Around Definability in AECs: Logics and Patterns

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Helsinki Logic Seminar / 14 September 2022

Universidad Nacional de Colombia / Bogotá

Contents

Definability issues (I): sentences axiomatizing AECs

Definability issues (II): patterns for AECs?

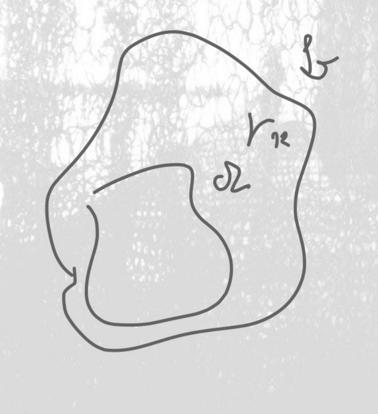
In February, our last quotation was...

If the Greeks were so attached to geometry, wasn't it that they thought by tracing lines, with no words? However (or maybe just because of that?) [they produced] a perfect axiomatic! Euclid's Postulates, construction. Limiting what one is allowed to trace.

Simone Weil, Cahier III

AECs: why so much stability theory?

And our first question was the title of this slide!



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AECs: why so much stability theory?

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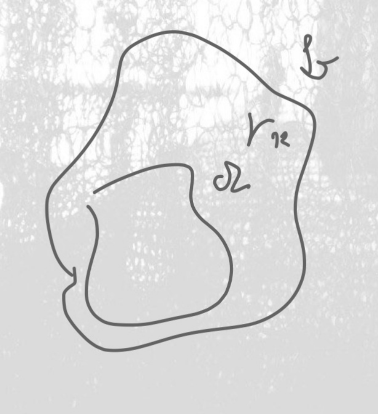


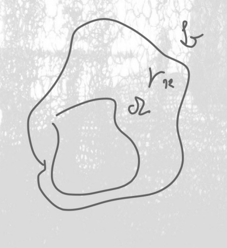
In AECs, we replace from the outset the initial extreme emphasis on φ, T, compactness by more **semantical** notions: $\prec_{\mathcal{K}}$, f a morphism, $f \in Aut(\mathbb{C})$, etc.

φ

T

$\mathsf{T}_0 \subseteq^{\text{fin}} \mathsf{T}$

$\vdots$



φ

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$T_0 \subseteq^{\text{fin}} T$

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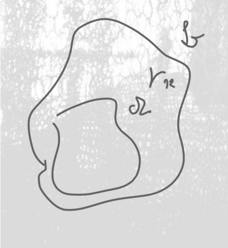
emphasis shift
towards 1980



φ

T

$T_0 \subseteq^{\mathrm{fin}} T$

: Instead of extracting $\prec$, f, etc. from T, φ, we turn $\prec$, f a strong embedding into the primitive notions!

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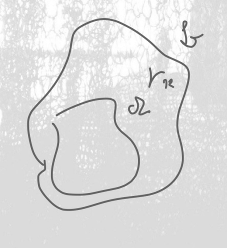

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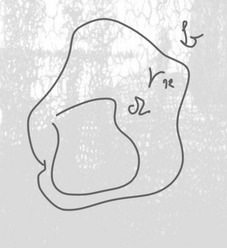
subgroup

subring

pure subring

strong substructure



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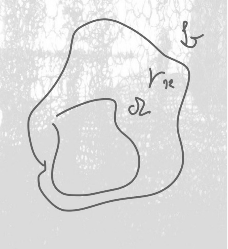

$\mathfrak{A} \prec_{\mathcal{K}} \mathfrak{B}$

"perfect" extension, algebraically closed, etc.

AEC - the axioms, briefly

Fix $\mathcal{K}$ be a class of τ-structures, $\prec_{\mathcal{K}}$ a binary relation on $\mathcal{K}$.

Definition

$(\mathcal{K}, \prec_{\mathcal{K}})$ is an abstract elementary class iff

- $\mathcal{K}$, $\prec_{\mathcal{K}}$ are **closed under isomorphism**,
- $M, N \in \mathcal{K}$, $M \prec_{\mathcal{K}} N \Rightarrow M \subset N$,
- $\prec_{\mathcal{K}}$ is a partial order,
- (**TV**) $M \subset N \prec_{\mathcal{K}} \bar{N}, M \prec_{\mathcal{K}} \bar{N} \Rightarrow M \prec_{\mathcal{K}} N$,
- ($\searrow$**LS**) There is some $\kappa = LS(\mathcal{K}) \geq \aleph_0$ such that for every $M \in \mathcal{K}$, for every $A \subset |M|$, there is $N \prec_{\mathcal{K}} M$ with $A \subset |N|$ and $\|N\| \leq |A| + LS(\mathcal{K})$,
- (**Unions of $\prec_{\mathcal{K}}$-chains**) A union of an arbitrary $\prec_{\mathcal{K}}$-chain in $\mathcal{K}$ belongs to $\mathcal{K}$, is a $\prec_{\mathcal{K}}$-extension of all models in the chain and is the sup of the chain.

And really, a lot of examples (and model theory)

Natural constructions in Mathematics are examples of AEC (or metric AEC)

1. Complete first order theories
2. Various classes axiomatizable in $L_{\omega_1,\omega}$ or $L_{\kappa\omega}$.

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1. Complete first order theories
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3. Metric AEC - stability theory started by Hirvonen and Hyttinen, Usvyatsov, and continued by Zambrano and V.
4. Metric AECs and connections with operator algebras (Hirvonen, Hyttinen)
5. Model Theory of Modules (Mazari-Armida)

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5. Model Theory of Modules (Mazari-Armida)
6. AECs of C^*-algebras (Argoty, Berenstein, V.)
7. Zilber analytic classes (pseudoexponentiation)
8. Classes of ACVF?

Elusive, “embedded” definability?

Those in the know of stability/classification theory of AECs are well aware that the inner workings depend very strongly on handling

- Indiscernible sequences and EM models,
- (Versions of) Morley Omitting Types [to transfer saturation, to transfer categoricity],
- and of course, many variants of “forking independence”, very specially versions of “splitting” (sometimes called weak definability)!

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So, the question of finding the right notion of definability responsible for all these inner workings seems important, on both technical bases (and on more philosophical grounds!)...

Axiomatizing an AEC: attempts (old and new)

- Shelah, V. 2020,
- Leung 2021.

Fix $(\mathcal{K}, \prec_{\mathcal{K}})$ an AEC with $\mathrm{LS}(\mathcal{K}) = \kappa$. We also assume all models in $\mathcal{K}$ are of cardinality $\geq \kappa$.

Earlier results:

- **Shelah's Presentation Theorem:** $\mathcal{K}$ is $\mathrm{PC}_{\kappa, 2^\kappa}$.

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- **Shelah-Vasey:** If $\mathsf{LS}(\mathcal{K}) = \aleph_0$, $\mathcal{K}$ is $\aleph_0$-stable and has the $\aleph_0$-AP, and $\mathsf{I}(\aleph_0, \mathcal{K}) \leq \aleph_0$ <u>then</u> $\mathcal{K}$ is $\mathsf{PC}_{\aleph_0}$.

More

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- **Shelah, V.** (2022). A better bound: we reduce the complexity of the sentence to $\mathbb{L}_{(2^\kappa)^+,\kappa^+}$, in the original vocabulary!

2020

Shelah-V.

$\mathcal{K} = \mathrm{Mod}\,(\psi_{\mathcal{K}})$

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(The $\omega \cdot \omega$ refers to quantification of an EF game of length $\omega \cdot \omega$)

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better logic,

In 2022, a better bound: in $\mathbb{L}_{(2^\kappa)^+,\kappa^+}$

better bound, but use of $\forall x_0 \exists y_0 \ldots \forall x_i \exists x_i \ldots$, $i < \omega \cdot \omega$

2020

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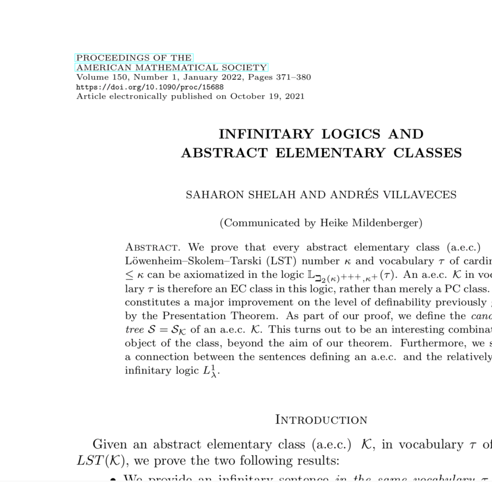

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 150, Number 1, January 2022, Pages 371–380
https://doi.org/10.1090/proc/15688
Article electronically published on October 19, 2021

INFINITARY LOGICS AND
ABSTRACT ELEMENTARY CLASSES

SAHARON SHELAH AND ANDRÉS VILLAVECES

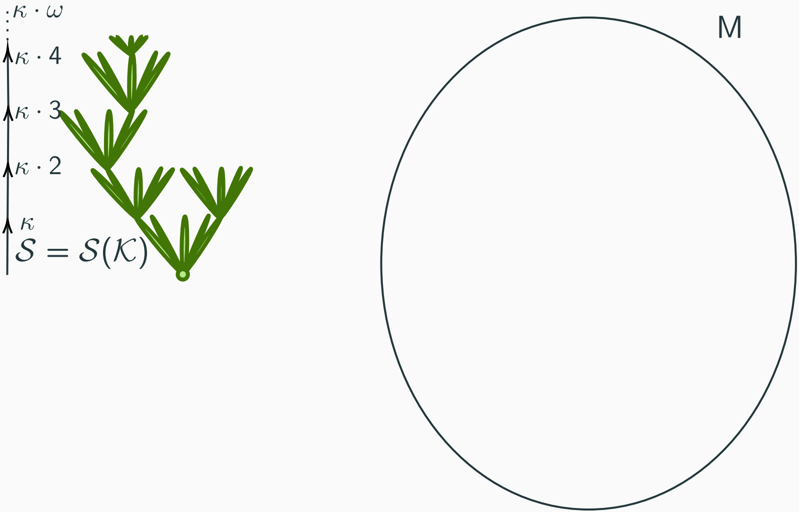
(Communicated by Heike Mildenberger)

ABSTRACT. We prove that every abstract elementary class (a.e.c.)
Löwenheim–Skolem–Tarski (LST) number κ and vocabulary τ of cardi
$\leq \kappa$ can be axiomatized in the logic $\mathbb{L}_{\beth_2(\kappa)^{+++},\kappa^+}(\tau)$. An a.e.c. $\mathcal{K}$ in voc
lary τ is therefore an EC class in this logic, rather than merely a PC class.
constitutes a major improvement on the level of definability previously
by the Presentation Theorem. As part of our proof, we define the *canc*
tree $\mathcal{S} = \mathcal{S}_{\mathcal{K}}$ of an a.e.c. $\mathcal{K}$. This turns out to be an interesting combina
object of the class, beyond the aim of our theorem. Furthermore, we s
a connection between the sentences defining an a.e.c. and the relatively
infinitary logic L^1_λ.

INTRODUCTION

Given an abstract elementary class (a.e.c.) $\mathcal{K}$, in vocabulary τ of
$LST(\mathcal{K})$, we prove the two following results:

Testing an arbitrary L-structure M against $\mathcal{S}_{\mathcal{K}}$



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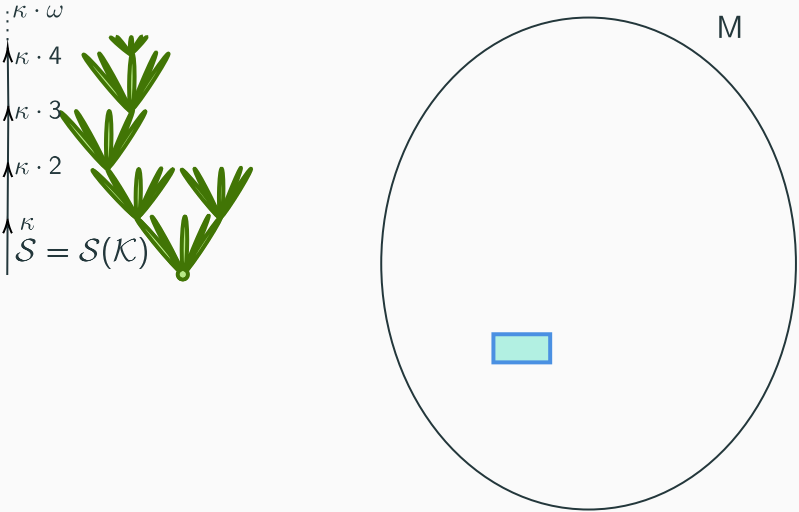

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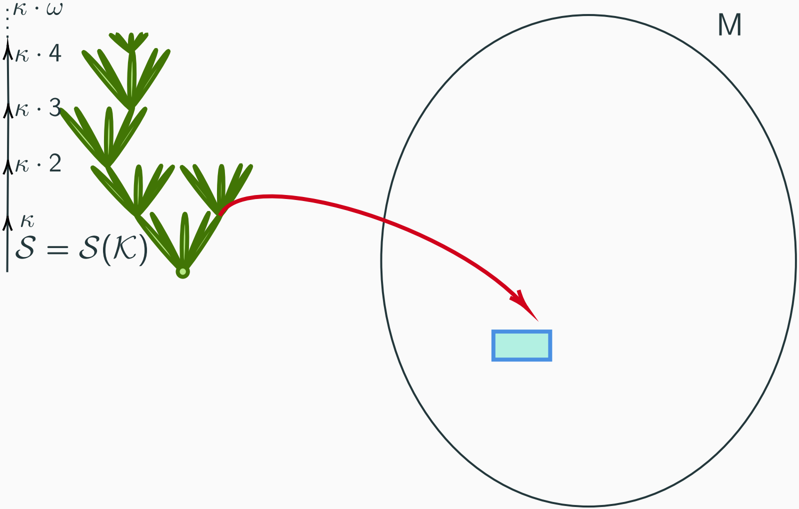

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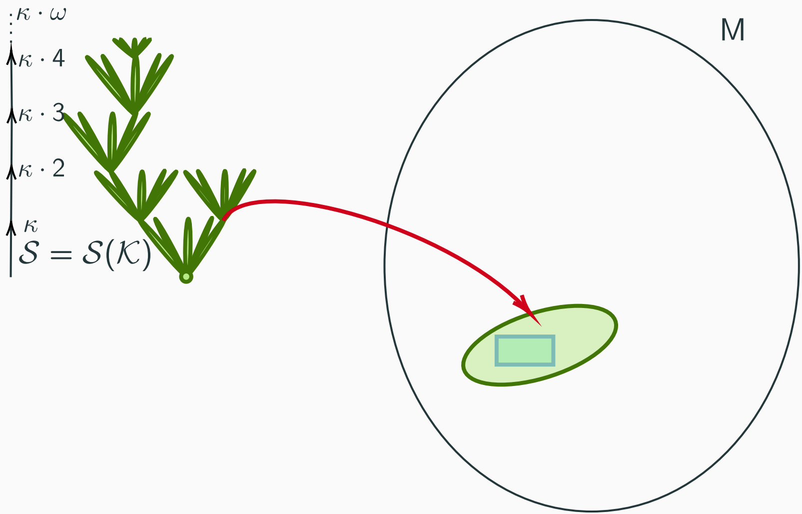

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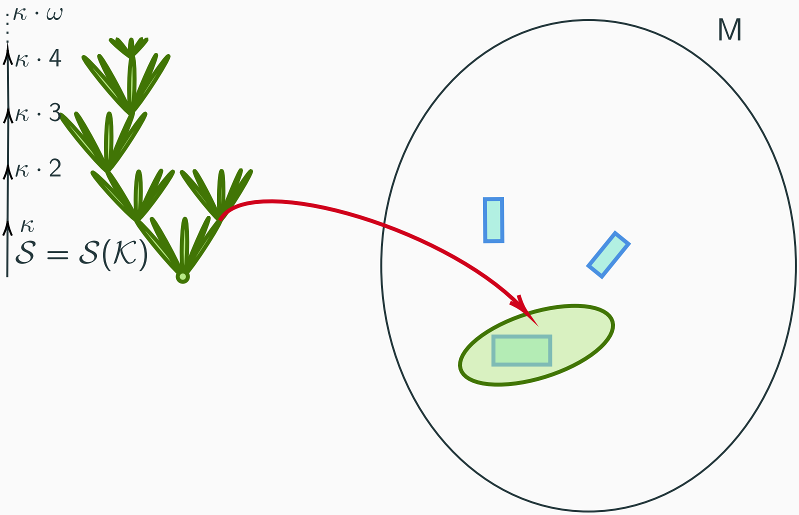

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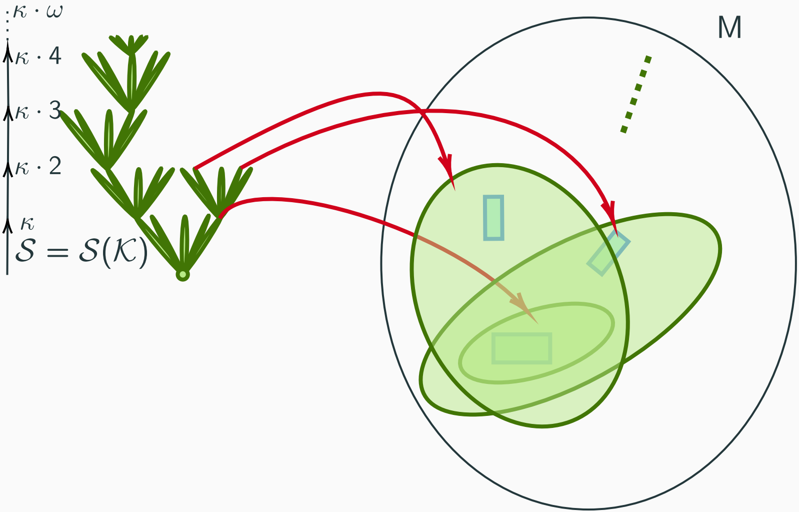

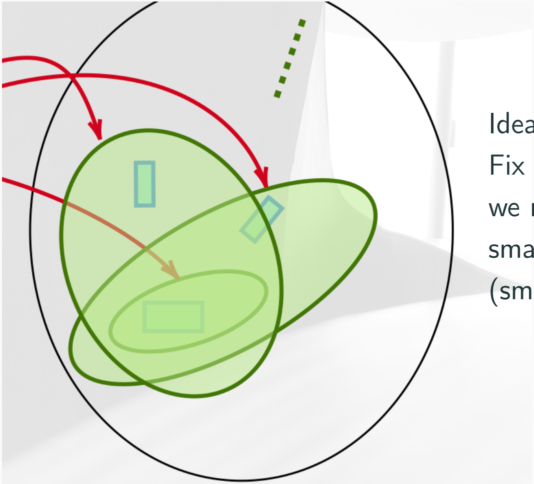

Idea of our axiomatization:

Fix an L-structure M. How can we realize M as a direct limit of small models $N \in \mathcal{K}$?

(small = size $\kappa = LS(\mathcal{K})$)

Realizing an arbitrary model as a limit

$$M = \lim\{N \subseteq M | N \in \mathcal{K}\}???$$

(Of course, we need a lot of constraints!)

Towards this goal

We use the canonical tree of $\mathcal{K}$: models of size $\kappa = \mathsf{LS}(\mathcal{K})$, with universes

$$\kappa, \kappa + \kappa, \kappa + \kappa + \kappa, \ldots$$

and a whole "system of $\prec_{\mathcal{K}}$-elementary embeddings" between those models:

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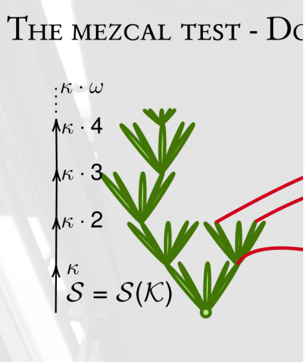


$\mathcal{S}_{\mathcal{K}}$: the canonical tree of $\mathcal{K}$.

In $\mathcal{S}_{\mathcal{K}}$, $\mathrm{N}_1 \lhd \mathrm{N}_2$ iff $\mathrm{N}_1 \prec_{\mathcal{K}} \mathrm{N}_2$.

We now use syntax to…

…to "test" the model M **- the test membership in** $\mathcal{K}$

M **must "pass"** $\beth_2(\kappa)^{++} + 2$ **tests (in 2020), or just** $\alpha < (2^\kappa)^+$ **tests (in 2021)**

$I_2(\kappa)^{++}+2$ (2020) / $\alpha < (2^\kappa)^+$ (2021)

sentences, "approximating" $\mathcal{K}$:

$\varphi_{0,0} = \top$

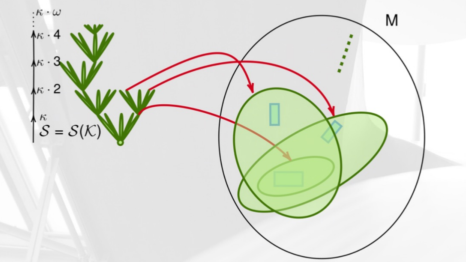
$\varphi_{1,0}$ iterate the "test" against the tree $\mathcal{S}_\alpha$

$\vdots$

$\varphi_{\alpha,0}$

$\vdots$

The mezcal test - Does $M \in \mathcal{K}$?



Formulas $\varphi_{M,\gamma,n}(\bar{x}_n)$

For M in the canonical tree $\mathcal{S}$ at level n, a formula with $\kappa \cdot n$ free variables, defined by induction on γ.

- $\gamma = 0$: $\varphi_{0,0} = \top$ ("truth"). If $n > 0$,

$$\varphi_{M,0,n} := \bigwedge \mathrm{Diag}^n_\kappa(M),$$

the atomic diagram of M in $\kappa \cdot n$ variables.

- γ limit: Then

$$\varphi_{M,\gamma,n}(\bar{x}_n) := \bigwedge_{\beta<\gamma} \varphi_{M,\beta,n}(\bar{x}_n).$$

- $\gamma = \beta + 1$: Then $\varphi_{M,\gamma,n}(\bar{x}_n)$ is the $L_{\lambda^+,\kappa^+}(\tau)$ formula

$$\forall \bar{z}_{[\kappa]} \bigvee_{\substack{N \succ_{\mathcal{K}} M \\ N \in \mathcal{S}_{n+1}}} \exists \bar{x}_{=n} \left[\varphi_{N,\beta,n+1}(\bar{x}_{n+1}) \wedge \bigwedge_{\alpha<\alpha_n[N]} \bigvee_{\delta \in S[N]} z_\alpha = x_\delta \right]$$

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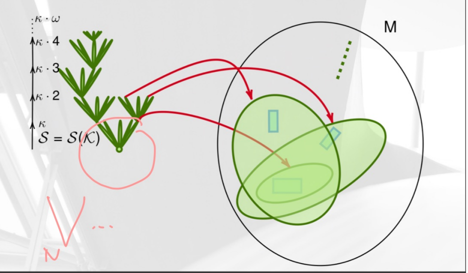

$$\varphi_{1,0} : \forall z \bigvee_{N \in \mathcal{S}_1} \exists \bar{x}_1 \left[\varphi_{N,0,1}(\bar{x}_1) \wedge \bigwedge_{\alpha<\kappa} \bigvee_{\delta<\kappa} z_\alpha = x_\delta \right]$$

size κ

a copy of N

the copy covers z

FORMULAS $\varphi_{M,\gamma,n}(\bar{x}_n)$

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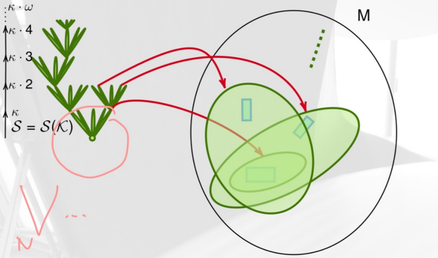
THE MEZCAL TEST - DOES $M \in \mathcal{K}$?



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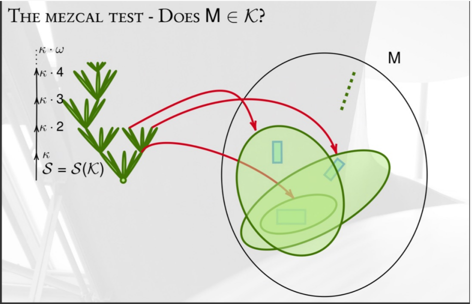
size κ — "copy of N" — covers z

$M \models \varphi_{1,0}$ if it may "be covered" by levels $N \in \mathcal{K}$, of size κ"

$$\varphi_{1,0}: \forall z \bigvee_{N \in \mathcal{S}_1} \exists \bar{x}_1 \Big[\varphi_{N,0,1}(\bar{x}_1) \wedge \bigwedge_{\alpha<\kappa} \bigvee_{\delta<\kappa} z_\alpha = x_\delta \Big]$$

size κ — version of N — covers z

" $\mathcal{S}_1$ covers M "

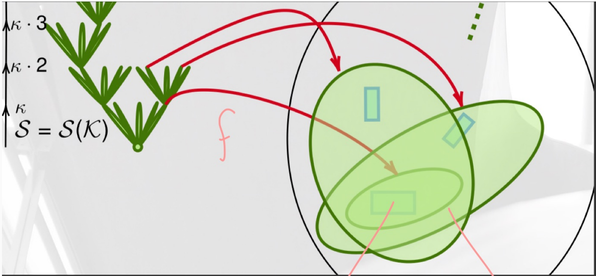


$$M \models \varphi_{2,0} = \forall z \bigvee_{N \in \mathcal{S}_1} \exists \bar{x}_2 \Big[\varphi_{N,1,2}(\bar{x}_2) \wedge \text{“} z \subseteq \bar{x}_2 \text{”} \Big]$$

κ ?

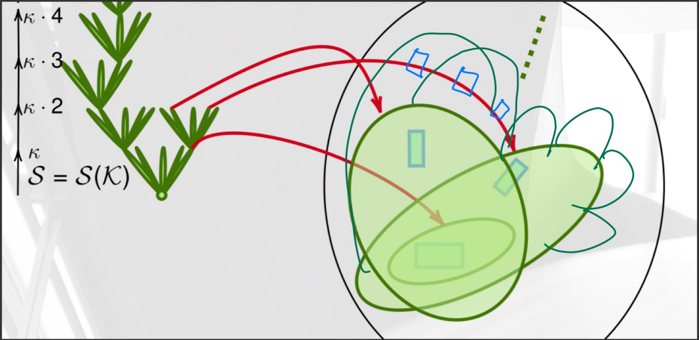
Covers and then Covers

$$\forall z \bigvee_{N \in \mathcal{S}_1} \exists \bar{x}_2 \Big[\forall z' \bigvee_{\substack{N' > N \\ N' \in \mathcal{S}_2}} \exists \bar{x}_1 \Big((\bar{x}_2 {}^\frown \bar{x}_\kappa) \wedge z' \subset \bar{x}_2 {}^\frown \bar{x}_\kappa \Big) \wedge z \subset \bar{x}_2 \Big]$$



Covering notions, refining …

$$\varphi_{2,0} \begin{cases} \forall z \text{ some } N \in \mathcal{S}_1 \text{ covers } z \text{ (via } f) \\ \forall z' \text{ some } N' \in \mathcal{S}_2 \ \ N' \geq_{\mathcal{K}} N \text{ covers } z' \end{cases}$$



$\varphi_{3,0}$: better cover yet ...

Problem: M is big !

As this way of covering may be insufficient, we iterate transfinitely:

$M \models \varphi_{\omega+1,0}$:

$\forall N \bigvee$ covers N BUT $M \models \varphi_{\omega,N,1}(\dots)$ → info of depth ω

$N \in \aleph_1$

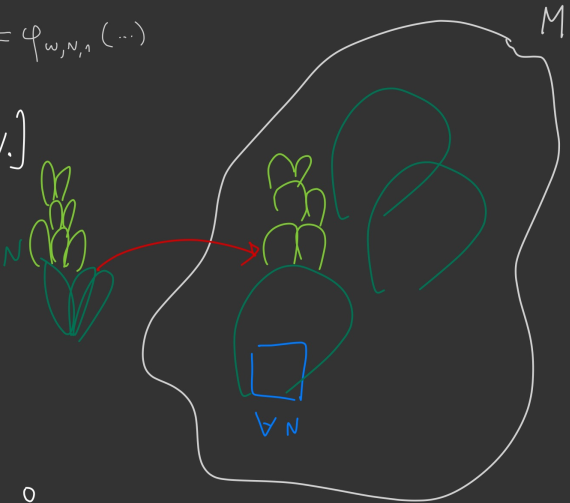
$\forall N \bigvee_{N \in \mathfrak{K}_1}$ cov. N But $M \models \varphi_{\omega,N,1}(\ldots)$

Theorem [Shelah, V.]

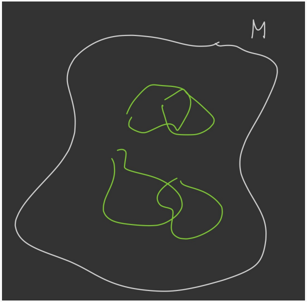
$M \in \mathcal{K}$

$\Updownarrow$

$M \models \varphi_{\beth_2(\kappa)^{++}+2,\, 0}$



Key Idea



Inside M (because of the sentences $\varphi_{\alpha,0}$ it satisfies), there are "densely" many models of size κ, from the class $\mathcal{K}$.

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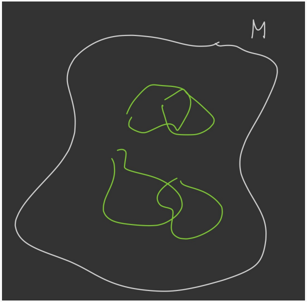

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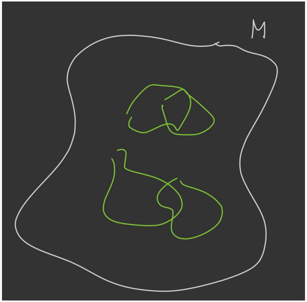

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However…, as $M \models \varphi_{\beth_2(\kappa)^+ + 2,0}$… the system will also turn out to be a $\prec_{\mathcal{K}}$-directed system!

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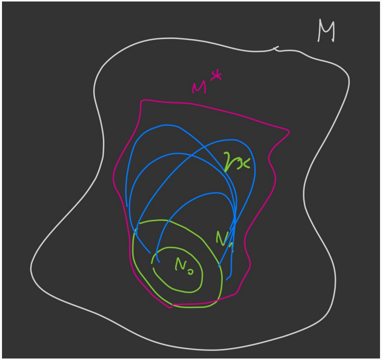

Two combinatorial arguments:

- In 2020, using Komjáth-Shelah's partition relation for well-founded trees.
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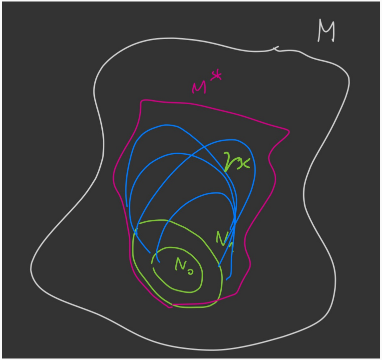
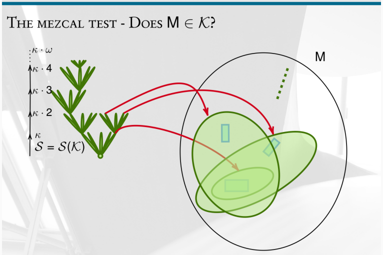


Two combinatorial arguments:

- In 2020, using Komjáth-Shelah's partition relation for well-founded trees.
- In 2021, we reduced complexity

Assuming $N_0 \not\prec_{\mathcal{K}} N_1$, using the tree $\mathcal{S}_{\mathcal{K}}$ and the fact that $M \models \varphi_{\alpha,0}$, we build a **tree of models** converging to the same model - by the axioms of AEC's we may conclude that $N_0 \prec_{\mathcal{K}} N_1$!

Steps:



- Build the tree $\mathcal{S}_{\mathcal{K}}$ (ω levels $\kappa \cdot$ n, n $< \omega$)

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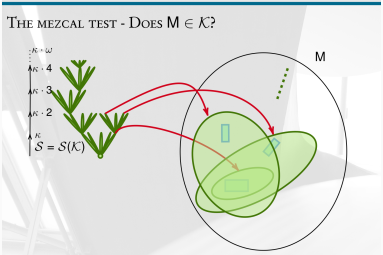

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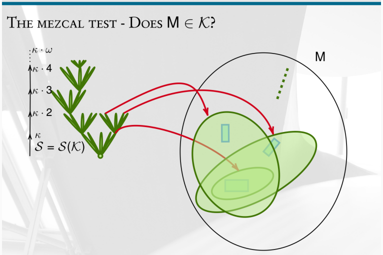
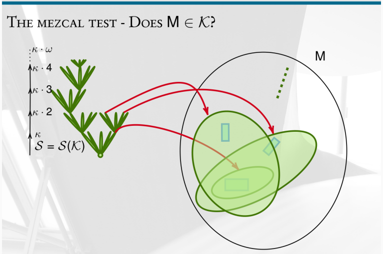


- Build the tree $\mathcal{S}_{\mathcal{K}}$ (ω levels $\kappa \cdot$ n, n $< \omega$)
- Build sentences $\varphi_{0,0}, \varphi_{1,0}, \dots, \varphi_{\alpha,0}, \dots$ capturing ever more "history" of embeddings
- M $\models \varphi_{\alpha,0}$ for α "high enough" implies (by very non-trivial combinatorics) that M is a $\prec_{\mathcal{K}}$-direct limit of small models from the class $\mathcal{K}$!

Leung's strategy:



Leung's strategy has similarities, but he replaces the combinatorics by the game quantifier

$$\forall x_0 \exists y_0 \forall x_1 \exists y_1 \ldots \forall x_i \exists y_i \ldots$$

of length $\omega \cdot \omega$.

New Issues:

- The axiomatization shows new aspects of the AEC $\mathcal{K}$, such as:
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- The axiomatization shows new aspects of the AEC $\mathcal{K}$, such as:
- Well-tuned complexity of $\mathcal{K}$,
- Connections with categoricity and stability (NIP),
- Logical properties controlling $\psi_{\mathcal{K}}$,
- Behaviour of $\prod_{i \in I} \mathfrak{A}_i / \mathcal{U}$ in terms of the logic,
- Bi-interpretability in AECs (Galois theory),
- $\mathcal{K}$'s behaviour in forcing extensions.

The sentence axiomatizing an AEC $\mathcal{K}$ with Löwenheim-Skolem-Tarski number κ has been reduced to the logic $\mathsf{L}_{(2^\kappa)^+,\kappa^+}$ (I am thankful to Jouko for a clarification of the reduction!).

On improving the bounds

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The same proof yields more, as we shall see:

A syntactic "Tarski-Vaught test" for AECs,
Definability of types (with Nájar)

Theorem

Let $\mathcal{K}$ be an a.e.c., $\tau = \tau(\mathcal{K}) \leq \kappa = \mathrm{LST}(\mathcal{K})$, $\lambda = \beth_2(\kappa)^{++}$. Then, given τ-models $M_1 \subseteq M_2$, the following are equivalent:

1. $M_1 \prec_{\mathcal{K}} M_2$
2. *if* $\bar{a} \in {}^{\kappa\geq}(M_1)$ *then there are* $\bar{b}$, N *and* f *such that*
    - 2.1 $\bar{b} \in {}^{\kappa\geq}(M_1)$ *and* $N \in \mathcal{S}_1$
    - 2.2 $\mathrm{Rang}(\bar{a}) \subseteq \mathrm{Rang}(\bar{b})$
    - 2.3 f *is an isomorphism from* N *onto* $M_1 \restriction \mathrm{Rang}(\bar{b})$
    - 2.4 $M_2 \models \varphi_{N,\lambda+1,1}[\langle f(a^*_\alpha) \mid \alpha < \kappa \rangle]$.

The semantic-syntactic correspondence for types

Recently, N. Nájar has started a path toward using our sentences to recapture types syntactically, building on earlier work by Vasey:

[1] See 2.3.15 in Infinitary Stability Theory. Arch.Math.Log., 55(3), 567-592.

The semantic-syntactic correspondence for types

Recently, N. Nájar has started a path toward using our sentences to recapture types syntactically, building on earlier work by Vasey:

Theorem (The semantic-syntactic correspondence[1])
Let $\mathcal{K}$ be an AEC with AP, JEP and MMN. The following are equivalent.

1. $\mathcal{K}$ *is* $\mathrm{LS}(\mathcal{K})^+$*-tame and* $\mathrm{LS}(\mathcal{K})^+$*-type short.*
2. *The application* $\mathrm{p} \mapsto \mathrm{p}^{\mathrm{s}}$ *is a bijection,*

where

Definition
Let $\mathrm{N}, \mathrm{M} \in \mathcal{K}$ be such that $\mathrm{M} \prec_{\mathcal{K}} \mathrm{N}$ and let $\bar{\mathrm{a}} \in |\mathrm{N}|$ $\mathrm{p} = \text{ga-tp}(\bar{\mathrm{b}})$.
If $\mathrm{p} = \text{ga-tp}(\bar{\mathrm{a}}/\mathrm{M}; \mathrm{n})$, then we define $\mathrm{p}^{\mathrm{s}} := \text{qf-tp}(\bar{\mathrm{a}}/\mathrm{M}; \mathrm{N})$.

[1] See 2.3.15 in Infinitary Stability Theory. Arch.Math.Log., 55(3), 567-592.

The Internal Logic of an AEC

A natural project: finding the internal logic of an AEC. On the face of it, it would seem that an AEC is about a generalized sentence, not about a logic per se. However, the fact they support so many constructions from stability theory (towers of models, structural control by [Galois] types, type omission, minimal pairs, stability spectrum, canonical forking notions for stable AECs, group configuration, etc.) raises the question of finding the natural internal logic of the AEC.

We have now embarked on this large scale project.

Two Internal Logics of an AEC

$$\mathbb{L}_{\mathcal{K}}^{1,\mathsf{aec}} < \mathbb{L}_{\mathcal{K}}^{2,\mathsf{aec}}$$

The two logics

$$\mathbb{L}^{1,\mathsf{aec}}_{\mathcal{K}} < \mathbb{L}^{2,\mathsf{aec}}_{\mathcal{K}}$$

$\psi_{\mathcal{K}} \in \mathbb{L}^{1,\mathsf{aec}}_{\mathcal{K}}$,
fragment of $\mathbb{L}_{(2^\kappa)^+,\kappa^+}$ containing $\psi_{\mathcal{K}}$ (Shelah-V. 2021)

$\psi_{\mathcal{K}} \in \mathbb{L}^{2,\mathsf{aec}}_{\mathcal{K}}$,
second order interpretability of $\mathcal{K}$ (Shelah-V. in progress)

We close $\mathbb{L}_{(2^\kappa)^+,\omega}$ under $\forall x$, $\exists x$, $\bigwedge_{i<2^\kappa} \psi_i$, $\neg$ and $\psi_{\mathcal{K}}$.

This can very easily define well-ordering!

("Non-well orders" form an AEC, of very low "Scott rank", in a natural way!)

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Other possibilities:

- Removing $\neg$ from $\mathbb{L}^{1,\mathsf{aec}}_{\mathcal{K}}$?
- Comparing/adapting $\mathbb{L}^{1}_{\kappa}$?
- Developing stability theory for $\mathbb{L}^{1}_{\kappa}$?
- Transfer stability theory to $\mathbb{L}^{1,\mathsf{aec}}_{\mathcal{K}}$?
- Omitting Types for these logics ?

Plan

Definability issues (I): sentences axiomatizing AECs

Why so much stability theory in AECs?

Axiomatizing AECs: attempts old and new

On the Internal Logic of an AEC

Definability issues (II): patterns for AECs?

The pattern language: constructions and obstructions

On the existence of “cores”

AECs with good closures

Definability patterns: some features

Around 2019, Hrushovski starts work on "definability patterns" (for first order theories T, or slightly more general contexts).

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- Finding models for this theory, and proving their canonicity,
- And going in many directions from here (Galois/Lascar, Ramsey, etc.)

The pattern language: first obstruction

(Simplifying to one sort, although the real power of the pattern language becomes visible when allowing different sorts to interact!), given $M \models T$, $\mathcal{L}$ consists of predicates Def_t, $t = (\varphi_1, \ldots, \varphi_n; \alpha)$, interpreted in $S = S(M)$ as

$$\mathrm{Def}_t^S = \{(p_1, \ldots p_n) : \forall a \in \alpha(M) \bigvee_{1 \leq i \leq n} (\varphi_i(x, a) \in p_i)\}.$$

For $n = 1$, the predicate $\mathrm{Def}_{\varphi;\alpha}$ captures those 1-types of T for which α acts as a (partial) definition scheme for φ.

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First obstruction: Which formulas to use for definitions???

Possible workarounds

The pattern theory $\mathcal{T}$ of T is the set of all (local) primitive universal $\mathcal{L}$-sentences true in S(M) for some $M \models T$.

Galois-types have very good behaviour in AECs... However, the collection of all Galois-types is not necessarily well-equipped with a "standard" topology!

Definability (of types) has been treated (by Shelah and others) in a weak, abstract way in AECs through non-splitting. Shelah even calls non-splitting extensions in the NIP theories context weakly definable types.

We may use sentences of the logics $\mathbb{L}^{1,\mathrm{aec}}$, to test syntactic definability patterns to build $\mathcal{L}$. This has strong shortcomings, but ...

Galois Morleyizations

Vasey in 2016 introduced "Galois Morleyizations" for AECs. Essentially, expanding L by adding predicates for all Galois types (orbits). He proved under "tameness" assumptions that part of the content of an AEC $\mathcal{K}$ may be read functorially from a SYNTACTIC counterpart of the AEC $\mathcal{K}$. In particular, stable AECs have canonical forking relations defined both semantically and syntactically. So far, there is (as far as I have seen) no study of definability of types in that context. But that should enter the picture...

Abstract cores

A core for T is an $\mathcal{L}$-structure $\mathcal{J}$ such that

- For any (orbit-bounded) $M \models T$, there is an $\mathcal{L}$-embedding

$$j : \mathcal{J} \to S(M)..$$

- For any j as above, there is a retraction $r : S(M) \to \mathcal{J}$ such that $r \circ j = Id_{\mathcal{J}}$.

Cores exist, are unique up to isomorphism. Aut($\mathcal{J}$) has a natural locally compact topology (basic closed sets of the form $W(R : a, b) = \{g : Def_t(ga_1, \ldots, ga_n, b_1, \ldots, b_m)\}$

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Calibrating the existence of such cores for additional contexts is doable: choice of logic or plain selection of predicates behaving as if coming from a concrete definability pattern.

The core of j?

Example: the core of j (the j-mapping), as axiomatized by Boris Zilber and Adam Harris in $\mathbb{L}_{\omega_1,\omega}$

$$((\mathbb{H},\sigma)_{\sigma\in\Gamma}, \mathrm{j}, (\mathbb{C},+,\cdot,0,1))$$

is an interesting case for study (here, the quasiminimality of the structure, plus the axiomatization in $\mathbb{L}_{\omega_1,\omega}$ are key).

Galois groups

AECs with good closures cl (generalizing acl but also quasiminimality) have been studied, with various topologies (joint work with Ghadernezhad). We have proved a version of the small index property for these (generalizing a result of Lascar and Shelah); the SIP is a key step toward a construction of the Lascar group in FO theories.

Thank you for your attention!

There is a Proustian view of the world. It is defined, first and foremost, by what it excludes: neither brute matter nor wilful spirit. Neither physics nor philosophy. Philosophy presupposes direct statements and explicit meanings, stemming from a truth-seeking spirit. Physics supposes an objective and unambiguous matter, subject to the conditions of reality. We go wrong when we believe in facts; only signs exist. We go wrong when we believe in truth; only interpretations exist. The sign is an always equivocal, implicit and implied meaning. "Throughout my existence, I had followed a path inverse to those of those peoples who only after having considered characters as mere successions of symbols start using them as a phonetic writing." [La prisonnière, Proust]

G. Deleuze - Proust and signs