Completeness as (TOPol) Reconstruction (of LOGic)

from Stone to Lurie

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Our topics

The main theme: reconstruction

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For example…

Examples of Reconstruction

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- Reconstruction of theories or bi-interpretability classes of models from their automorphism groups.

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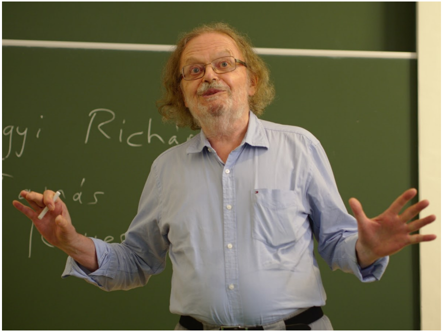
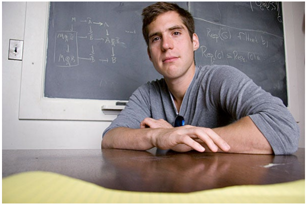
- Reconstruction of varieties from their homotopy groups,
- Reconstruction of theories or bi-interpretability classes of models from their automorphism groups.

In the second situation, the key to reconstruction is in…capturing the topology of the automorphism groups from the pure algebraic structure!

Recovering SYNTax from SEMANTics





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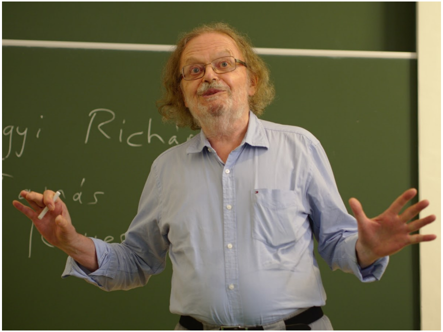
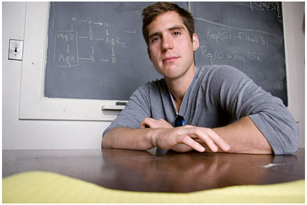
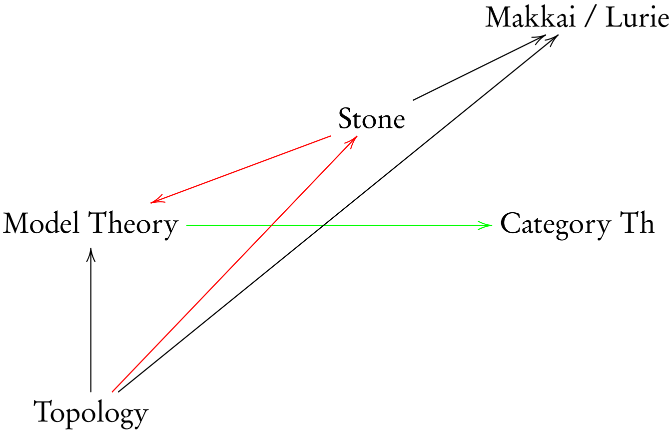




Stone, Makkai and more recently Lurie achieve exactly that, in three different contexts.

Reconstruction(s)



Our topics

Reconstruction of Syntax from Semantics: Stone

Reconstructing Syntax from Semantics: Makkai and Lurie
- Ultrafilters, ultrafunctors
- Properties of ultrafilters and ultrafunctors - in categories
- Makkai / Lurie

Dualities

Some language…

We first revisit a very classical theorem, albeit with a somewhat modernized :) language…

(Lawvere, Makkai, Reyes, Lurie…)
We use Lurie's notation from his recent (2019) paper Ultracategories

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- Given a boolean algebra $\mathsf{B} = (B, \wedge, \vee, (\cdot)^c, 0, 1)$, its spectrum $\mathsf{Hom_{BAlg}(B, \{0, 1\})}$ is the set of homomorphisms $h : B \to \{0, 1\}$.
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- BAlg is the category of boolean algebras whose morphisms are boolean algebra homomorphisms.
- $\mathsf{Spec}(\mathsf{B})$ is a subset of $\prod_{x \in \mathsf{B}} \{0, 1\}$, the set of all functions from B into $\{0, 1\}$; thus, $\mathsf{Spec}(\mathsf{B})$ has a topology naturally induced by the product topology. This topology depends functorially on the boolean algebra B.

Stone's Duality Theorem - 1936

The construction $\mathsf{B} \mapsto \mathsf{Spec}(\mathsf{B})$ determines a fully faithful embedding

$$\mathsf{Spec} : \mathsf{BAlg}^{\mathsf{op}} \to \mathsf{Top}$$

from the (opposite) category of boolean algebras into the category of topological spaces. The essential image of this functor is the full subcategory $\mathsf{Stone} \subseteq \mathsf{Top}$ whose objects are Stone spaces (i.e., compact Hausdorff topological totally disconnected topological spaces).

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Stone's theorem also provides an equivalence between syntax and semantics in propositional logic.

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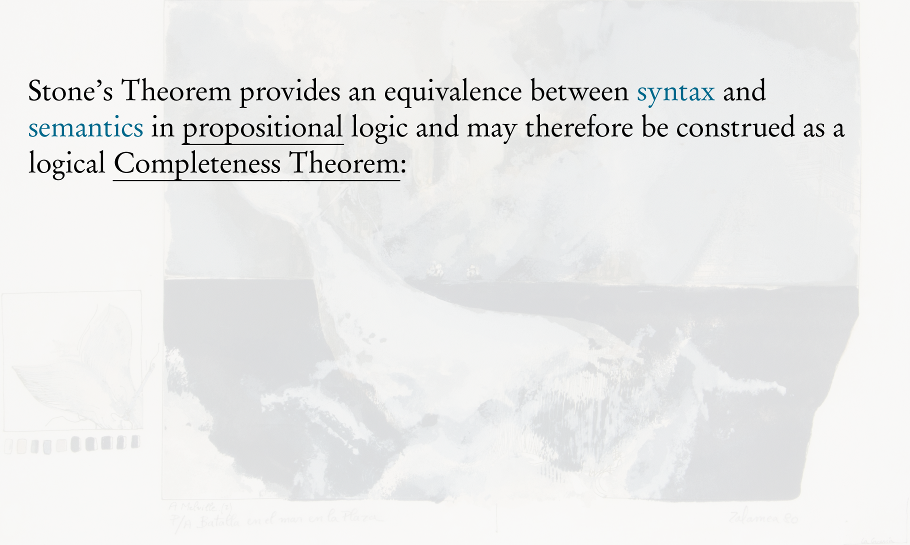
Stone's Theorem provides an equivalence between syntax and semantics in propositional logic and may therefore be construed as a logical Completeness Theorem:



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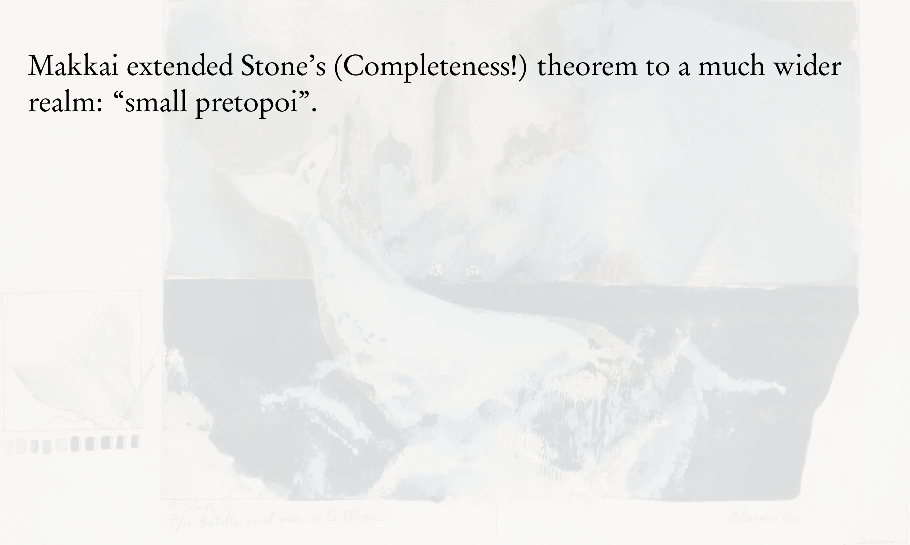
The theorem implies that every boolean algebra may be reconstructed from two data:

- its collection of models $\mathsf{Spec(B)}$, and
- its topology.

A complicated statement…but the same idea

Makkai extended Stone's (Completeness!) theorem to a much wider realm: "small pretopoi".



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We would like to grasp a few items from all this:

- What are these definitions?
- How does topology play into reconstruction?
- How is this a "Completeness Theorem"?

Classical constructions / Convenient notations

In order to catch better what Makkai (and much later Lurie) mean, it is worth recalling some more classical constructions: the Stone-Čech compactification and ultraproducts. (Lurie's notation!)

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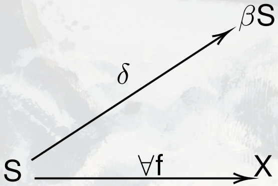
For every $\mathsf{s} \in \mathsf{S}$, we have the "Dirac delta ultrafilter" based on s: the homomorphism

$$\delta_{\mathsf{s}} : \mathcal{P}(\mathsf{S}) \to \{0, 1\} \text{ given by } \delta_{\mathsf{s}}(\mathsf{I}) = \begin{cases} 1 & \text{si } \mathsf{s} \in \mathsf{I} \\ 0 & \text{si } \mathsf{s} \notin \mathsf{I}. \end{cases}$$

Recalling some basic topology

The "Dirac function" $\delta : \mathsf{S} \to \beta\mathsf{S}$ given by $\mathsf{s} \mapsto \delta_{\mathsf{s}}$ is injective.
In basic topology, $\beta\mathsf{S}$ is the "universal" compactification of S:

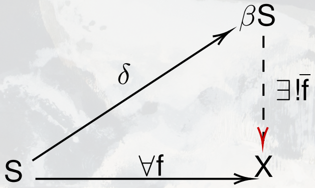
> Given X a Hausdorff compact space, and $\mathsf{f} : \mathsf{S} \to \mathsf{X}$ a function, there exists a unique continuous function $\overline{\mathsf{f}} : \beta\mathsf{S} \to \mathsf{X}$ such that $\overline{\mathsf{f}} \circ \delta = \mathsf{f}$.



Recalling some basic topology

The "Dirac function" $\delta : S \to \beta S$ given by $s \mapsto \delta_s$ is injective. In basic topology, βS is the "universal" compactification of S:

Given X a Hausdorff compact space, and $f : S \to X$ a function, there exists a unique continuous function $\bar{f} : \beta S \to X$ such that $\bar{f} \circ \delta = f$.

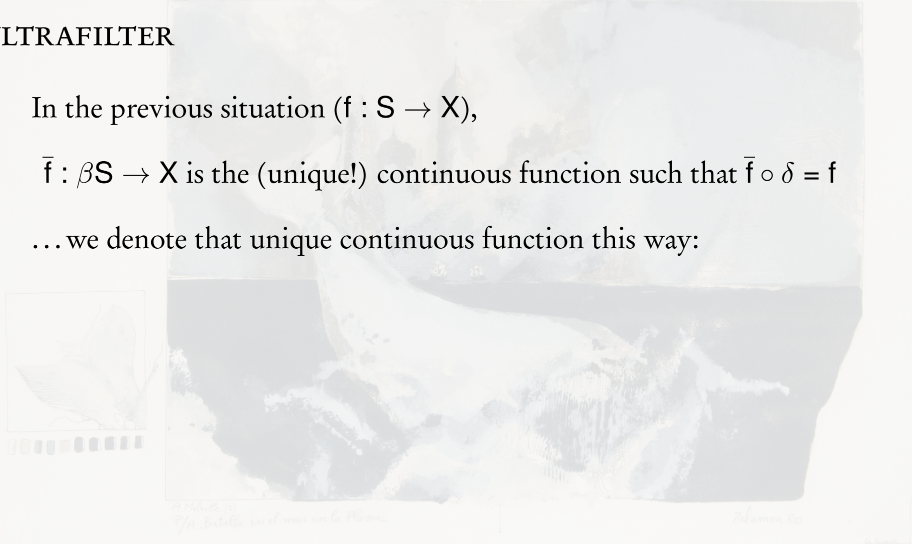


Integrating a function $\mathsf{f} : \mathsf{S} \to \mathsf{X}$ with respect to a ultrafilter

In the previous situation ($\mathsf{f} : \mathsf{S} \to \mathsf{X}$),

$\bar{\mathsf{f}} : \beta\mathsf{S} \to \mathsf{X}$ is the (unique!) continuous function such that $\bar{\mathsf{f}} \circ \delta = \mathsf{f}$

…we denote that unique continuous function this way:



INTEGRATING A FUNCTION $f : S \to X$ WITH RESPECT TO A ULTRAFILTER

In the previous situation ($f : S \to X$),

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…we denote that unique continuous function this way:

$$\overline{f}(\mu) = \int_S f(s) d\mu.$$

So, <u>every</u> function f from a set S into a Hausdorff compact space X may be "integrated" with respect to a <u>ultrafiter</u> $\mu \in \beta S$; this produces an element $\int_S f(s) d\mu \in X$

Integral over ultrafilters, and the topology of βS

- X's topology determines the integral:
  - $\int_S f(s) d\delta_t = f(t)$
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  - $\int_S f(s) d\delta_t = f(t)$
  - $\int_S f(s) d\mu$ depends continuously on $\mu \in \beta S$
- If $f : S \to X$ has a dense image, then the topology of X may be recovered from the data

$$\mu \mapsto \int_S f(s) d\mu$$

(as any continuous function between Hausdorff spaces $\beta S \to X$ is a quotient)

Coding

So, from f and μ, through the construction

$$(\mathsf{f}, \mu) \mapsto \int_{\mathsf{S}} \mathsf{f}(\mathsf{s})\mathsf{d}\mu$$

we may recover the topology of X.

Our topics

Makkai - Generalizing the previous

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We may bring duality to wider contexts (predicate calculus, coherent logic, etc.) but the previous needs to be generalized further.

Let T be a theory (or even a small pretopos $\mathcal{C}$) and $\{\mathsf{M_s}\}_{\mathsf{s}\in\mathsf{S}}$ a collection of models of T (or in $\mathcal{C}$) with indices in a set S.

We build the ultraproduct of the models $\mathsf{M_s}$ by a ultrafilter μ:

Ultraproducts, in our "new" language

Given a family $\{M_s\}_{s \in S}$ of non-empty sets, a ultrafilter μ over S provides an equivalence relation $\sim_\mu$ on the product $\prod_{s \in S} M_s$

$$(x_s)_{s \in S} \sim_\mu (y_s)_{s \in S} \iff \mu\left(\left\{s \in S \mid x_s = y_s\right\}\right) = 1.$$

The ultraproduct of $\{M_s\}_{s \in S}$ along μ is the quotient

$$\int_S M_s d\mu := (\prod_{s \in S} M_s)/ \sim_\mu$$

(And similarly for models of T ... or of $\mathcal{C}$)

The construction $\{\mathsf{M_s}\}_{\mathsf{s}\in\mathsf{S}} \mapsto \int_{\mathsf{S}} \mathsf{M_s} \mathsf{d}\mu$

Given a ultrafilter in $\beta\mathsf{S}$, the construction $\{\mathsf{M_s}\}_{\mathsf{s}\in\mathsf{S}} \mapsto \int_{\mathsf{S}} \mathsf{M_s} \mathsf{d}\mu$ yields a functor

$$\int_{\mathsf{S}} (\bullet)\mathsf{d}\mu : \mathsf{Mod(T)}^{\mathsf{S}} \to \mathsf{Mod(T)}.$$

The construction $\{M_s\}_{s\in S} \mapsto \int_S M_s d\mu$

Given a ultrafilter in βS, the construction $\{M_s\}_{s\in S} \mapsto \int_S M_s d\mu$ yields a functor

$$\int_S (\bullet) d\mu : \mathrm{Mod}(T)^S \to \mathrm{Mod}(T).$$

These functors (along with two natural transformations linking them) are the key to the ultrastructure of the category $\mathrm{Mod}(T)$ ($\mathrm{Mod}(\mathcal{C})$).

UltraFun

Let F be a functor between two categories $\mathcal{M}$ and $\mathcal{N}$. A ultra-structure on F is a family of isomorphisms

$$\mathsf{F}(\int_{\mathsf{S}} \mathsf{M_s} \mathsf{d}\mu) \approx \int_{\mathsf{S}} \mathsf{F}(\mathsf{M_s})\mathsf{d}\mu$$

with indices in collections $\{\mathsf{M_s}\}_{\mathsf{s}\in\mathsf{S}}$ and ultrafilters $\mu \in \beta\mathsf{S}$ (with certain coherence properties).

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A ultrafunctor is a functor F with ultrastructure on top.

Before looking at Makkai's statement (and the recent one by Lurie), let us dig a bit deeper in the properties of ultrafilters and ultrafunctors, in categories.
First, we may build objects $\int_{\mathsf{S}} \mathsf{M}_{\mathsf{s}} \mathsf{d}\mu$ in a category $\mathcal{M}$ whenever it admits products and (small) fibered colimits. In that case:

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> Given a collection of objects $\{M_s\}_{s\in S}$ in $\mathcal{M}$ and given a ultrafilter μ on S,
>
> $$\int_S M_s d\mu = \varinjlim_{\mu(S_0)=1} \Big(\prod_{s\in S_0} M_s\Big).$$
>
> This is the categorical ultraproduct of $\{M_s\}_{s\in S}$ along μ.

Basic properties of ultraproducts

The construction $\{M_s\}_{s \in S} \mapsto \int_S M_s d\mu$ satisfies:

- For any collection of objects $\{M_s\}_{s \in S}$ in $\mathcal{M}$ and δ_{s_0} the Dirac ultrafilter associated to a $s_0 \in S$, there exists a canonical iso

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- If $\{N_t\}_{t\in T}$ is a collection of objects with indices in T and $\nu_\bullet = \{\nu_s\}_{s\in S}$ is a collection of ultrafilters with indices in S, there exists a canonical transform

$$\Delta_{\mu,\nu_\bullet} : \int_T N_t d(\int_S \nu_s d\mu) \to \int_S (\int_T N_t d\nu_s) d\mu;$$

(the Fubini transform) - here $\int_S \nu_s d\mu$ is the ultrafilter on T given by $\int_S \nu_s d\mu(T_0) = \mu(\{s \in S \mid \nu_s(T_0) = 1\})$.

But there is (much) more…

For instance, the pushforward $f_*\mu$ of a ultrafilter μ on S by $f : S \to T$, given by

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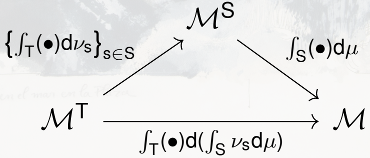

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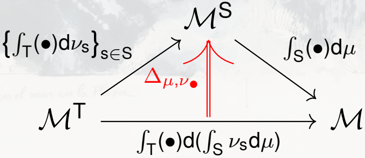

Makkai, now

Makkai's Conceptual Strong Completeness Theorem

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There are various proofs; recently (2019), Lurie gave a quite strong generalization.

Lurie, 2019

Lurie's (Conceptual) Completeness Theorem

Given $\mathcal{C}$ a small pretopos, its associated coherent topos $\mathsf{Shv}(\mathcal{C})$ is equivalent to the category of left ultrafunctors $\mathsf{Fun}^{\mathsf{LUlt}}(\mathsf{Mod}(\mathcal{C}), \mathsf{Set})$:

$$\mathsf{Shv}(\mathcal{C}) \longleftrightarrow \mathsf{Fun}^{\mathsf{LUlt}}(\mathsf{Mod}(\mathcal{C}), \mathsf{Set})$$

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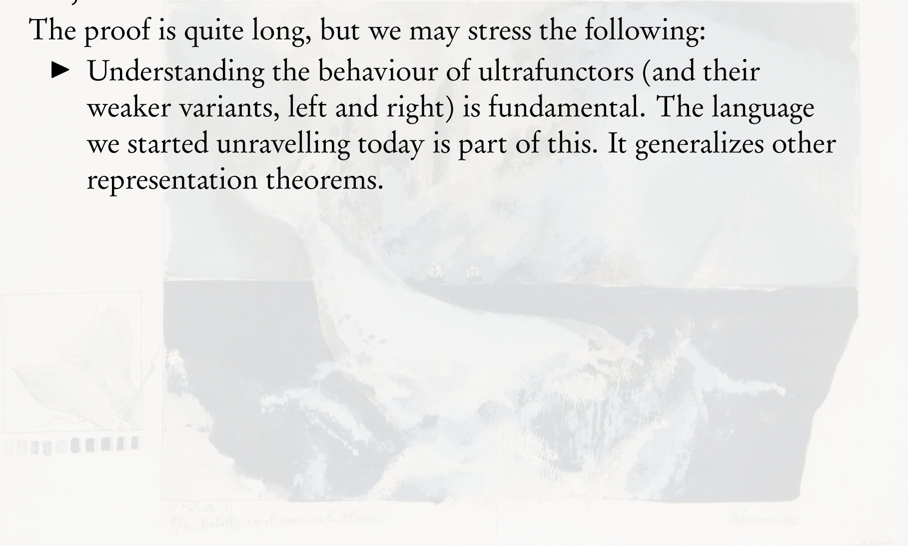

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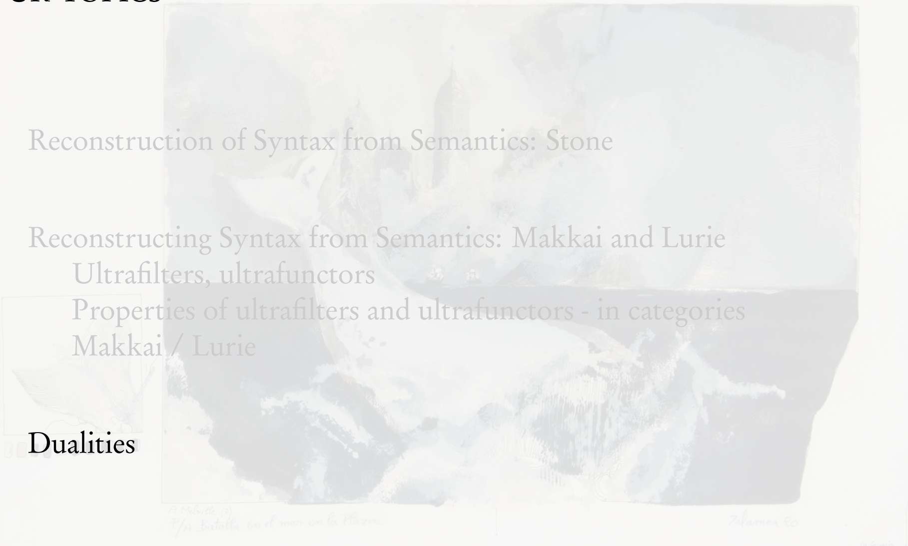
- Many categories have NO products but DO HAVE ultraproducts (and ultrafunctors). The constructions depend on "categorical hulls" that allow to pin down the ultraproducts. An important example: categories of models of theories.
- Lurie (personal communication) claims to be searching the "natural logic of ∞-categories" through these completeness theorems.

Our topics



Reconstruction of Syntax from Semantics: Stone

Reconstructing Syntax from Semantics: Makkai and Lurie
- Ultrafilters, ultrafunctors
- Properties of ultrafilters and ultrafunctors - in categories
- Makkai / Lurie

Dualities

Questions / panorama

We now discuss a bit some of the panorama formed between logic and topology.

Other situations / questions

- Reconstruction of structures M. It almost never works! However, knowing the topology of $\mathsf{Aut}(\mathsf{M})$ (in case M is not rigid), one may recover…the “bi-interpretability class of M and (in some lucky situations) even $\mathsf{Th}(\mathsf{M})$. (The SIP world.) With Ghadernezhad, we have used closure classes to show that (for example) Zilber’s Complex Pseudo-exponentiation may be reconstructed from its automorphisms (this is still an open problem for classical exponentiation).

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- With Shelah, we now have a combinatorial reconstruction (using quite strong infinitary logic) of "abstract elementary classes" (classes where reflection phenomena between models replace axiomatizations). There is no topological proof.

Muito obrigado pela vossa atenção!

Figure: *La integral del silencio* - R. Matta - 1990