S}_n$; this is a tree with ω levels under $\prec_{\mathcal{K}}$ (equivalenty under $\subseteq$).


$\mathcal{S}(\mathcal{K})$
$\kappa \cdot \omega$
$\kappa \cdot 4$
$\kappa \cdot 3$
$\kappa \cdot 2$
κ
$\mathcal{S} = \mathcal{S}(\mathcal{K})$
$\mathcal{S}_3$
$\mathcal{S}_2$
$\mathcal{S}_1$

FORMULAS $\varphi_{M,\gamma,n}(\bar{x}_n)$

For M in the canonical tree $\mathcal{S}$ at level n, a formula with $\kappa \cdot n$ free variables, defined by induction on γ.

- $\gamma = 0$: $\varphi_{0,0} = \top$ ("truth"). If $n > 0$,

$$\varphi_{M,0,n} := \bigwedge \mathrm{Diag}^n_\kappa(M),$$

  the atomic diagram of M in $\kappa \cdot n$ variables.

- γ limit: Then

$$\varphi_{M,\gamma,n}(\bar{x}_n) := \bigwedge_{\beta<\gamma} \varphi_{M,\beta,n}(\bar{x}_n).$$

- $\gamma = \beta + 1$: Then $\varphi_{M,\gamma,n}(\bar{x}_n)$ is the $L_{\lambda^+,\kappa^+}(\tau)$ formula

$$\forall \bar{z}_{[\kappa]} \bigvee_{\substack{N \succ_{\mathcal{K}} M \\ N \in \mathcal{S}_{n+1}}} \exists \bar{x}_{=n} \left[\varphi_{N,\beta,n+1}(\bar{x}_{n+1}) \wedge \bigwedge_{\alpha<\alpha_n[N]} \bigvee_{\delta \in S[N]} z_\alpha = x_\delta \right]$$

Testing the class against the tree - Does $\mathsf{M} \in \mathcal{K}$?

So we have sentences $\varphi_{\gamma,0}$, for $\gamma < \lambda^+$, such that $i < j < \lambda^+$ implies $\varphi_j \to \varphi_i$. These sentences are better and better approximations of the aec $\mathcal{K}$; they describe how small models of the class embed into arbitrary ones.
Let us take a closer look at low levels:

The catch (beginnings)

When does $\mathsf{M} \models \varphi_{1,0}$?

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That is, for every subset Z of M of size $\leq \kappa$ some model N in the tree (level 1, of size κ) embeds into M, covering Z.

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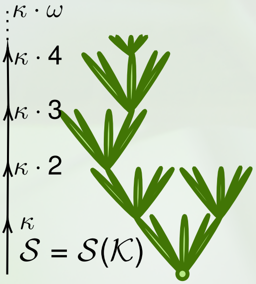
This is slightly more complicated to unravel:

$\forall \bar{z}_{[\kappa]} \bigvee_{N \in \mathcal{M}_1} \exists \bar{x}_{=1} \left[\varphi_{N,1,1}(\bar{x}_1) \wedge \bigwedge_{\alpha < \alpha_0[N]} \bigvee_{\delta \in S[N]} z_\alpha = x_\delta \right]$

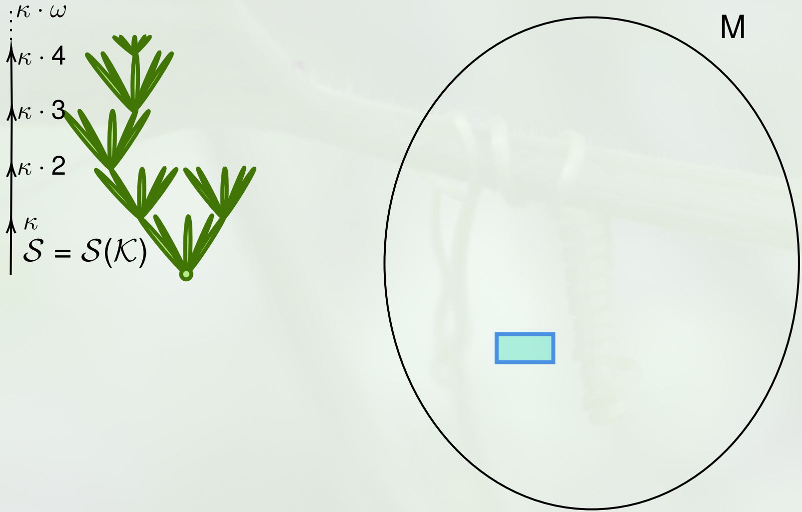
For every subset Z of M of size $\leq \kappa$ some model N in the tree (at level 1) M is such that $M \models \varphi_{N,1,1}$, through some "image of N" covering Z...

for all $Z' \subset M$ of size κ there is some $N' \succ_{\mathcal{K}} N$ in the canonical tree, at level 2, extending N, such that some tuple $\bar{x}_{=2}$ from M covers Z' and is the "image" of N' by an embedding

The mezcal test - Does $\mathsf{M} \in \mathcal{K}$?



The mezcal test - Does M $\in \mathcal{K}$?



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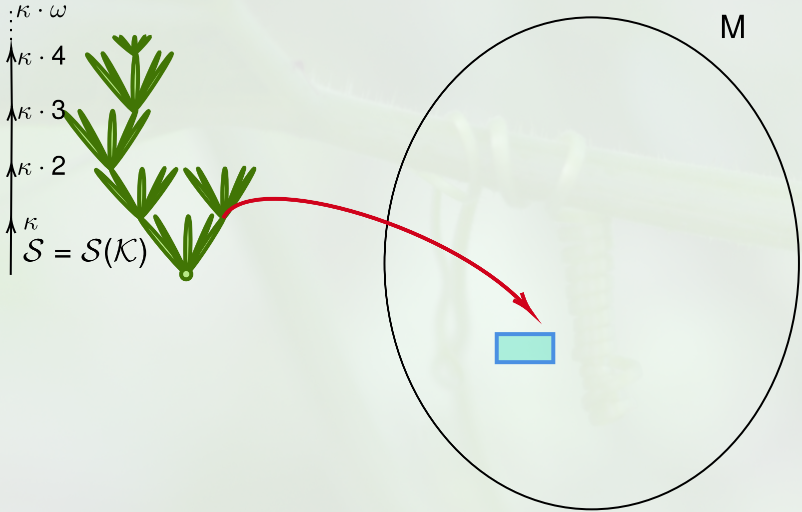

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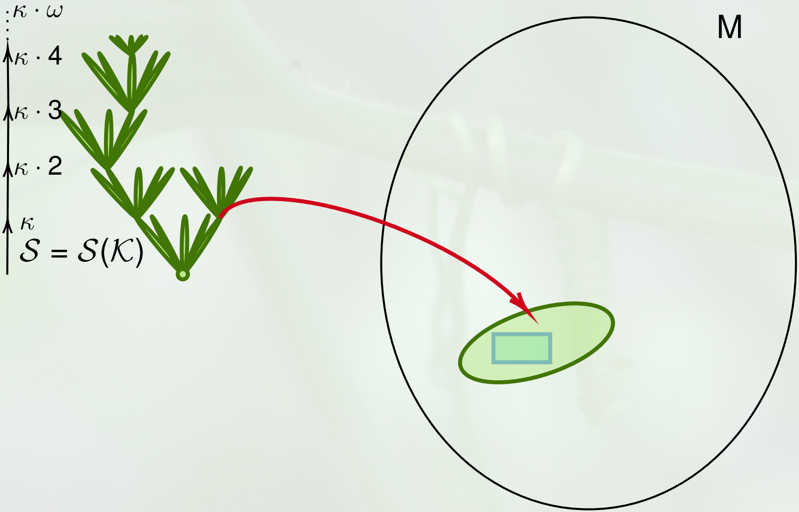

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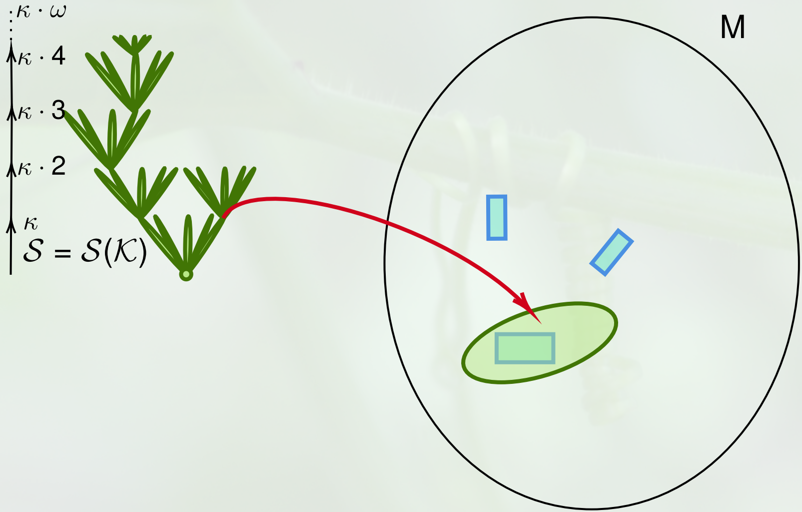

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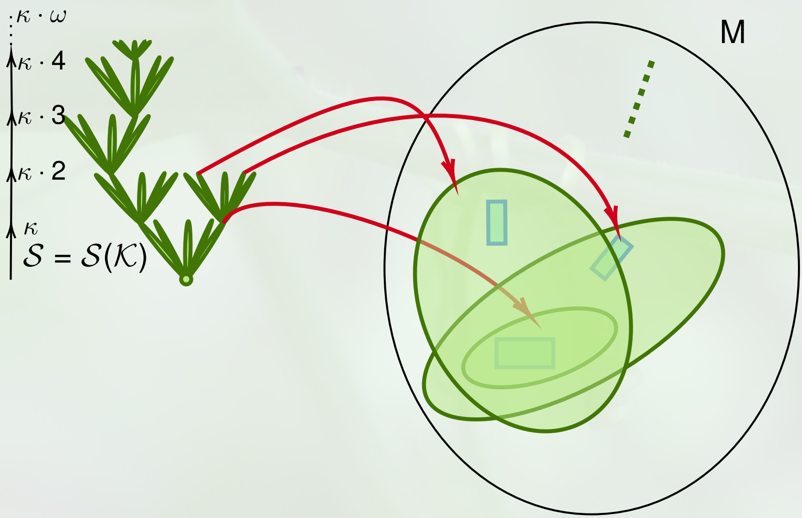


Theorem

$M \in \mathcal{K}$ *implies* $M \models \varphi_{\gamma,0}$ *for each* $\gamma < \lambda^+$

Theorem

$\mathsf{M} \models \varphi_{\beth_2(\kappa)^+ +2,0}$ *implies* $\mathsf{M} \in \mathcal{K}$

This much harder implication requires understanding the tree of possible embeddings of small models; a partition property for well-founded trees due to Komjath and Shelah is the key…

The tree property enables us to "reconstruct" M (satisfying $\varphi_{\lambda+2,0}$ as a limit of models of size κ, in the class $\mathcal{K}$).
With this we can

- define "quantificational depth" of an aec (variants of Baldwin-Shelah (building on Mekler and Eklöf) give examples of high quantificational depth)...
- get definability of the "strong submodel relation" $\prec_{\mathcal{K}}$ ...and genuine variants of a Tarski-Vaught test
- a grip on biinterpretability of AECs...

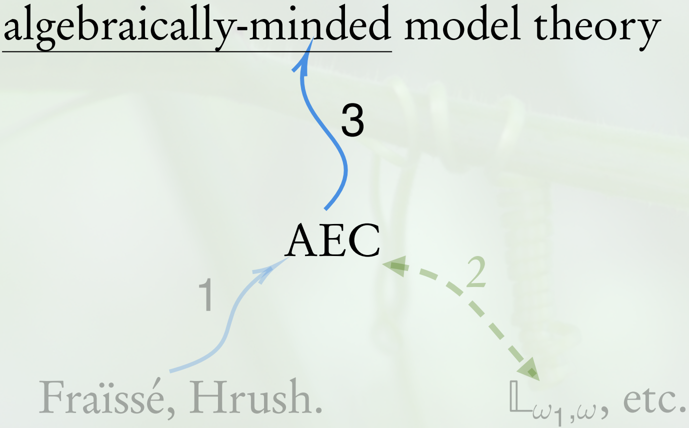
Plan

Closures, amalgams, AECs
- Various origins
- The definition of AECs
- The Presentation Theorem - Galois types - stability

Axiomatizing an AEC: new results

A case for study: locally finite groups
- Old issues and a somewhat different perspective
- Definability schemes

Various Origins of AECs



1975: THE CALL FOR "ALGEBRAICALLY-MINDED"...

[Sh: 54a]

THE LAZY MODEL-THEORETICIAN'S GUIDE TO STABILITY

by

Saharon SHELAH *
The Hebrew University of Jerusalem
Université Catholique de Louvain

§ 0. *INTRODUCTION*

The main aim of this article is to make propaganda for [S 1]; hence novelty is not the intention; there are many explanations, definitions and theorems and few proofs. No previous knowledge of stability is required, but then you have to take some statements on faith. We have in mind mainly those who are interested in algebraically-minded model theory, i.e. in generic models, the class of e-closed (= existentially closed) models and universal-homogeneous models rather than elementary classes and saturated models. So our main point is that though stability theory was developed for the latter context, almost everything goes through in the wider context

Locally finite groups; Hall's theorem

A group G is locally finite - lf if every finitely generated subgroup is finite.
G is an existentially closed lf group (also called universal locally finite in the literature) if G is a lf group and for any two finite groups $K \subseteq L$ and embedding $f : K \to G$, f can be extended to $g : L \to G$.

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The class $\mathcal{K}_{\mathsf{lf}}$ of lf groups is a non-first-order axiomatizable aec (it is also a so-called "homogeneous diagram"), with $\prec_{\mathcal{K}} = \subseteq$. The subclass $\mathcal{K}_{\mathsf{exlf}}$ consists of the existentially closed lf groups in $\mathcal{K}_{\mathsf{lf}}$.

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Hall (1959) proved that every lf group can be extended to an exlf group ("Hall's universal lf group").

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Hall (1959) proved that every lf group can be extended to an exlf group ("Hall's universal lf group").
$G^{\oplus} := \{ f \in \mathrm{Sym}(G) \mid \exists K \subseteq^{\mathrm{fin}} G\, [a \in G \to aK = f(a)K] \}$.
(and iterate ω times!)

Issues with naturality and cardinality

The "Hall extension" is good: canonical, and every automorphism of G may be extended to an automorphism of $G^{\oplus}$.
But the cardinality jumps: $|G^{\oplus}| = 2^{|G|}$.

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The "Hall extension" is good: canonical, and every automorphism of G may be extended to an automorphism of $\mathsf{G}^{\oplus}$.
But the cardinality jumps: $|\mathsf{G}^{\oplus}| = 2^{|\mathsf{G}|}$.
Also, if $\mathsf{G}_1 \subseteq \mathsf{G}_2 \dots$ there is no information on the connection between $\mathsf{G}_1^{\oplus}$ and $\mathsf{G}_2^{\oplus}$.
Shelah (in [Sh:312]) proposes a way to deal with these two issues.

NON-STRUCTURE RESULTS IN $\mathcal{K}_{\mathsf{exlf}}$

1. Grossberg-Shelah (1983): If $\lambda = \lambda^{\aleph_0}$, then no $\mathsf{G} \in \mathcal{K}_{\mathsf{exlf}}$ of cardinality λ is universal in the class.
2. Although $\mathcal{K}_{\mathsf{exlf}}$ is $\aleph_0$-categorical, it is far from being uncountably categorical (Macintyre-Shelah, 1976): 2^λ many non-isomorphic models if $\lambda > \aleph_0$.
3. Natural question: what about complete members of $\mathcal{K}_{\mathsf{exlf}}$ — groups with no non-inner automorphisms?
4. Hickin (1978): in $\aleph_1$ there is a complete locally finite group…really $2^{\aleph_1}$ such groups.
5. Giorgetta-Shelah (1980s): many more non-structure results…so, hopeless story?

Definability schemes

Shelah approaches this "globally" and aims at findind a "stable core" inside an extremely unstable class. For this, he proposes abstract definability schemes.

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Definition 0.6. We say that $p = \mathrm{tp}_{\mathrm{bs}}(\bar{a}, G, H) \in \mathbf{S}^n_{\mathrm{bs}}(G)$ does not split over $K \subseteq G$ when for every $m < \omega$ and $\bar{b}_1, \bar{b}_2 \in {}^m G$ satisfying $\mathrm{tp}_{\mathrm{bs}}(\bar{b}_1, K, G) = \mathrm{tp}_{\mathrm{bs}}(\bar{b}_2, K, G)$ we have $\mathrm{tp}_{\mathrm{bs}}(\bar{b}_1 {}^\frown \bar{a}, K, H) = \mathrm{tp}_{\mathrm{bs}}(\bar{b}_2 {}^\frown \bar{a}, K, H)$.

Definition 0.7. 1) Let $\mathbf{D}(\mathbf{K}) = \bigcup_n \mathbf{D}_n(\mathbf{K})$, where $\mathbf{D}_n(\mathbf{K}) = \{\mathrm{tp}_{\mathrm{bs}}(\bar{a}, \emptyset, M) : \bar{a} \in {}^n M \text{ and } M \in \mathbf{K}\}$.
2) Assume[1] $p(\bar{x})$ is a k-type, that is, $\bar{x} = \langle x_\ell : \ell < k \rangle$ and for some $p'(\bar{x})$ we have $p(\bar{x}) \subseteq p'(\bar{x}) \in \mathbf{D}_k(\mathbf{K})$ and $m < \omega$. We let $\mathbf{D}_{p(\bar{x}),m}(\mathbf{K}) = \mathbf{D}_m(p(\bar{x}), \mathbf{K})$ be the set of $q(\bar{x}, \bar{y}) \in \mathbf{D}_{k+m}(\mathbf{K})$ such that $q(\bar{x}, \bar{y}) \supseteq p(\bar{x})$, which means that there is $M \in \mathbf{K}$ and $\bar{a} \in {}^k M$ realizing $p(\bar{x})$ and $(\bar{a}, \bar{b})$ realizing $q(\bar{x}, \bar{y})$ in M, i.e. $\ell g(\bar{a}) = k, \ell g(\bar{b}) = m$ and $\bar{a} {}^\frown \bar{b}$ realizes $q(\bar{x}, \bar{y})$.
3) In part (2) let $\mathbf{D}_{p(\bar{x})}(\mathbf{K}) = \cup\{\mathbf{D}_m(p(\bar{x}), \mathbf{K}) : m < \omega\}$.

Remark 0.8. Below $\mathfrak{s} \in \Omega_{n,k}[\mathbf{K}]$ is a scheme to fully define a type $q(\bar{z}) \in \mathbf{S}^n_{\mathrm{bs}}(M)$ for a given parameter $\bar{a} \in {}^k M$ such that $q(\bar{z})$ does not split over $\bar{a}$. Sometimes $\mathfrak{s}$ is not unique but if, e.g., $M \in \mathbf{K}_{\mathrm{exlf}}$ it is.

Definition 0.9. 1) Let $\Omega[\mathbf{K}]$ be the set of schemes, i.e. $\cup\{\Omega_{n,k}[\mathbf{K}] : k, n < \omega\}$ where $\Omega_{n,k}[\mathbf{K}]$ is the set of (k,n)-schemes $\mathfrak{s}$ which means, see below.
1A) We say $\mathfrak{s}$ is a (k,n)-scheme when for some $p(\bar{x}) = p_{\mathfrak{s}}(\bar{x}_{\mathfrak{s}})$ with $\ell g(\bar{x}_{\mathfrak{s}}) = k$, (and $k_{\mathfrak{s}} = k(\mathfrak{s}) = k, n_{\mathfrak{s}} = n(\mathfrak{s}) = n$) we have:

- (a) $\mathfrak{s}$ is a function with domain $\mathbf{D}_{p(\bar{x})}(\mathbf{K})$ such that for each m it maps $\mathbf{D}_{p(\bar{x}),m}(\mathbf{K})$ into $\mathbf{D}_{k+m+n}(\mathbf{K})$
- (b) if $s(\bar{x}, \bar{y}) \in \mathbf{D}_{p(\bar{x}),m}(\mathbf{K})$ and $r(\bar{x}, \bar{y}, \bar{z}) = \mathfrak{s}(s(\bar{x}, \bar{y}))$ then $r(\bar{x}, \bar{y}, \bar{z}) {\restriction} (k+m) = s(\bar{x}, \bar{y})$; that is, if $(\bar{a}, \bar{b}, \bar{c})$, i.e. $\bar{a} {}^\frown \bar{b} {}^\frown \bar{c}$, realizes $r(\bar{x}, \bar{y}, \bar{z})$ in $M \in \mathbf{K}$ so $k = \ell g(\bar{a}), m = \ell g(\bar{b}), n = \ell g(\bar{c})$, then $\bar{a} {}^\frown \bar{b}$ realizes $s(\bar{x}, \bar{y})$ in M; see 1.2(1)

[1]This is used to define the set $\mathfrak{S}$ of schemes; for this section the case $p(\bar{x}) = p'(\bar{x})$ is enough

Canonical closure for locally finite groups

And gets "canonicity" of a new version of the old Hall closure of locally finite groups by using the definability schemes.

§ 0(C). **The Results.** In particular (in the so-called first avenue, see below):

Theorem 0.11. *Let λ be any cardinal $\geq |\mathfrak{S}|$.*
1) For every $G \in \mathbf{K}^{\mathrm{lf}}_{\leq\lambda}$ there is $H_G \in K^{\mathrm{exlf}}_{\lambda}$ which is λ-full over G (hence over any $G' \subseteq G$; see Definition 1.15) and $\mathfrak{S}$-constructible over it (see 1.19).
2) If $H \in \mathbf{K}^{\mathrm{lf}}_{<\lambda}$ is λ-full over $G(\in \mathbf{K}^{\mathrm{lf}}_{\leq\lambda})$ then H_G from above can be embedded into H over G, see 1.23(4).

This is proved by 1.23 + §2. So in some sense H_G is prime over G, that is, it is prime but not among the members of $\mathbf{K}^{\mathrm{exlf}}_{\lambda}$, i.e. for a different class. Still we would like to have canonicity so uniqueness. There are some additional avenues helpful toward this.

The second avenue tries to get results which are nicer by assuming $\mathfrak{S}$ is so called symmetric which is the parallel of being stable in this context. Under this assumption we prove the existence of a canonical closure of a locally finite group to an exlf one. This is done in 1.12, 1.13.

The third avenue is without assuming "$\mathfrak{S}$ is symmetric" but using a more complicated construction, for which we have similar, somewhat weaker results using

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Conclusions/Questions:

- Understand non-stable AECs is now under way (Grossberg and Mazari-Armida have made strong generalizations of Kim-Pillay to simple AECs, and I have work in progress with Shelah and with my student Nájar on dependent AECs - and dependent diagrams).

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Conclusions/Questions:

- Understand non-stable AECs is now under way (Grossberg and Mazari-Armida have made strong generalizations of Kim-Pillay to simple AECs, and I have work in progress with Shelah and with my student Nájar on dependent AECs - and dependent diagrams).
- Definability schemes have not been used (so far) very much elsewhere. They seem to hold a key for the specific case of dependent **homogeneous diagrams** (results by Kaplan, Lavi and Shelah, more results on the way)…

For your attention, thank you!

From Chía, Colombia - for the IC+QM London joint Logic/Model Theory Seminar
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