ZFC vs HoTT - ¿a possible crisis in foundations of mathematics?

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Topics

- New foundations?
- The first crisis: Why ZFC? What does ZFC achieve?
  - The continuum problem - properties of the reals
  - Cohen's times
  - Cohen's forcing: Expanding the universe?
  - Nombres (secciones) - control del genérico
  - The Cohen model - control
- HoTT + UF (Homotopy Type Theory + Univalent Foundations)
  - Type Theories
  - Univalence - Synthetic and Analytic
  - Conclusions/Beginnings

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In 2006 a new crisis is looming, after Voevodsky's work.

Voevodsky - A story of complicated back-and-forths

Voevodsky (1966-2017) in his short life received many accolades: Fields Medal in 2002 (age 36), for his new cohomology theories for algebraic varieties. He proved conjectures by Milnor and Bloch-Kato, connecting K-theoretic groups of fields, and Galois cohomology.

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- 2009: univalent models ... computational verification

Two different kinds of pathologies: in Objects…

The previous crisis, a century ago, was mainly a crisis on the (ontological) status of mathematical **objects:**

- What are real numbers? What's their cardinality, really?
- What are sets, really? - Russell's Paradox
- How do we "anchor" the reals, the continuum, on the natural numbers?
- What's the nature of Cantor's infinities?
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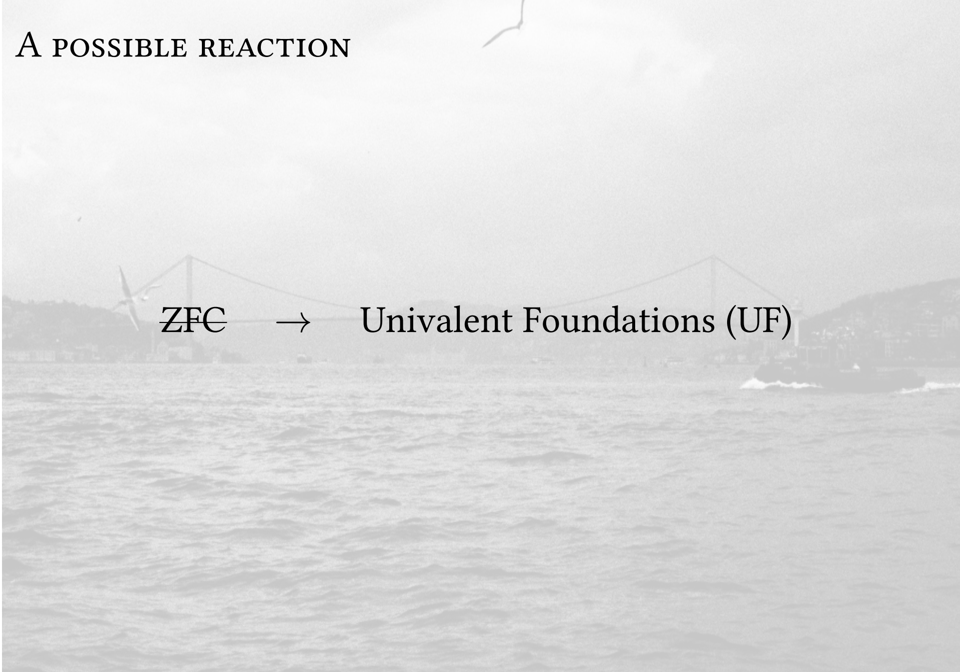
Voevodsky unveiled a crisis of mathematical **proofs,** rather than objects!

- How do we weed out a wrong lemma?
- How to reduce our creation to something “programmable” - so as to avoid mistakes?
- How to ground mathematics (at least, the part close to Voevodsky’s work) in a different way, close to the two previous questions?

A possible reaction



~~ZFC~~ $\rightarrow$ Univalent Foundations (UF)

Set Theory, seen from outside

In 2016, during an event in Bielefeld where ZFC was being compared with UF. Set Theorist Mirna Džamonja summarized the view from outside of set theory:

- ZFC axioms plus possible the existence of large cardinals
- Important for foundations of mathematics, since many classical notions may be axiomatized in set theory and can be represented in von Neumann's cumulative hierarchy.
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In a volume from 2015 celebrating the work of the logician Jouko Väänänen, his colleague Roman Kossak summarized the absolute success of ZFC: **a measure of the success of these foundations is that mathematicians do not care about these matters anymore**. (I would add mathematical logicians to Kossak's claim!)

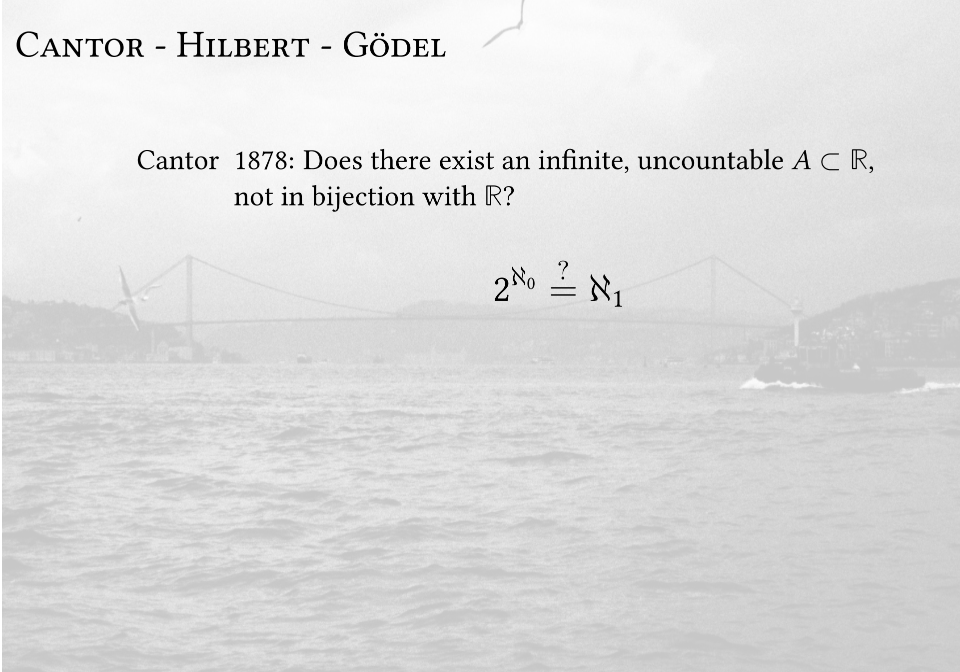
Set Theory, seen from Within

- By Gödel's Incompleteness Theorem, it is better to concentrate on what can be done in ZFC and study what cannot, through independence proofs (Shelah, The Future of Set Theory)
- Continue the search for "natural" axioms in addition to ZFC, with the hope of proving questions such as CH, etc. (Gödel's program, the California Set Theory school, Woodin, etc.)
- Use axioms such as $V = L$, etc. (too restrictive)
- Forcing Axioms (MA, MM, PFA, etc.) - Magidor, Viale, etc.
- Look for structural features of the continuum through detailed combinatorial analysis, or pcf theory, or careful use of ultraproducts, etc.

How crises unfold, and whence they come

We may (as mathematicians!) look at the kind of questions that gave rise to the crisis of a century ago and what kind of responses appeared (and keep appearing):

- In Set Theory, the role of the Continuum Problem and some of its answers.
- In HoTT+UF, the role of Grothendieck's question on deformations, deformations of deformations, etc.



Cantor - Hilbert - Gödel

Cantor 1878: Does there exist an infinite, uncountable $A \subset \mathbb{R}$, not in bijection with $\mathbb{R}$?

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Hilbert 1900: First problem: prove (or refute) the Continuum Hypothesis.

Gödel 1940: The Continuum Hypothesis cannot be refuted: in L, a "model of ZFC", the Continuum Hypothesis holds. Key point: the <u>rigidity</u> of L (Condensation).

Really capturing the reals

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Theorem

(Cantor, 189x) *If $(X, <) \models$ (1) and there exists a countable dense $D \subset X$ such that $(D, <) \approx (\mathbb{Q}, <)$ then $(X, <) \approx (\mathbb{R}, <)$.*

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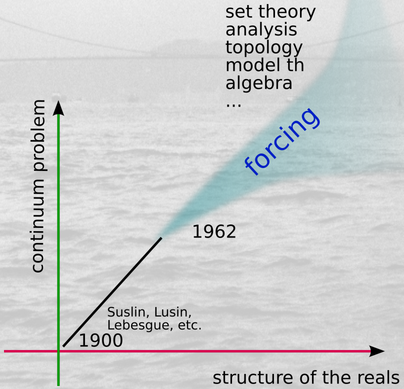
Suslin's Hypothesis (SH): yes. But...SH is independent!



Set Theory Zeitgeist c. 1960 / the wild sixties

- Dana Scott: No measurable cardinals in L.
- Azriel Lévy: $L(\mathbb{R})$ - relative constructibility.
- Alfred Tarski: model theory with a very strong structural emphasis in Berkeley. The "West Coast" style.
- Michael Morley: categoricity transfer - combinatorial set theory used in model theory (thesis supervisor: S. Mac Lane).
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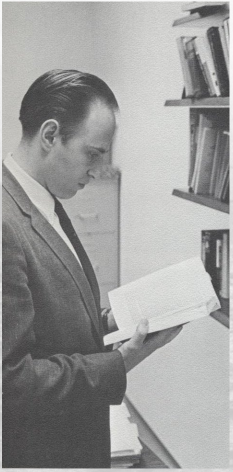
There were no techniques for constructing models of ZFC beyond L and levels of the von Neumann hierarchy!

Paul Cohen (1934-2007)



THE INDEPENDENCE OF THE CONTINUUM HYPOTHESIS

By Paul J. Cohen*

DEPARTMENT OF MATHEMATICS, STANFORD UNIVERSITY

Communicated by Kurt Gödel, September 30, 1963

This is the first of two notes in which we outline a proof of the fact that the Continuum Hypothesis cannot be derived from the other axioms of set theory, including the Axiom of Choice. Since Gödel[3] has shown that the Continuum Hypothesis is consistent with these axioms, the independence of the hypothesis is thus established. We shall work with the usual axioms for Zermelo-Fraenkel set theory,[2] and by Z-F we shall denote these axioms without the Axiom of Choice, (but with the Axiom of Regularity). By a model for Z-F we shall always mean a collection of actual sets with the usual ϵ-relation satisfying Z-F. We use the standard definitions[3] for the set of integers ω, ordinal, and cardinal numbers.

Theorem 1. *There are models for Z-F in which the following occur:*

(1) *There is a set a, $a \subseteq \omega$ such that a is not constructible in the sense of reference 3, yet the Axiom of Choice and the Generalized Continuum Hypothesis both hold.*

(2) *The continuum (i.e., $\mathcal{P}(\omega)$ where $\mathcal{P}$ means power set) has no well-ordering.*

(3) *The Axiom of Choice holds, but $\aleph_1 \neq 2^{\aleph_0}$.*

(4) *The Axiom of Choice for countable pairs of elements in $\mathcal{P}(\mathcal{P}(\omega))$ fails.*

Only part 3 will be discussed in this paper. In parts 1 and 3 the universe is well-ordered by a single definable relation. Note that 4 implies that there is no simple

Angus MacIntyre: “(Cohen) had done work that should long outlast our times. For mathematical logic, and the broader culture that surrounds it, his name belongs with that of Gödel. Nothing more dramatic than their work has happened in the history of the subject.”

Adding reals - adjoining roots

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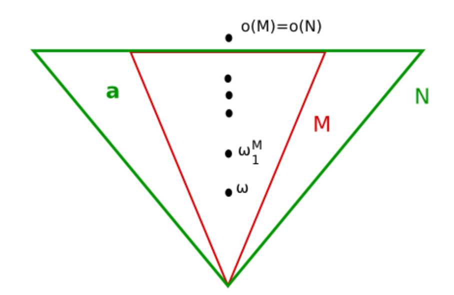
Let M be a countable transitive model of ZFC. In principle, we could have $M \models CH$ (as maybe…so far… $ZFC \vdash CH$).



ω_1^M, ω_2^M, etc. are countable ordinals in V.

Adding reals or roots

In V:

- ♣ Fix $\vec{a} = \langle a_\xi | \xi < \omega_2^M \rangle$ a sequence of ω_2^M different reals (or subsets of ω, or elements of ${}^\omega 2$, which are not in M).
- ♣ Add $\vec{a}$ to M... to get $N = M[\vec{a}]$. Then

$$N \models c \geq \omega_2$$

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Not much! But we should certainly add whatever is constructible from $\vec{a}$. For example, we should have that $\{\xi | (a_\xi)^2 > 8\} \in N$.

Adding reals or roots

Allegory:

Pick a field (e.g. $\mathbb{Q}$ or $\mathbb{F}_n$)	Extend it to $\mathbb{Q}[\pi]$ or $\mathbb{Q}[\sqrt{2}]$	We don't have $\mathbb{Q}[\pi] = \mathbb{Q} \cup \{\pi\}$	Close it under...

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Start with $M \models ZFC + CH$	Add $\aleph_2$ new reals ($\vec{a}$)	We are far from a model of ZFC	What does that mean in our case?

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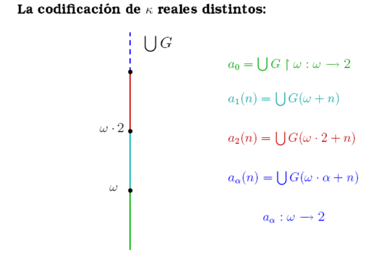
Same ordinals: $M \cap On = N \cap On$.

Same cardinals: if $\alpha \in M \cap On$ and $(\alpha \text{ is a cardinal})^M$ then $(\alpha \text{ is a cardinal})^N$. In N: $\neg \exists \beta < \alpha (f : \beta \xrightarrow{onto} \alpha)$.

ADDING κ-MANY DIFFERENT REALS

To get $N \models \neg CH$, we use $\mathbb{P} = Fn(\aleph_2^M, 2)$. So, $N \models 2^{\aleph_0} \geq \aleph_2$.
As $(\mathbb{P} \text{ is } ccc)^M$, N and M have the same cardinals.

$\bigcup G$ codifies a κ-sequence of different reals $\vec{a}_G$.



Why are all those $\aleph_2$ reals different? By genericity: if $\alpha < \beta < \aleph_2$, the set

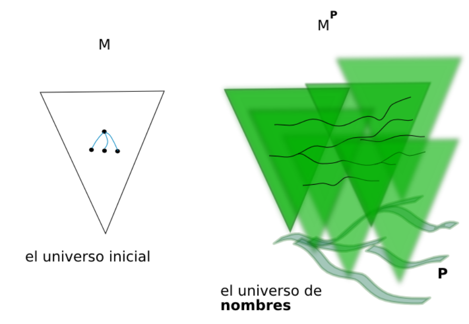
$$E_{\alpha\beta} = \{p \in \mathbb{P} | \exists n[p(\omega \cdot \alpha + n) \neq p(\omega \cdot \beta + n) \wedge \omega \cdot \alpha + n, \omega \cdot \beta + n \in \mathrm{dom}(p)]\}$$

is dense.

The extension must be generic

Cohen, on genericity: *thus* **a** *must have certain special properties…Rather than describe* **a** *directly, it is better to examine the various properties of* **a** *and determine which are desirable and which are not. The chief point is that we do not wish* **a** *to contain "special" information about M, which can only be seen from the outside…The* **a** *which we construct will be referred to as a "generic" set relative to M. The idea is that all the properties of* **a** *must be "forced" to hold merely on the basis that* **a** *behaves like a "generic" set in M. This concept of deciding when a statement about* **a** *is "forced" to hold is the key point of the construction.*

The universe of names $M^{\mathbb{P}}$



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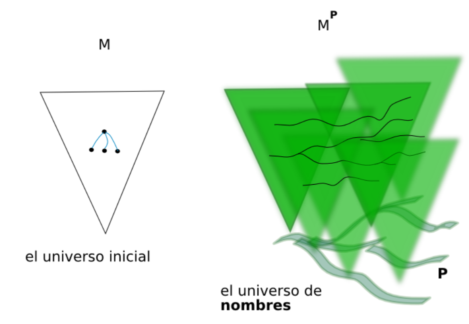


Definition ($\mathbb{P}$-names)

τ is a $\mathbb{P}$-name iff τ is a relation such that if $\langle \sigma, q \rangle \in \tau$, σ is a $\mathbb{P}$-name and $q \in \mathbb{P}$. $V^{\mathbb{P}}$ is the class of all $\mathbb{P}$-names.

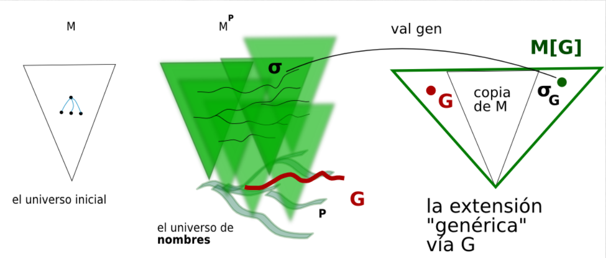
The generic model $M[G]$ - two steps!

Definición (The generic extension)

Given a filter G, a $\mathbb{P}$-name τ,

$\tau_G = val(\tau, G) = \{\sigma_G | \langle \sigma, q \rangle \in \tau \wedge q \in G\}$.

$M[G] = \{\tau_G | \tau \in M, \tau \text{ is a } \mathbb{P}\text{-name}\}$.



Truth and Definibility Lemmas

How does one control what is forced? A larger part of the technical construction goes through the tools to control in M what is forced to hold in a model $M[G]$. This is not a trivial step: in the interesting cases, $G \notin M$.

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The control of "what's forced" is achieved through a "forcing logic" very close to sheaves over partially ordered sets, and through a "Generic Model Theorem": the Definability Lemma roughly says that $\Vdash$ is definable in M.

($p \Vdash \varphi$ as a predicate of two variables can**not** be defined in M, for much the same reason that $M \models \varphi$ is not definable in M [Tarski]).

The Truth Lemma says exactly how truth $\models$ works in the generic extension $M[G]$.

Forcing theorems - iterated forcing and MA

During the first years after Cohen's work, Solovay and Silver at Berkeley refined the theory and turned it into a tool of enormous power to build models of many interesting statements.
Additionally, Martin captured many questions of the early 20th century in terms of a unique "forcing axiom" (called Martin's Axiom), with many interesting consequences under the failure of the Continuum Hypothesis.
The idea to prove the Consistency of Martin's Axiom used Iterated Forcing (just as forcing but controlling the expansions of the universe after (transfinite) iterations of forcing). This is not trivial: at limit stages many objects may appear that could alter seriously the properties of the universes involved.

Some additional directions

- (Kennedy, Magidor, Väänänen - current work): Logics in between first and second order, with generalized quantifiers, and their connection with inner models of set theory.
- (Woodin, 1999) Ω-logic (this would imply that CH is false)
- (Woodin, 2010) Ultimate-L (if it exists, CH is true)
- (Dzamonja, Väänänen - currently) $\beth_\omega$-compactness of the "chain" logic $L^{ch}_{\kappa,\kappa}$ with κ strong limit singular - this would imply the Singular Cardinal Hypothesis (Shelah) for those cardinals.
- (Shelah, 1995) A theory that allows in ZFC to provide robust answer to the Continuum Hypothesis, with a flavor akin to localizations in number theory: **pcf** theory.
- (Väänänen, Villaveces - current work): Logics in between first and second order, associated to singular cardinal and some control of interpolation.

A problem due to Grothendieck

Shulman motivated the subject of Homotopy Type Theory via this Grothendieck problem:

> …*the study of n-truncated homotopy types (of semisimplicial sets, or of topological spaces) [should be] essentially equivalent to the study of so-called n-groupoids. . . . This is expected to be achieved by associating to any space (say) X its "fundamental n-groupoid" $\prod_n(X)$. The obvious idea is that* 0*-objects of* $\prod_n(X)$ *should be the points of* X, 1*-objects should be "homotopies" or paths between points,* 2*-objects should be homotopies between* 1*-objects, etc.*

Grothendieck, 1983

Types vs sets - Extensionality

Recall the Axiom of Extensionality of set theory:

$$\textbf{Ext: } \forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \to x = y)$$

The role of **equality** is in some sense over-defined. Mixed with the Axiom of Choice and the Excluded Middle Principle, working in ZFC, we get the extremely powerful theory we use in a large part of mathematics…

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but we (seem to) lose the possibility of thinking equality as equivalence.
We thus mainly abandon the idea of extensionality.

Dependent Type Theory (Martin-Löf)

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and we replace first order logic by a type theory. The origins date back to Russell, but in the 1960s Per Martin-Löf constructed (for reasons that were back then linked to the study of randomness and probability) his **dependent type theory**. This may also be regarded (de Bruijn) as a computational language:

Dependent Type Theory (Martin-Löf)

We thus mainly abandon the idea of extensionality…

and we replace first order logic by a type theory. The origins date back to Russell, but in the 1960s Per Martin-Löf constructed (for reasons that were back then linked to the study of randomness and probability) his **dependent type theory**. This may also be regarded (de Bruijn) as a computational language:

Basic expressions (contexts) are of the form

$$\textbf{term : Type}$$

for example, $x : \mathbb{N}$ o $y : \mathbb{R}$...

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And from contexts we may pass on to proofs or **judgments** according to certain rules:

$$x : \mathbb{N}, y : \mathbb{R} \vdash (5x^2 + 2, x \times y) : \mathbb{N} \times \mathbb{R}$$

Classifying Category

We may integrate all of these items into the (so called) "Classifying" Category (of contexts) **Ctx**. In this category:

- Objects are contexts, and
- Morphisms are **judgments** $\vdash$.

$$\Gamma \vdash a : A$$

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This looks like…a deduction ???

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$$\Gamma \vdash a : A$$

we think of a as a morphism from Γ into A.
And when writing

$$\Gamma \vdash B \quad \textbf{type}$$

we think of B as a family of contexts. In this sense

- Contexts with variables are **types** and
- those with no variables (after substitution by constants) are **morphisms**.

Dependent Type Theory centers on **how** to treat those substitutions.

Deductions - A Proof System

There are several other rules and constructions:

Type Format	Notation	(special case)
Inhabitant	$a : A$	
Dependent type	$x : A \vdash B(x)$	
Sigma (sum)	$\sum_{(x:A)} B(x)$	$A \times B$
Pi (product)	$\prod_{(x:A)B(x)}$	$A \to B$
Coproduct	$A + B$	
Identity	$Id_A(a, b)$, $a = b$	
Universe	U	
Base	Nat, Bool, 1, 0	

and extensionality axioms on morphismos...

$$\frac{\Gamma \vdash A \quad \textbf{type}; \Gamma; n : A \vdash B \quad \textbf{type}}{\Gamma \vdash \prod_{(n:a)} B \vdash \textbf{type}}$$

Connection with propositional logic and sets

The format numerator (premise) / denominator (conclusion) is very close to formal languages and was implemented for proof assistants. The most famous of them in this context is **Coq** (No excluded middle!)

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Type	V/F	Set	Morph	$\cdots$	n-types	$\cdots$

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In this sense type theory could seem to be a generalization of set theory, but this is NOT the case (set theory does not merely consist of its objects but also of axioms, and first order logic - Džamonja).

Identity - non-identity - ¿two identities?

In Martin-Löf's theory the type $Id_A(x, y)$ captures the idea that **the propositions** x and y are equivalent. Therefore we may prove their equivalence in the system.

Identity - non-identity - ¿two identities?

In Martin-Löf's theory the type $Id_A(x, y)$ captures the idea that **the propositions** x and y are equivalent. Therefore we may prove their equivalence in the system.

All this opens the way to **two different kinds** of identity:

- **Definitional** Identity $A = B$ (types) and $x = y : A$ for objects of a given type, and
- **Propositional** Identity $Id_A(x, y)$.

Clearly, definitional identity implies propositional identity; the converse usually does not hold.

Identity vs deformation

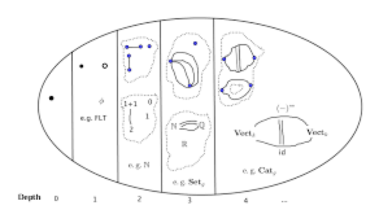
- Martin-Löf: some ways of perceiving the problem of the two identities add further judgments but may have Russell-like paradoxes (Girard)…

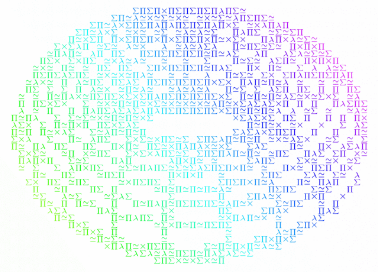
Identity vs deformation

- Martin-Löf: some ways of perceiving the problem of the two identities add further judgments but may have Russell-like paradoxes (Girard)…
- enter Voevodsky with his Univalence Axiom: $Id = Eq$.

Building carefully the Univalence Axiom is quite delicate (a frequent reading of this is **the univalence type is inhabited**).

Standard model





In topology: ∞-groupoids

The so-called ∞-groupoids of homotopy theory end up providing the first "standard model" of Hott+UF. Voevodsky basically

- Interprets the category **Ctx** as the homotopic category of (Kan) simplicial complexes,

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The so-called ∞-groupoids of homotopy theory end up providing the first "standard model" of Hott+UF. Voevodsky basically

- Interprets the category **Ctx** as the homotopic category of (Kan) simplicial complexes,
- Types are then spaces and morphisms,
- But most importantly, propositional equality is now **exactly** homotopic equivalence, and the stratified structure is an ∞-groupoid.

Voevodsky's Theorem

Theorem (Voevodsky - 2012)

If we assume the existence of two inaccessible cardinals, then it is consistent that the category $sSet/W$ forms a model of Martin-Löf's type theory together with the Univalence Axiom.

Notes:

- Two inaccessible cardinals is stronger than ZFC but very weak among axiomatics of set theory.
- In the proof he uses the Excluded Middle at the level of Propositions (1) and the Axiom of Choice at the level of Sets (2).

Some more recent results in synthetic homotopy theory

The following facts of Algebraic Topology have been proved (and verified) in HoTT:

- $\pi_1(S^1) = \mathbb{Z}$ (Shulman, Licata)
- $\pi_k(S^n) = 0$ if $k < n$ (Brunerie, Licata)
- $\pi_n(S^n) = \mathbb{Z}$ (Brunerie, Licata)
- The exact sequence of a fibration (Voevodsky)
- The van Kampen Theorem (Shulman)
- Covering Spaces Theory (Hou)
- ...

An "internal" critical position: Lurie

Two critiques to the project:

- From outside: almost all theorems are algebraic topology - less so of algebraic geometry, and really very few are of other disciplines.
- More internally: Jacob Lurie in various posts shows scepticism –the sort of scepticism of one of the great specialists of Higher Topos Theory– and asks tough questions to HoTT proponents. For instance, computing the fundamental group of $S^1 \times S^1$ took them years of work.

Pluralist perspective...

Džamonja's mathematical pluralist perspective may be summarized as follows:

- Univalent Foundations are really a foundation for the constructive part of mathematics –the key point was to note the connection between homotopy theory and type
- The use of proof assistants (Coq, Agda) may formalize an important part of mathematics, and verify proofs.

Pluralist perspective…

- (Voevodsky) HoTT+UF is consistent modulo the consistency of ZFC.
- Set Theory is still the important standard of consistency (Voevodsky, Logic Colloquium 2013).

Summarizing...

- **Set Theory:** foundations for an important part of mathematics in a format consistent with usual practice.

Mirna Džamonja

SUMMARIZING...

- **Set Theory:** foundations for an important part of mathematics in a format consistent with usual practice.
- **Category Theory:** a way of modelling parts of mathematics that depend on proper classes and where universal properties are essential –such as algebraic geometry.

Mirna Džamonja

Summarizing…

- **Set Theory:** foundations for an important part of mathematics in a format consistent with usual practice.
- **Category Theory:** a way of modelling parts of mathematics that depend on proper classes and where universal properties are essential –such as algebraic geometry.
- **Univalent Foundations:** a novel way of <u>discussing</u> proofs –obviously a central and very important topic.

Mirna Džamonja

Bourbaki - a "historical" vision of the rôle of ZFC

Bourbaki in the first volume of Théorie des ensembles says:

.5cm

> *We know that, logically speaking, it is possible to derive all current mathematics from a unique source, set theory. When doing this, we do not pretend to write in stone a law; some day may come when mathematicians will reason in a completely different way that is not formalizable in the language we adopt here, and according to some, recent progress in homology suggests that day is not far along. In that case we will have to enlarge the syntax, even if it is not necessary to completely change the language. The future of mathematics will decide.*

Axiom Systems: Constitutions?

- Rami Grossberg (in private communications) has described ZFC as “a good constitution”: it may perhaps last long but it does not have to be eternal.
- In light of the recent proofs and announcements (Mochizuki on the **abc** Conjecture in 2014, controversy with Scholze and Stix), and the lack of informed consensus on the matter among the mathematical community at large, Voevodsky’s worry is particularly relevant.
- A the time of writing (2021) the mathematical community has no clarity of **where** the formalization of IUTT (Interuniversal Teichmüller Theory) takes place. Scholze and Stix’s isolating a key lemma and reducing it to more classical algebraic geometry is a possibility, but the stakes seem open.

Grazie!

Fondamente Nuove - Apollonio Domenichini - 1770