Axiomatizations of abstract elementary classes and natural logics for model theory:

The role of partition relations.

Andrés Villaveces - *Universidad Nacional de Colombia - Bogotá*

Toronto Set Theory Seminar - February 2021

Contents

A Combinatorial Meeting Point

Capturing an Abstract Elementary Class

Comparing Two Infinitary Logics

A Combinatorial Meeting Point

In November 2018, there was an interesting **combinatorial** coincidence:

Café Léa, Rue Pascal / Rue Claude-Bernard

A Combinatorial Meeting Point

In November 2018, there was an interesting **combinatorial** coincidence:

- I was beginning to use a partition theorem on well-ordered trees (due to Komjáth and Shelah) in joint work with **Shelah** to axiomatize abstract elementary classes, and

Café Léa, Rue Pascal / Rue Claude-Bernard

A Combinatorial Meeting Point

Café Léa, Rue Pascal / Rue Claude-Bernard

In November 2018, there was an interesting **combinatorial** coincidence:

- I was beginning to use a partition theorem on well-ordered trees (due to Komjáth and Shelah) in joint work with **Shelah** to axiomatize abstract elementary classes, and
- Jouko **Väänänen**, who was working with Boban **Veličković** in a variant of Shelah's logic L^1_κ and simultaneously with me on a weakening of the same logic L^1_κ, realized during a last day meeting in the café that it was exactly that same partition theorem that was the "missing piece" for an argument they were building with Boban…

Ordinals and order types form Ramsey classes…

Before stating Komjáth-Shelah, let us just remember that **cardinals** and **order types** form Ramsey classes (using here an informal notion of “Ramsey Class”):

Ordinals and order types form Ramsey classes…

Before stating Komjáth-Shelah, let us just remember that **cardinals** and **order types** form Ramsey classes (using here an informal notion of “Ramsey Class”):

- $\mu^+ \rightarrow (\mu^+)^1_\mu$

Ordinals and order types form Ramsey classes...

Before stating Komjáth-Shelah, let us just remember that **cardinals** and **order types** form Ramsey classes (using here an informal notion of "Ramsey Class"):

- $\mu^+ \to (\mu^+)^1_\mu$
- Given an order type φ and a cardinal μ, there is some order type ψ such that

$$\psi \to (\varphi)^1_\mu.$$

…However, scattered order types do not!

Of course, one might ask whether many other important classes (of orders, e.g.) are “Ramsey”.

…However, scattered order types do not!

Of course, one might ask whether many other important classes (of orders, e.g.) are “Ramsey”.
For instance, scattered order types **do not** form a Ramsey class!

…However, scattered order types do not!

Of course, one might ask whether many other important classes (of orders, e.g.) are “Ramsey”.
For instance, scattered order types **do not** form a Ramsey class!
[An order type φ is scattered iff $\eta \not\leq \varphi$, where $\eta = \mathrm{o.t.}[(\mathbb{Q}, <)]$; this means there is no order-preserving embedding from the rationals into a partially ordered set of order type φ.]

. . . However, scattered order types do not!

Of course, one might ask whether many other important classes (of orders, e.g.) are “Ramsey”.

For instance, scattered order types **do not** form a Ramsey class!

[An order type φ is scattered iff $\eta \not\leq \varphi$, where $\eta = \mathsf{o.t.}[(\mathbb{Q}, <)]$; this means there is no order-preserving embedding from the rationals into a partially ordered set of order type φ.]

There exists some scattered order type (s.o.t.) ϕ such that for every s.o.t. ψ, we have

$$\psi \not\to (\phi)^1_\omega.$$

A positive result: Komjáth-Shelah

Although s.o.t.'s do not outright form a Ramsey class, Komjáth and Shelah proved in 2003 a beautiful theorem giving a weaker form[1]:

Theorem

For every s.o.t. ϕ and every cardinal μ there exists a s.o.t. ψ such that

$$\psi \to [\phi]^1_{\mu,\omega}$$

Here, $\psi \to [\phi]^1_{\mu,\omega}$ means that, given an ordered set of (scattered) order type ψ, given a coloring $F : S \to \mu$, there exists a countable subset $X \subseteq \mu$ such that $f^{-1}(X)$ contains a subset of o.t. ϕ. (Homogeneity of the coloring is spread on ω-many colors forming a subset of the wanted order type.)

[1] P. Komjáth, S. Shelah: A Partition Theorem for Scattered Order Types, Combinatorics, Probability and Computing, 12(2003), 621–626.

Scattered orders - Hausdorff Characterization

Hausdorff characterized scattered order types as the smallest class containing $0, 1$ and closed under well-ordered sums and reverse well-ordered sums.

Scattered orders - Hausdorff Characterization

Hausdorff characterized scattered order types as the smallest class containing $0, 1$ and closed under well-ordered sums and reverse well-ordered sums.

This is very useful. As an example, it allows us to check that for **every scattered** $(S, <)$ **with o.t.** ϕ **there is** $f : S \to \omega$ **such that** $f^{-1}(n)$ **has no subset of o.t.** $(\omega^* + \omega)^n$. So,

$$\phi \not\to (1 + (\omega^* + \omega) + (\omega^* + \omega)^2 + \cdots)^1_\omega.$$

(Illustrate proof on “blackboard”.)

The crucial (and most useful) lemma: partitioning well-founded trees

On the way to their proof, Komjáth and Shelah prove an even more interesting (!) lemma, a partition relation on well-founded trees: For any α let $\mathsf{FS}(\alpha)$ be the tree of all descending sequences of elements of α. We use $\mathsf{len}(\mathsf{s})$ to denote the length of $\mathsf{s} \in \mathsf{FS}(\alpha)$.

Lemma (Komjáth-Shelah 2003)

Assume that α *is an ordinal and* μ *a cardinal. Set* $\lambda = (|\alpha|^{\mu^{\aleph_0}})^+$. *Suppose* $\mathsf{T} = \mathsf{FS}(\lambda^+)$ *and* $\mathsf{F} : \mathsf{T} \to \mu$. *Then there is a subtree* $\mathsf{T}^* = \{(\delta^{\mathsf{s}}_0, \ldots, \delta^{\mathsf{s}}_{\mathsf{n}}) : \mathsf{s} = (\mathsf{s}_0, \ldots, \mathsf{s}_{\mathsf{n}}) \in \mathsf{FS}(\alpha)\}$ *of* T *and a function* $\mathsf{c} : \omega \to \mu$ *such that for all* $\mathsf{s} \in \mathsf{T}^*$ *we have* $\mathsf{F}(\mathsf{s}) = \mathsf{c}(\mathsf{len}(\mathsf{n}))$.

Crucial point: given α an ordinal, μ a cardinal, if we color a **large enough** well founded tree (of descending sequences of ordinals) into μ many colors, we may extract a subtree "of size $|\alpha|$" **where colors only depend** on the length of the sequence.

Representing scattered order-types

Let α be an ordinal, let
$\mathsf{H}(\alpha)$ denote the set of functions $\mathsf{f} : \alpha \to \{-1, 0, 1\}$ such that

$$|\mathsf{D}(\mathsf{f})| < \aleph_0,$$

where $\mathsf{D}(\mathsf{f}) = \{\beta < \alpha \mid \mathsf{f}(\beta) \neq 0\}$.
Let $\mathsf{f} \prec \mathsf{g}$ iff $\mathsf{f}(\beta) < \mathsf{g}(\beta)$ where β is the maximum ordinal where f and g differ.

Lemma

Use Hausdorff: enough to show that if ϕ_1, ϕ_2 can be embedded into some $\mathsf{H}(\alpha)$, then ANY well-ordered sum or reverse well-ordered sum of ϕ_1, ϕ_2 can be. Enough to show that $\mathsf{H}(\alpha) \times \beta \to \mathsf{H}(\alpha + \beta)$ and $\mathsf{H}(\alpha) \times \beta^* \to \mathsf{H}(\alpha + \beta)$.

Representing scattered order-types

Let α be an ordinal, let
$\mathsf{H}(\alpha)$ denote the set of functions $\mathsf{f} : \alpha \to \{-1, 0, 1\}$ such that

$$|\mathsf{D}(\mathsf{f})| < \aleph_0,$$

where $\mathsf{D}(\mathsf{f}) = \{\beta < \alpha \mid \mathsf{f}(\beta) \neq 0\}$.
Let $\mathsf{f} \prec \mathsf{g}$ iff $\mathsf{f}(\beta) < \mathsf{g}(\beta)$ where β is the maximum ordinal where f and g differ.

Lemma

- $\mathsf{H}(\alpha)$ *is scattered, for every* α.

Use Hausdorff: enough to show that if ϕ_1, ϕ_2 can be embedded into some $\mathsf{H}(\alpha)$, then ANY well-ordered sum or reverse well-ordered sum of ϕ_1, ϕ_2 can be. Enough to show that $\mathsf{H}(\alpha) \times \beta \to \mathsf{H}(\alpha + \beta)$ and $\mathsf{H}(\alpha) \times \beta^* \to \mathsf{H}(\alpha + \beta)$.

Representing scattered order-types

Let α be an ordinal, let
$\mathsf{H}(\alpha)$ denote the set of functions $\mathsf{f} : \alpha \to \{-1, 0, 1\}$ such that

$$|\mathsf{D}(\mathsf{f})| < \aleph_0,$$

where $\mathsf{D}(\mathsf{f}) = \{\beta < \alpha \mid \mathsf{f}(\beta) \neq 0\}$.
Let $\mathsf{f} \prec \mathsf{g}$ iff $\mathsf{f}(\beta) < \mathsf{g}(\beta)$ where β is the maximum ordinal where f and g differ.

Lemma

- $\mathsf{H}(\alpha)$ *is scattered, for every* α.
- *If* ϕ *is a s.o.t., then* ϕ *can be embedded into some* $(\mathsf{H}(\alpha), \prec)$.

Use Hausdorff: enough to show that if ϕ_1, ϕ_2 can be embedded into some $\mathsf{H}(\alpha)$, then ANY well-ordered sum or reverse well-ordered sum of ϕ_1, ϕ_2 can be. Enough to show that $\mathsf{H}(\alpha) \times \beta \to \mathsf{H}(\alpha + \beta)$ and $\mathsf{H}(\alpha) \times \beta^* \to \mathsf{H}(\alpha + \beta)$.

From well-founded trees to scattered order types

To get that for every s.o.t. ϕ, for every cardinal μ there is a s.o.t. ψ such that $\psi \to [\phi]^1_{\mu,\omega}\ldots$
First, now enough to prove that given α, μ there is some λ such that

$$\mathsf{H}(\lambda^+) \to [\mathsf{H}(\alpha)]^1_{\mu,\omega}.$$

Pick λ as in the lemma: $\lambda = \left(|\alpha|^{\mu^{\aleph_0}}\right)^+$ and let $\mathsf{G} : \mathsf{H}(\lambda^+) \to \mu$ be a coloring. From this, build a coloring F of $\mathsf{FS}(\lambda^+)$ …and use the lemma to get an α-subtree $\mathsf{x}(\mathbf{s} \mid \mathbf{s} \in \mathsf{FS}(\alpha))$ such that

$$\mathsf{F}\,(\mathsf{x}(\mathsf{s}(0)), \mathsf{x}(\mathsf{s}(0), \mathsf{s}(1)), \ldots, \mathsf{x}(\mathsf{s}(0), \ldots, \mathsf{s}(n))) = \mathsf{c}(n).$$

Conclude by building from this an embedding from $\mathsf{H}(\alpha) \to \mathsf{H}(\lambda^+)$

…

Plan

- A Combinatorial Meeting Point
- Capturing an Abstract Elementary Class
- Comparing Two Infinitary Logics
  - Shelah's logic L^1_κ
  - Approximations from above: chain logic, …

AEC - THE AXIOMS, BRIEFLY

Fix $\mathcal{K}$ be a class of τ-structures, $\prec_{\mathcal{K}}$ a binary relation on $\mathcal{K}$.

Definition

$(\mathcal{K}, \prec_{\mathcal{K}})$ is an abstract elementary class iff

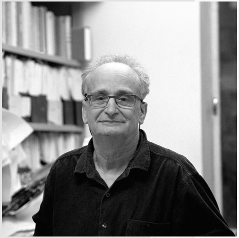
- $\mathcal{K}$, $\prec_{\mathcal{K}}$ are **closed under isomorphism,**
- $\mathsf{M}, \mathsf{N} \in \mathcal{K}$, $\mathsf{M} \prec_{\mathcal{K}} \mathsf{N} \Rightarrow \mathsf{M} \subset \mathsf{N}$,
- $\prec_{\mathcal{K}}$ is a partial order,
- (**TV**) $\mathsf{M} \subset \mathsf{N} \prec_{\mathcal{K}} \bar{\mathsf{N}}, \mathsf{M} \prec_{\mathcal{K}} \bar{\mathsf{N}} \Rightarrow \mathsf{M} \prec_{\mathcal{K}} \mathsf{N}$,
- ($\searrow$**LS**) There is some $\kappa = \mathsf{LS}(\mathcal{K}) \geq \aleph_0$ such that for every $\mathsf{M} \in \mathcal{K}$, for every $\mathsf{A} \subset |\mathsf{M}|$, there is $\mathsf{N} \prec_{\mathcal{K}} \mathsf{M}$ with $\mathsf{A} \subset |\mathsf{N}|$ and $\|\mathsf{N}\| \leq |\mathsf{A}| + \mathsf{LS}(\mathcal{K})$,
- (**Unions of $\prec_{\mathcal{K}}$-chains**) A union of an arbitrary $\prec_{\mathcal{K}}$-chain in $\mathcal{K}$ belongs to $\mathcal{K}$, is a $\prec_{\mathcal{K}}$-extension of all models in the chain and is the $\sup$ of the chain.

Examples

Natural constructions in Mathematics are examples of AEC (or metric AEC)

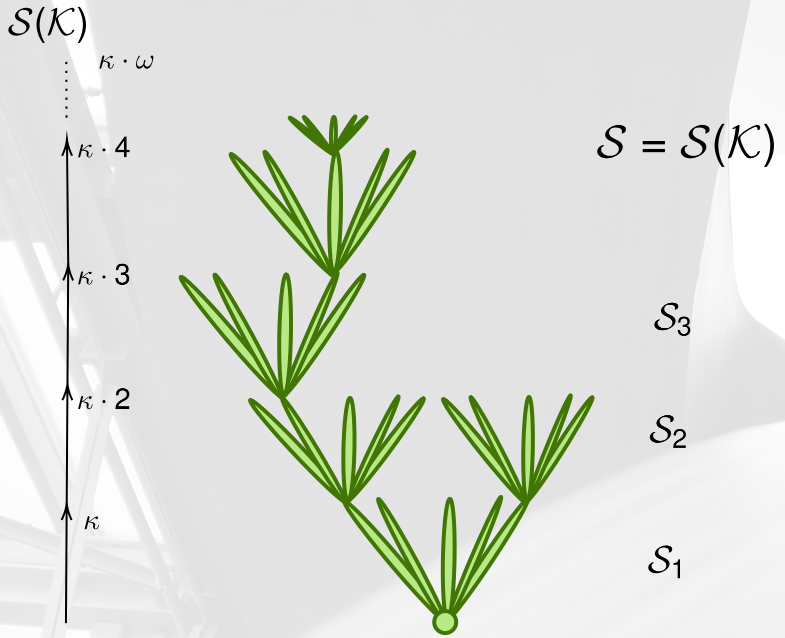
1. Complete first order theories
2. Various classes axiomatizable in $\mathsf{L}_{\omega_1,\omega}$ or $\mathsf{L}_{\kappa\omega}$.
3. Covers of Abelian algebraic groups, classes of modules (Mazari-Armida).
4. Metric (continuous) AECs - stability theory started by Hirvonen and Hyttinen, Usvyatsov, and continued by Zambrano and V.; Eagle, Tall, Iovino, Caicedo, Hamel have recent work related to these.
5. Gelfand triples (Zambrano, V.)
6. AECs of C^*-algebras (Argoty, Berenstein, V.)
7. Zilber analytic classes (pseudoexponentiation)
8. "Hart-Shelah"-like examples (Baldwin, Kolesnikov, Shelah, V. 2021)
9. New: dependent (NIP) AECs (with Shelah)

The canonical tree of an a.e.c.



This is joint work with Saharon Shelah.
Fix an a.e.c. $\mathcal{K}$ with vocabulary τ and $\mathsf{LS}(\mathcal{K}) = \kappa$.
Let $\lambda = \beth_2(\kappa + |\tau|)^+$.
The **canonical tree** of $\mathcal{K}$:

- $\mathcal{S}_\mathsf{n} := \{\mathsf{M} \in \mathcal{K} \mid$ for some $\bar{\alpha} = \bar{\alpha}_\mathsf{M}$ of length n, M has universe $\{\mathsf{a}^*_\alpha \mid \alpha \in \mathsf{S}_{\bar{\alpha}[\mathsf{M}]}\}$ and $\mathsf{m} < \mathsf{n} \Rightarrow \mathsf{M} \restriction \mathsf{S}_{\bar{\alpha}\restriction \mathsf{m}[\mathsf{M}]} \prec_\mathcal{K} \mathsf{M}\}$ (and $\mathcal{S}_0 = \{\mathsf{M}_{\mathsf{empt}}\}$),
- $\mathcal{S} = \mathcal{S}_\mathcal{K} := \bigcup_\mathsf{n} \mathcal{S}_\mathsf{n}$; this is a tree with ω levels under $\prec_\mathcal{K}$ (equivalenty under $\subseteq$).


$\mathcal{S}(\mathcal{K})$
$\kappa \cdot \omega$
$\kappa \cdot 4$
$\kappa \cdot 3$
$\kappa \cdot 2$
κ
$\mathcal{S} = \mathcal{S}(\mathcal{K})$
$\mathcal{S}_3$
$\mathcal{S}_2$
$\mathcal{S}_1$

Formulas $\varphi_{M,\gamma,n}(\bar{x}_n)$

For M in the canonical tree $\mathcal{S}$ at level n, a formula with $\kappa \cdot n$ free variables, defined by induction on γ.

- $\gamma = 0$: $\varphi_{0,0} = \top$ ("truth"). If $n > 0$,

$$\varphi_{M,0,n} := \bigwedge \mathrm{Diag}^n_\kappa(M),$$

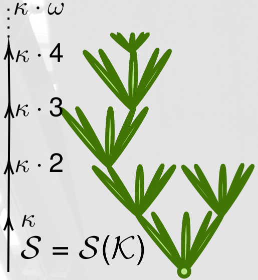
  the atomic diagram of M in $\kappa \cdot n$ variables.
- γ limit: Then

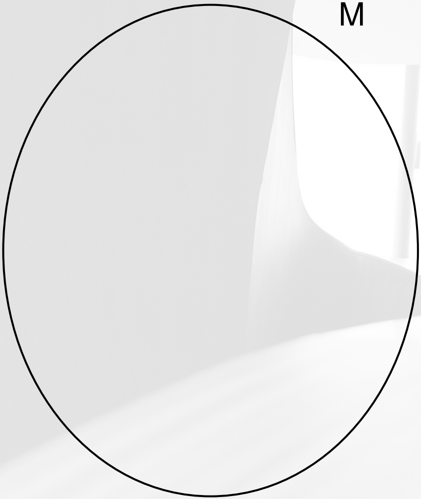
$$\varphi_{M,\gamma,n}(\bar{x}_n) := \bigwedge_{\beta<\gamma} \varphi_{M,\beta,n}(\bar{x}_n).$$

- $\gamma = \beta + 1$: Then $\varphi_{M,\gamma,n}(\bar{x}_n)$ is the $L_{\lambda^+,\kappa^+}(\tau)$ formula

$$\forall \bar{z}_{[\kappa]} \bigvee_{\substack{N \succ_{\mathcal{K}} M \\ N \in \mathcal{S}_{n+1}}} \exists \bar{x}_{=n} \left[\varphi_{N,\beta,n+1}(\bar{x}_{n+1}) \wedge \bigwedge_{\alpha<\alpha_n[N]} \bigvee_{\delta \in S[N]} z_\alpha = x_\delta \right]$$

Testing the class against the tree - Does $M \in \mathcal{K}$?





So we have sentences $\varphi_{\gamma,0}$, for $\gamma < \lambda^+$, such that $i < j < \lambda^+$ implies $\varphi_j \to \varphi_i$. These sentences are better and better approximations of the aec $\mathcal{K}$; they describe how small models of the class embed into arbitrary ones.
Let us take a closer look at low levels:

The catch (beginnings)

When does $\mathsf{M} \models \varphi_{1,0}$?

The catch (beginnings)

When does $\mathsf{M} \models \varphi_{1,0}$?
When in M,
$$\forall \bar{z}_{[\kappa]} \bigvee_{N \in \mathcal{M}_1} \exists \bar{x}_{=0} \left[\varphi_{N,0,1}(\bar{x}_1) \wedge \bigwedge_{\alpha < \alpha_0[N]} \bigvee_{\delta \in S[N]} z_\alpha = x_\delta \right]$$

The catch (beginnings)

When does $M \models \varphi_{1,0}$?
When in M,

$$\forall \bar{z}_{[\kappa]} \bigvee_{N \in \mathcal{M}_1} \exists \bar{x}_{=0} \left[\varphi_{N,0,1}(\bar{x}_1) \wedge \bigwedge_{\alpha < \alpha_0[N]} \bigvee_{\delta \in S[N]} z_\alpha = x_\delta \right]$$

That is, for every subset Z of M of size $\leq \kappa$ some model N in the tree (level 1, of size κ) embeds into M, covering Z.

The catch (beginnings)

When does $M \models \varphi_{1,0}$?
When in M,
$\forall \bar{z}_{[\kappa]} \bigvee_{N \in \mathcal{M}_1} \exists \bar{x}_{=0} \left[\varphi_{N,0,1}(\bar{x}_1) \wedge \bigwedge_{\alpha<\alpha_0[N]} \bigvee_{\delta \in S[N]} z_\alpha = x_\delta\right]$

That is, for every subset Z of M of size $\leq \kappa$ some model N in the tree (level 1, of size κ) embeds into M, covering Z.

When does $M \models \varphi_{2,0}$?

THE CATCH (BEGINNINGS)

When does $M \models \varphi_{1,0}$?
When in M,

$$\forall \bar{z}_{[\kappa]} \bigvee_{N \in \mathcal{M}_1} \exists \bar{x}_{=0} \left[\varphi_{N,0,1}(\bar{x}_1) \wedge \bigwedge_{\alpha < \alpha_0[N]} \bigvee_{\delta \in S[N]} z_\alpha = x_\delta \right]$$

That is, for every subset Z of M of size $\leq \kappa$ some model N in the tree (level 1, of size κ) embeds into M, covering Z.

When does $M \models \varphi_{2,0}$?
When in M,

$$\forall \bar{z}_{[\kappa]} \bigvee_{N \in \mathcal{M}_1} \exists \bar{x}_{=0} \left[\varphi_{N,1,1}(\bar{x}_1) \wedge \bigwedge_{\alpha < \alpha_0[N]} \bigvee_{\delta \in S[N]} z_\alpha = x_\delta \right]$$

This is slightly more complicated to unravel:

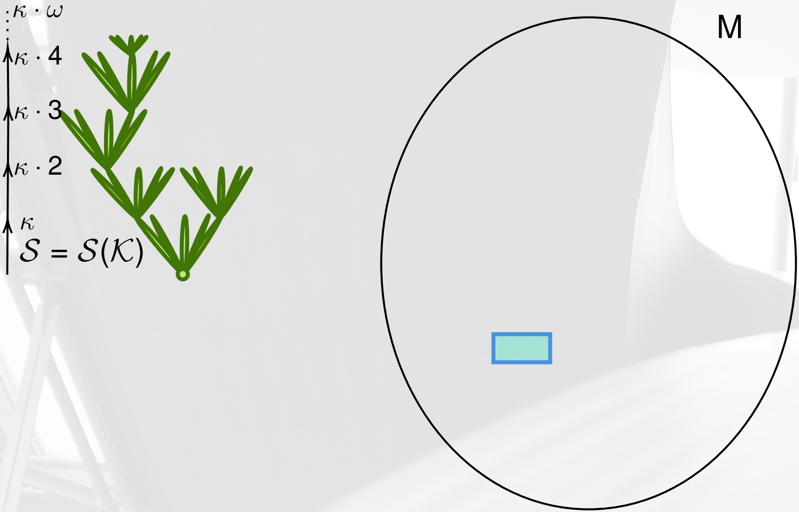
$\forall \bar{z}_{[\kappa]} \bigvee_{N \in \mathcal{M}_1} \exists \bar{x}_{=1} \left[\varphi_{N,1,1}(\bar{x}_1) \wedge \bigwedge_{\alpha < \alpha_0[N]} \bigvee_{\delta \in S[N]} z_\alpha = x_\delta \right]$

For every subset Z of M of size $\leq \kappa$ some model N in the tree (at level 1) M is such that $M \models \varphi_{N,1,1}$, through some "image of N" covering Z…

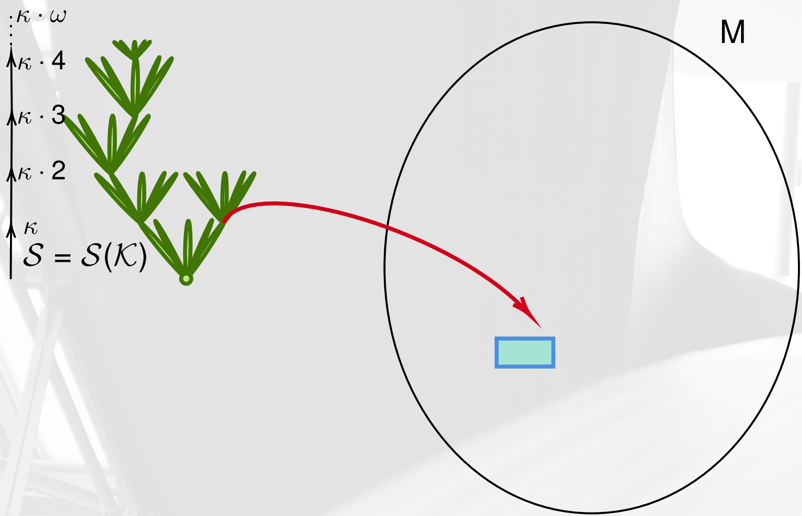
for all $Z' \subset M$ of size κ there is some $N' \succ_{\mathcal{K}} N$ in the canonical tree, at level 2, extending N, such that some tuple $\bar{x}_{=2}$ from M covers Z' and is the "image" of N' by an embedding

The mezcal test - Does $\mathsf{M} \in \mathcal{K}$?

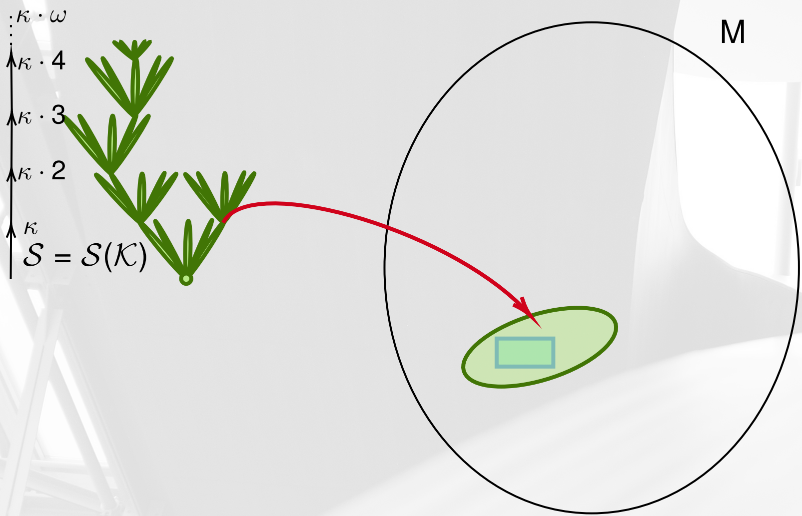
The mezcal test - Does M $\in \mathcal{K}$?



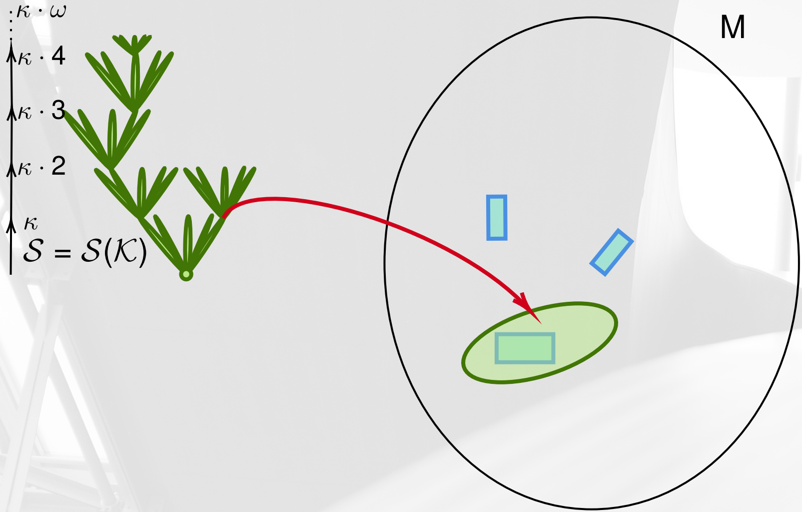
The mezcal test - Does M $\in \mathcal{K}$?



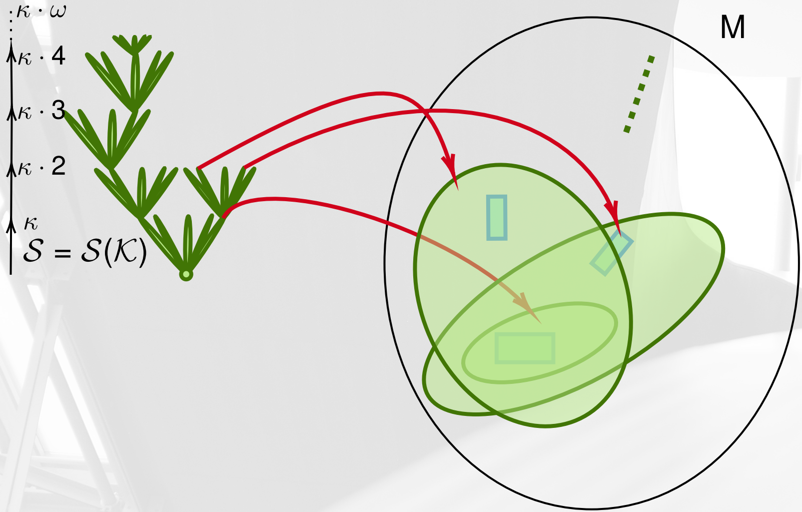
The mezcal test - Does $\mathsf{M} \in \mathcal{K}$?



THE MEZCAL TEST - DOES $\mathsf{M} \in \mathcal{K}$?



The mezcal test - Does $\mathsf{M} \in \mathcal{K}$?



Theorem

$\mathsf{M} \in \mathcal{K}$ *implies* $\mathsf{M} \models \varphi_{\gamma,0}$ *for each* $\gamma < \lambda^+$

Theorem

$\mathsf{M} \models \varphi_{\beth_2(\kappa)^+ + 2, 0}$ *implies* $\mathsf{M} \in \mathcal{K}$

This much harder implication requires understanding the tree of possible embeddings of small models; the partition property due to Komjáth and Shelah is the key…

The combinatorics behind: our by now old friend...

Theorem (Komjáth-Shelah (2003))

Let α *be an ordinal and* μ *a cardinal. Set* $\lambda = \left(|\alpha|^{\mu^{\aleph_0}}\right)^+$ *and let* $F(ds(\lambda^+)) \to \mu$ *be a colouring of the tree of finite descending sequences of ordinals* $< \lambda$*. Then there are an embedding* $\varphi : ds(\alpha) \to ds(\lambda)$ *and a function* $c : \omega \to \mu$ *such that for every* $\eta \in ds(\alpha)$ *of length* $n+1$

$$F(\varphi(\eta)) = c(n).$$

We apply it with number of colours μ equal to $\kappa^{|\tau|+\kappa} = 2^\kappa$; therefore $(2^\kappa)^{\aleph_0} = 2^\kappa$. We thus obtain a sequence $(\eta_n)_{n<\omega}$, $\eta_n \in ds(\lambda)$ such that:

$$k \leq m \leq n, \ell \in \{1,2\} \Rightarrow N^\ell_{\eta_m \restriction k} = N^\ell_{\eta_n \restriction k}.$$

The tree property enables us to "reconstruct" M (satisfying $\varphi_{\lambda+2,0}$ as a limit of models of size κ, in the class $\mathcal{K}$).
With this we can

- define "quantificational depth" of an aec (variants of Baldwin-Shelah (building on Mekler and Eklöf) give examples of high quantificational depth)…
- get definability of the "strong submodel relation" $\prec_{\mathcal{K}}$ …and genuine variants of a Tarski-Vaught test
- a grip on biinterpretability of AECs…

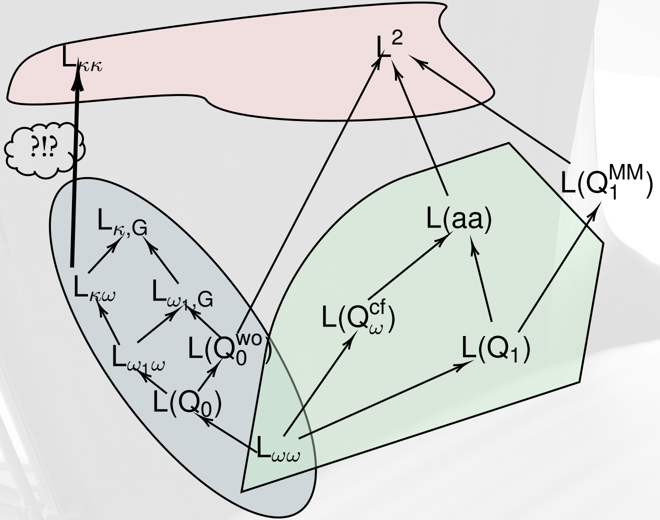
Plan

A Combinatorial Meeting Point

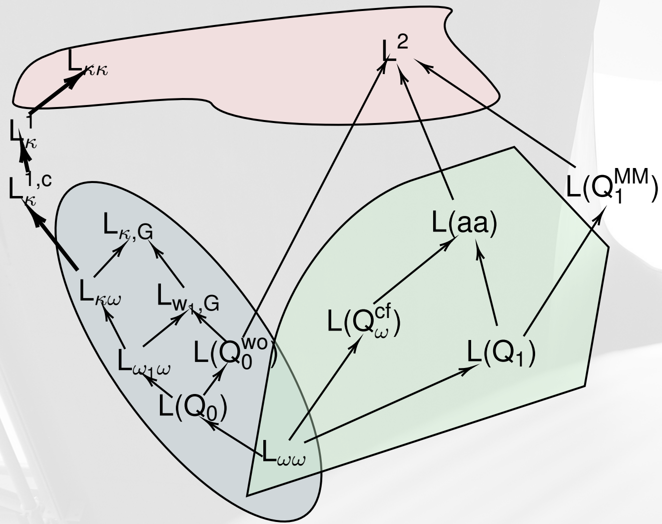
Capturing an Abstract Elementary Class

Comparing Two Infinitary Logics
- Shelah's logic L^1_κ
- Approximations from above: chain logic, …

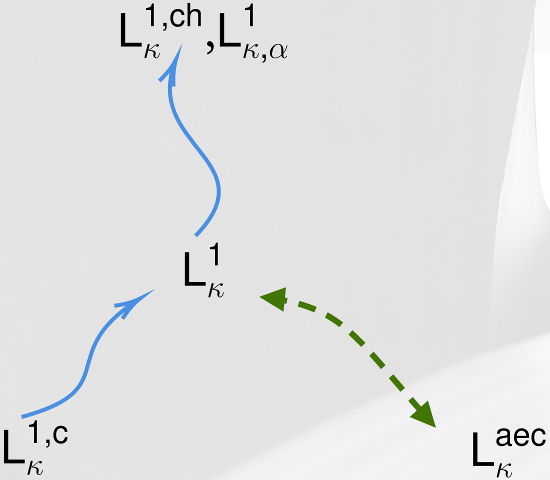
A (Väänänen) map of various infinitary logics



New Logics


L^2
$L_{\kappa\kappa}$
L^1_κ
$L^{1,c}_\kappa$
$L_{\kappa,G}$
$L_{\kappa\omega}$
$L_{w_1,G}$
$L(Q^{cf}_\omega)$
$L(aa)$
$L(Q_1^{MM})$
$L_{\omega_1\omega}$
$L(Q_0^{wo})$
$L(Q_1)$
$L(Q_0)$
$L_{\omega\omega}$

Close Up...



Close Up…


$L^{1,ch}_{\kappa}, L^{1}_{\kappa,\alpha}$
r
L^{1}_{κ}
Δ
?
$L^{1,c}_{\kappa}$
L^{aec}_{κ}

Interpolation

- Craig($L_{\kappa^+\omega}, L_{(2^\kappa)^+\kappa^+}$) (Malitz 1971).

Interpolation

- Craig($L_{\kappa^+\omega}, L_{(2^\kappa)^+\kappa^+}$) (Malitz 1971).
  If $\varphi \vdash \psi$, where φ is a τ_1-sentence and ψ is a τ_2-sentence and both are in $L_{\kappa^+\omega}$ then
  there exists $\chi \in L_{(2^\kappa)^+\kappa^+}(\tau_1 \cap \tau_2)$ such that
  $$\varphi \vdash \chi \vdash \psi.$$
- The original argument used "consistency properties". Other proofs have stressed the "Topological Separation" aspect of Interpolation.

So what about "balancing" Interpolation?

- Problem: Find L^* such that

$$\mathsf{L}_{\kappa^+\omega} \leq \mathsf{L}^* \leq \mathsf{L}_{(2^\kappa)^+\kappa^+}$$

and $\mathsf{Craig}(\mathsf{L}^*)$.

So what about "balancing" Interpolation?

- Problem: Find L^* such that

$$\mathsf{L}_{\kappa^+\omega} \leq \mathsf{L}^* \leq \mathsf{L}_{(2^\kappa)^+\kappa^+}$$

  and $\mathsf{Craig}(\mathsf{L}^*)$.

- Shelah, 2012: For singular strong limit κ of cofinality ω there is a logic L^1_κ such that

$$\bigcup_{\lambda<\kappa} \mathsf{L}_{\lambda^+\omega} \leq \mathsf{L}^1_\kappa \leq \bigcup_{\lambda<\kappa} \mathsf{L}_{\lambda^+\lambda^+}$$

  and $\mathsf{Craig}(\mathsf{L}^1_\kappa)$.

So what about "balancing" Interpolation?

- Problem: Find L^* such that

$$\mathsf{L}_{\kappa^+\omega} \leq \mathsf{L}^* \leq \mathsf{L}_{(2^\kappa)^+\kappa^+}$$

and $\mathsf{Craig}(\mathsf{L}^*)$.

- Shelah, 2012: For singular strong limit κ of cofinality ω there is a logic L^1_κ such that

$$\bigcup_{\lambda<\kappa} \mathsf{L}_{\lambda^+\omega} \leq \mathsf{L}^1_\kappa \leq \bigcup_{\lambda<\kappa} \mathsf{L}_{\lambda^+\lambda^+}$$

and $\mathsf{Craig}(\mathsf{L}^1_\kappa)$.

- Moreover, in the case $\kappa = \beth_\kappa$, the logic L^1_κ also has a Lindström-type characterization as the maximal logic with a peculiar strong form of undefinability of well-order.

A description of Shelah's logic L^1_κ

- Shelah's L^1_κ is not really defined as usual; rather, it is defined by declaring what its elementary equivalence relation is.

A description of Shelah's logic L^1_κ

- Shelah's L^1_κ is not really defined as usual; rather, it is defined by declaring what its elementary equivalence relation is.
- This elementary equivalence relation is given by an EF-game type equivalence.

A description of Shelah's logic L^1_κ

- Shelah's L^1_κ is not really defined as usual; rather, it is defined by declaring what its elementary equivalence relation is.
- This elementary equivalence relation is given by an EF-game type equivalence.
- Then…what is the syntax of Shelah's logic?

A description of Shelah's logic L^1_κ

- Shelah's L^1_κ is not really defined as usual; rather, it is defined by declaring what its elementary equivalence relation is.
- This elementary equivalence relation is given by an EF-game type equivalence.
- Then…what is the syntax of Shelah's logic?
- We describe two partial answers, one approaching from below (Väänänen-V.), the other one from above (Džamonja, Väänänen).

Shelah's game $G_\theta^\beta(M, N)$.

ANTI	ISO
$\beta_0 < \beta, \vec{a^0}$	
	$f_0 : \vec{a^0} \to \omega$, $g_0 : M \to N$ a p.i.
$\beta_1 < \beta_0, \vec{b^1}$	$f_1 : \vec{a^1} \to \omega$, $g_1 : M \to N$ a p.i., $g_1 \supseteq g_0$
$\vdots$	$\vdots$

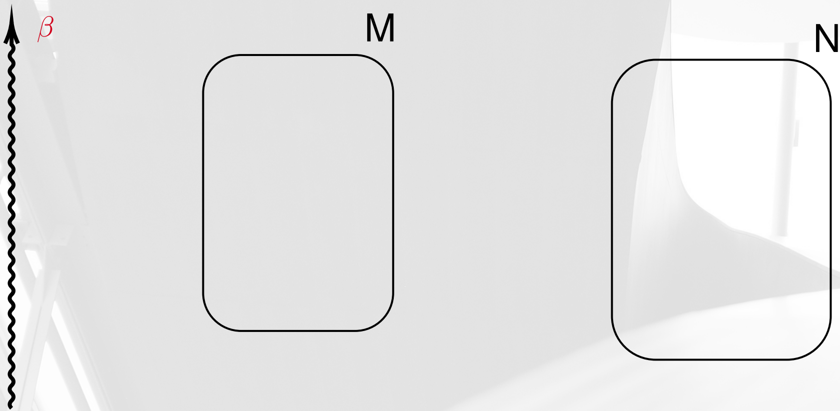
Constraints:

- $\text{len}(\vec{a^n}) \leq \theta$
- $f_{2n}^{-1}(m) \subseteq \text{dom}(g_{2n})$ for $m \leq n$.
- $f_{2n+1}^{-1}(m) \subseteq \text{ran}(g_{2n})$ for $m \leq n$.

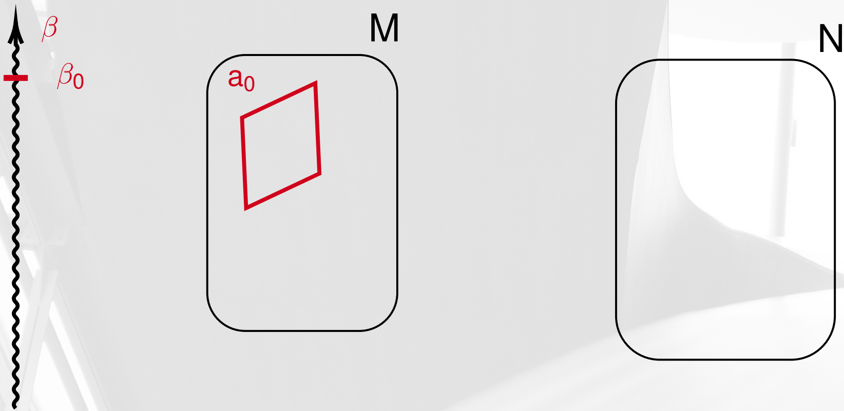
ISO wins if she can play all her moves, otherwise ANTI wins.

- $\mathsf{M} \sim_\theta^\beta \mathsf{N}$ iff ISO has a winning strategy in the game.
- $\mathsf{M} \equiv_\theta^\beta \mathsf{N}$ is defined as the transitive closure of $\mathsf{M} \sim_\theta^\beta \mathsf{N}$.
- A union of $\leq \beth_{\beta+1}(\theta)$ equivalence classes of $\equiv_\theta^\beta$ for some $\theta < \kappa$ and $\beta < \theta^+$ is called a sentence of L_κ^1.

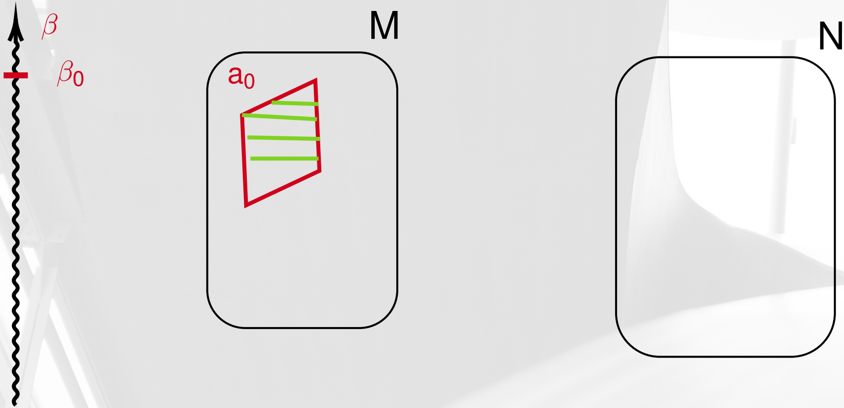
Shelah's game $G_\theta^\beta(M, N)$.



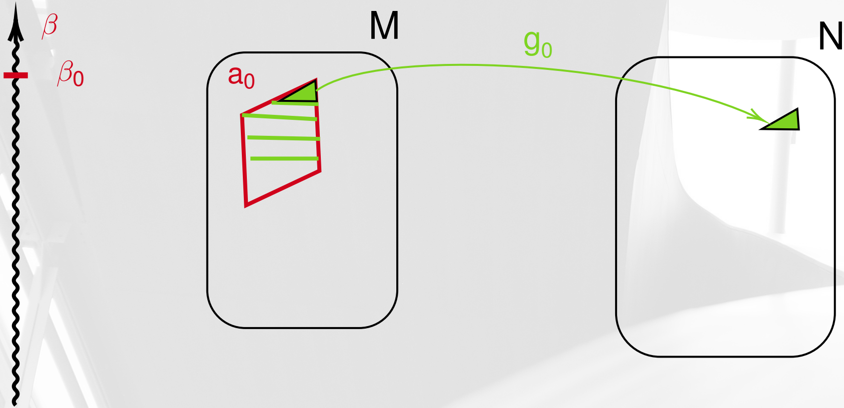
SHELAH'S GAME $G^{\beta}_{\theta}(M, N)$.



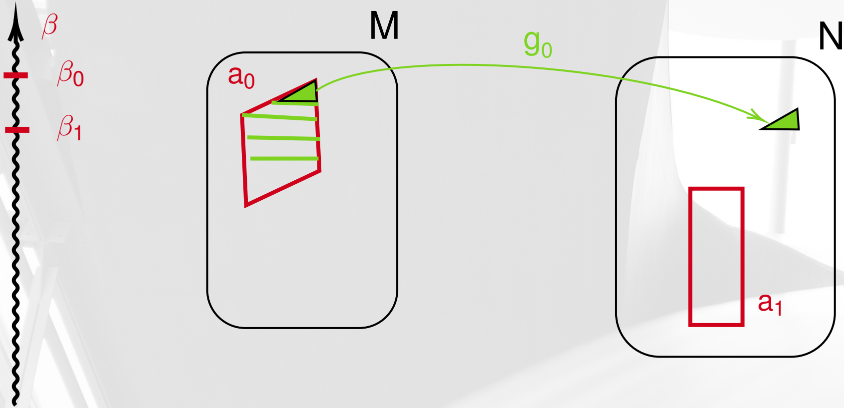
SHELAH'S GAME $G_\theta^\beta(M, N)$.



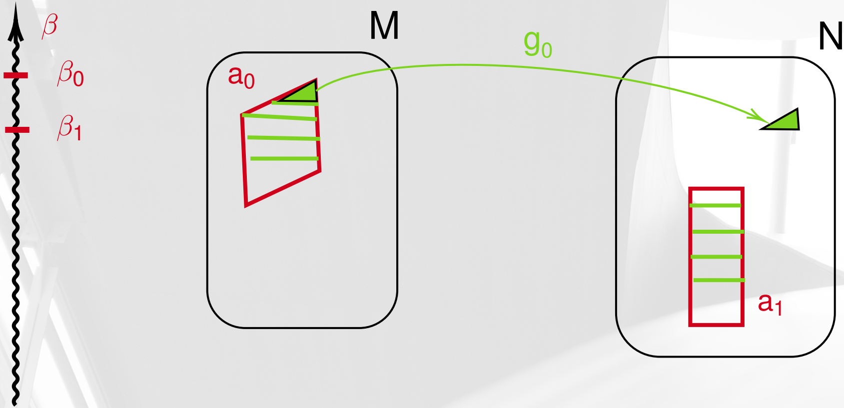
Shelah's game $G_\theta^\beta(M, N)$.



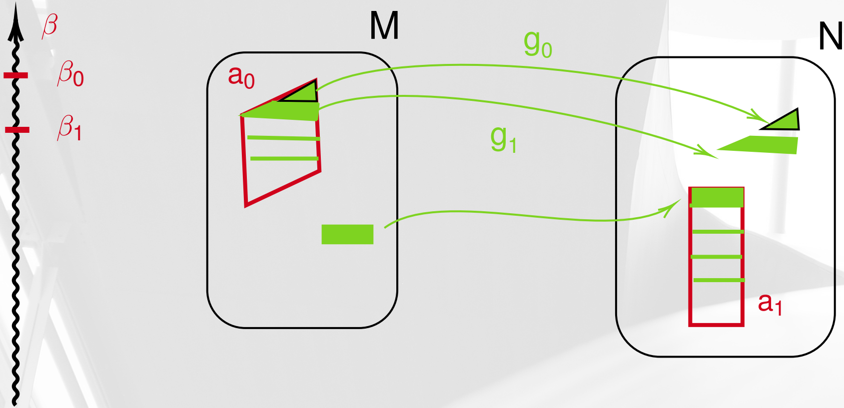
Shelah's game $G^{\beta}_{\theta}(M, N)$.



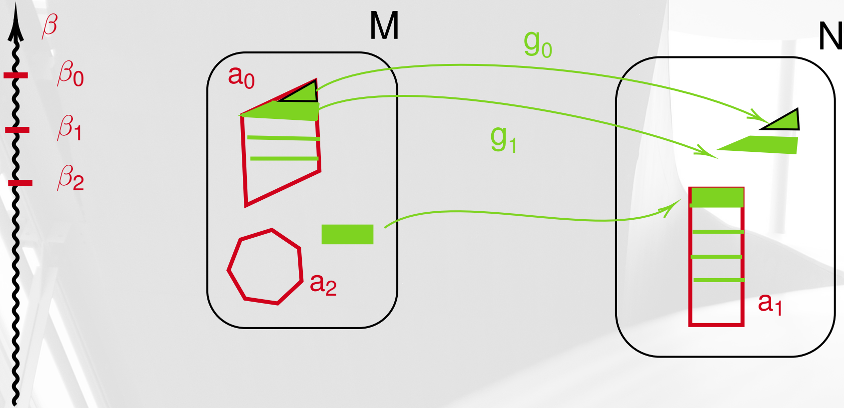
Shelah's game $G_\theta^\beta(M, N)$.



Shelah's game $G_\theta^\beta(M, N)$.



SHELAH'S GAME $G_\theta^\beta(M, N)$.



Musings on approximation from above

I: Chain logic $L^{1,ch}_{\kappa}$: Carol Karp

(This is recent work of Džamonja and Väänänen)

- Syntax: $L_{\kappa\kappa}$, κ singular strong limit of cof ω.
- Semantics in chain models ($M_0 \subseteq M_1 \subseteq \ldots$)
- $\exists\vec{x}\phi$ means $\exists\vec{x}((\bigvee_n \bigwedge_j x_j \in M_n) \wedge \phi)$
- Craig($L^{1,ch}_{\kappa}$) (E. Cunningham, 1975)
- $L_{\kappa\omega} < L^{1,ch}_{\kappa} < L_{\kappa\kappa}$
- $L^{1}_{\kappa} \leq L^{1,c}_{\kappa} < L_{\kappa\kappa}$
- "Chu-transform" (Chu-spaces) is used as a device to compare logics.

II: From above, a new game (other splittings)

- L^1_κ is robust, but the lack of proper syntax if problematic.
- Väänänen and Veličković define a deliberately stronger but simpler logic and then show that it is the same as L^1_κ, under conditions on κ.

The modified game $G^{1,\beta}_{\theta,\alpha}(M, N)$.

$\beta_0 < \beta, \vec{a^0}$	
	$f_0 : \vec{a^0} \to \alpha$, $g_0 : M \to N$ a p.i.
$\beta_1 < \beta_0, \vec{b^1}$	$f_1 : \vec{a^0} \cup \vec{b^1} \to \alpha$, $g_1 : M \to N$ a p.i., $g_1 \supseteq g_0$
$\vdots$	$\vdots$

Constraints:

- $\text{len}(\vec{a^n}) \leq \theta$, $\text{len}(\vec{b^n}) \leq \theta$.
- $f_{i+1}(x) < f_i(x)$ if $f_i(x) \neq 0$.
- $f^{-1}_{2n}(0) \subseteq \text{dom}(g_{2n})$ for $m \leq n$.
- $f^{-1}_{2n+1}(0) \subseteq \text{ran}(g_{2n})$ for $m \leq n$.

Player II wins if she can play all her moves, otherwise Player I wins.

From above, the Väänänen-Veličković variant of the game

- $\mathsf{G}^{1,\beta}_{\theta,\alpha}(\mathsf{M},\mathsf{N})$ is the EF-game of a logic $\mathsf{L}^{1}_{\theta,\alpha}$ up to the quantifier-rank β.
- If $\omega \leq \alpha \leq \alpha'$ and $\theta \leq \eta$, then $\mathsf{L}^{1}_{\theta} \leq \mathsf{L}^{1}_{\theta,\alpha} \leq \mathsf{L}^{1}_{\theta,\alpha'} \leq \mathsf{L}_{\eta^{+}\eta^{+}}$.
- If α is indecomposable, then "Player II has a winning strategy in $\mathsf{G}^{1,\beta}_{\theta,\alpha}(\mathsf{M},\mathsf{N})$" is transitive and $\mathsf{L}^{1}_{\kappa,\alpha}$ has a syntax (less clear than that of our $\mathsf{L}^{1,c}_{\kappa}$).

From above, the Väänänen-Veličković variant of the game

Theorem

If $\kappa = \beth_\kappa$ *and* α *is indecomposable, then* $\mathsf{L}^1_\kappa = \mathsf{L}^1_{\kappa,\alpha}$.

Comparison of the two games:

Trivially: If $\beta' \leq \beta$, $\theta' \leq \theta$ and $\alpha \leq \alpha'$, then

$$\mathsf{II} \uparrow \mathsf{G}^{1,\beta}_{\theta,\alpha}(\mathsf{A},\mathsf{B}) \Rightarrow \mathsf{II} \uparrow \mathsf{G}^{1,\beta'}_{\theta',\alpha'}(\mathsf{A},\mathsf{B}).$$

Theorem

For every β there is β^* such that

$$\mathsf{II} \uparrow \mathsf{G}^{1,\beta^*}_{2^\theta,\alpha}(\mathsf{A},\mathsf{B}) \Rightarrow \mathsf{II} \uparrow \mathsf{G}^{1,\beta}_{\theta,\omega}(\mathsf{A},\mathsf{B}).$$

Here if $\kappa = \beth_\kappa$ and $\beta < \kappa$, then $\beta^* < \kappa$. The proof uses…the same Komjáth-Shelah lemma we now have seen!

Thank you! ¡Gracias! Fié nzhinga!