On the Internal Logic of an Abstract Elementary Class

Andrés Villaveces

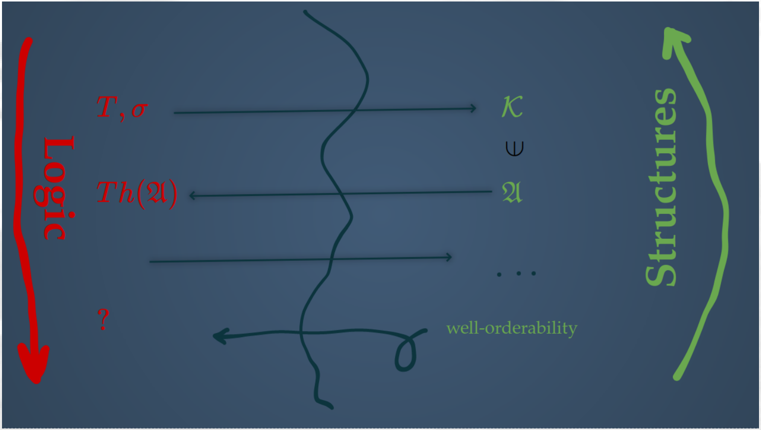
Helsinki Logic Seminar - February '22

Universidad Nacional de Colombia / Bogotá

Contents

Axiomatizing the un-axiomatized…

And studying limitations to possible axiomatizations



Given a model class $\mathcal{K}$

(given as some amalgamation class, or some AEC, or a Fraïssé class, or a Ramsey class, or a Hrushovski-Zilber approximation system...

the question of its definability in some logic may be instrumental...

Or. . .

> *If the Greeks were so attached to geometry, wasn't it that they thought by tracing lines, with no words? However (or maybe just because of that?) [they produced] a perfect axiomatic! Euclid's Postulates, construction. Limiting what one is allowed to trace.*

Simone Weil, Cahier III

Plan

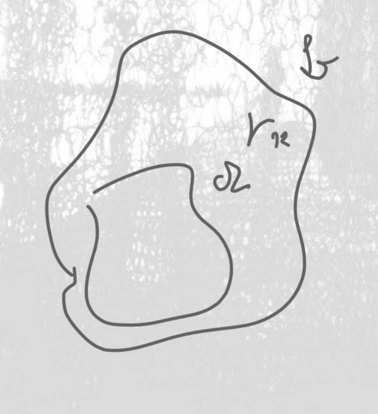
Axiomatizing the un-axiomatized

Why so much stability theory in AECs?

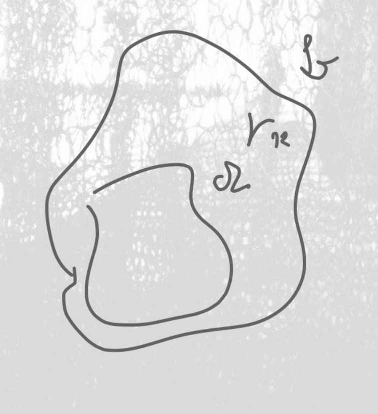
Axiomatizing AECs: attempts old and new

On the Internal Logic of an AEC

AECs: why so much stability theory?



In AECs, we replace from the outset the initial extreme emphasis on φ, T, compactness



In AECs, we replace from the outset the initial extreme emphasis on φ, T, compactness by more **semantical** notions:

$\prec_{\mathcal{K}}$, f a morphism,

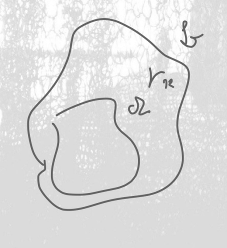
$f \in \mathrm{Aut}(\mathbb{C})$, etc.

φ

T

$T_0 \subseteq^{\text{fin}} T$

$\vdots$



φ

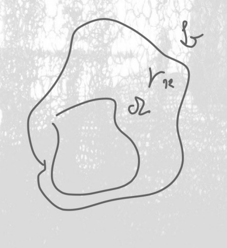
T

$T_0 \subseteq^{\text{fin}} T$

$\vdots$

emphasis shift

towards 1980



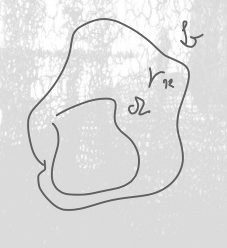
φ

T

$T_0 \subseteq^{\text{fin}} T$

: Instead of extracting $\prec$, f, etc. from T, φ, we turn $\prec$, f a strong embedding into the primitive notions!

emphasis shift towards 1980



φ

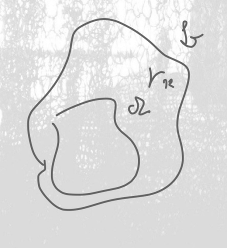
T

$T_0 \subseteq^{fin} T$

: Instead of extracting $\prec$, f, etc. from T, φ, we turn $\prec$, f a strong embedding into the primitive notions!

emphasis shift
towards 1980

subgroup
subring
pure subring
strong substructure



φ

T

$T_0 \subseteq^{\text{fin}} T$

: Instead of extracting $\prec$, f, etc. from T, φ, we turn $\prec$, f a strong embedding into the primitive notions!

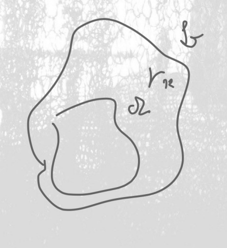
emphasis shift towards 1980

subgroup

subring

pure subring

strong substructure



$\mathfrak{A} \prec_{\mathcal{K}} \mathfrak{B}$

"perfect" extension, algebraically closed, etc.

AEC - the axioms, briefly

Fix $\mathcal{K}$ be a class of τ-structures, $\prec_{\mathcal{K}}$ a binary relation on $\mathcal{K}$.

Definition

$(\mathcal{K}, \prec_{\mathcal{K}})$ is an abstract elementary class iff

- $\mathcal{K}$, $\prec_{\mathcal{K}}$ are **closed under isomorphism**,
- $M, N \in \mathcal{K}$, $M \prec_{\mathcal{K}} N \Rightarrow M \subset N$,
- $\prec_{\mathcal{K}}$ is a partial order,
- (**TV**) $M \subset N \prec_{\mathcal{K}} \bar{N}, M \prec_{\mathcal{K}} \bar{N} \Rightarrow M \prec_{\mathcal{K}} N$,
- ($\searrow$**LS**) There is some $\kappa = LS(\mathcal{K}) \geq \aleph_0$ such that for every $M \in \mathcal{K}$, for every $A \subset |M|$, there is $N \prec_{\mathcal{K}} M$ with $A \subset |N|$ and $\|N\| \leq |A| + LS(\mathcal{K})$,
- (**Unions of $\prec_{\mathcal{K}}$-chains**) A union of an arbitrary $\prec_{\mathcal{K}}$-chain in $\mathcal{K}$ belongs to $\mathcal{K}$, is a $\prec_{\mathcal{K}}$-extension of all models in the chain and is the sup of the chain.

And really, a lot of examples (and model theory)

Natural constructions in Mathematics are examples of AEC (or metric AEC)

1. Complete first order theories
2. Various classes axiomatizable in $L_{\omega_1,\omega}$ or $L_{\kappa\omega}$.

And really, a lot of examples (and model theory)

Natural constructions in Mathematics are examples of AEC (or metric AEC)

1. Complete first order theories
2. Various classes axiomatizable in $L_{\omega_1,\omega}$ or $L_{\kappa\omega}$.
3. Metric AEC - stability theory started by Hirvonen and Hyttinen, Usvyatsov, and continued by Zambrano and V.
4. Metric AECs and connections with operator algebras (Hirvonen, Hyttinen)
5. Model Theory of Modules (Mazari-Armida)

And really, a lot of examples (and model theory)

Natural constructions in Mathematics are examples of AEC (or metric AEC)

1. Complete first order theories
2. Various classes axiomatizable in $L_{\omega_1,\omega}$ or $L_{\kappa\omega}$.
3. Metric AEC - stability theory started by Hirvonen and Hyttinen, Usvyatsov, and continued by Zambrano and V.
4. Metric AECs and connections with operator algebras (Hirvonen, Hyttinen)
5. Model Theory of Modules (Mazari-Armida)
6. AECs of C^*-algebras (Argoty, Berenstein, V.)
7. Zilber analytic classes (pseudoexponentiation)
8. Classes of ACVF?

And quite a bit of stability theory

Categoricity Transfer

Superstability

Stability (canonical forking)

Simplicity for some AECs

NTP_2 classes? (In process!)

Plan

Axiomatizing the un-axiomatized

Why so much stability theory in AECs?

Axiomatizing AECs: attempts old and new

On the Internal Logic of an AEC

Axiomatizing an AEC: attempts (old and new)

- Shelah, V. 2020,
- Leung 2021.

Fix $(\mathcal{K}, \prec_{\mathcal{K}})$ an AEC with $\mathrm{LS}(\mathcal{K}) = \kappa$. We also assume all models in $\mathcal{K}$ are of cardinality $\geq \kappa$.

Earlier results:

- **Shelah's Presentation Theorem:** $\mathcal{K}$ is $\mathrm{PC}_{\kappa,2^\kappa}$.

Axiomatizing an AEC: attempts (old and new)

- Shelah, V. 2020,
- Leung 2021.

Fix $(\mathcal{K}, \prec_{\mathcal{K}})$ an AEC with $\mathrm{LS}(\mathcal{K}) = \kappa$. We also assume all models in $\mathcal{K}$ are of cardinality $\geq \kappa$.

Earlier results:

- **Shelah's Presentation Theorem:** $\mathcal{K}$ is $\mathrm{PC}_{\kappa,2^\kappa}$. There is $\mathrm{L}' \supset \mathrm{L}$, and there is ψ a sentence in $\mathbb{L}_{2^{\kappa+},\omega}$, $|\mathrm{L}'| \leq 2^\kappa$ such that $\mathcal{K} = \{\mathrm{M} \restriction \mathrm{L} : \mathrm{M} \models \psi\}$,

Axiomatizing an AEC: attempts (old and new)

- Shelah, V. 2020,
- Leung 2021.

Fix $(\mathcal{K}, \prec_{\mathcal{K}})$ an AEC with $\mathsf{LS}(\mathcal{K}) = \kappa$. We also assume all models in $\mathcal{K}$ are of cardinality $\geq \kappa$.

Earlier results:

- **Shelah's Presentation Theorem:** $\mathcal{K}$ is $\mathsf{PC}_{\kappa, 2^\kappa}$. There is $\mathsf{L}' \supset \mathsf{L}$, and there is ψ a sentence in $\mathbb{L}_{{2^\kappa}^+, \omega}$, $|\mathsf{L}'| \leq 2^\kappa$ such that $\mathcal{K} = \{\mathsf{M} \restriction \mathsf{L} : \mathsf{M} \models \psi\}$,
- **Shelah-Vasey:** If $\mathsf{LS}(\mathcal{K}) = \aleph_0$, $\mathcal{K}$ is $\aleph_0$-stable and has the $\aleph_0$-AP, and $\mathsf{I}(\aleph_0, \mathcal{K}) \leq \aleph_0$ <u>then</u> $\mathcal{K}$ is $\mathsf{PC}_{\aleph_0}$.

More

- **Kueker:** if $\mathcal{K}$ is closed under $\equiv_{\infty,\omega_1}$-equivalence, L is countable, then there is an $L_{\infty,\omega}$-sentence axiomatizing $\mathcal{K}$,

More

- **Kueker:** if $\mathcal{K}$ is closed under $\equiv_{\infty,\omega_1}$-equivalence, L is countable, then there is an $\mathsf{L}_{\infty,\omega}$-sentence axiomatizing $\mathcal{K}$,
- **Shelah, V.** (2020): For arbitrary AECs $\mathcal{K}$, we provide a sentence in $\mathbb{L}_{\beth_2(\kappa)^{+3},\kappa^+}$, in the same vocabulary L of $\mathcal{K}$, axiomatizing $\mathcal{K}$,

More

- **Kueker:** if $\mathcal{K}$ is closed under $\equiv_{\infty,\omega_1}$-equivalence, L is countable, then there is an $L_{\infty,\omega}$-sentence axiomatizing $\mathcal{K}$,
- **Shelah, V.** (2020): For arbitrary AECs $\mathcal{K}$, we provide a sentence in $\mathbb{L}_{\beth_2(\kappa)^{+3},\kappa^+}$, in the same vocabulary L of $\mathcal{K}$, axiomatizing $\mathcal{K}$,
- **Leung** (2021): For arbitrary AECss $\mathcal{K}$, a sentence in $\mathbb{L}_{(2^\kappa)^+),\kappa^+}(\omega \cdot \omega)$, in the same vocabulary L of $\mathcal{K}$ but with a "game quantifier", axiomatizing $\mathcal{K}$,

More

- **Kueker:** if $\mathcal{K}$ is closed under $\equiv_{\infty,\omega_1}$-equivalence, L is countable, then there is an $\mathsf{L}_{\infty,\omega}$-sentence axiomatizing $\mathcal{K}$,
- **Shelah, V.** (2020): For arbitrary AECs $\mathcal{K}$, we provide a sentence in $\mathbb{L}_{\beth_2(\kappa)^{+3},\kappa^+}$, in the same vocabulary L of $\mathcal{K}$, axiomatizing $\mathcal{K}$,
- **Leung** (2021): For arbitrary AECss $\mathcal{K}$, a sentence in $\mathbb{L}_{(2^\kappa)^+),\kappa^+}(\omega \cdot \omega)$, in the same vocabulary L of $\mathcal{K}$ but with a "game quantifier", axiomatizing $\mathcal{K}$,
- **Shelah, V.** (in progress). A better bound: we reduce the complexity of the sentence to $\mathbb{L}_{(2^\kappa)^+,\kappa^+}$, in the original vocabulary!

2020

Shelah-V.

$\mathcal{K} = \mathsf{Mod}\left(\psi_{\mathcal{K}}\right)$

$\psi_{\mathcal{K}} \in \mathbb{L}_{\beth_2(\kappa)^{+3},\kappa^+}$

in vocabulary L

2021

Leung

$\mathcal{K} = \mathsf{Mod}\left(\psi_{\mathsf{Leung}}\right)$

$\psi_{\mathsf{Leung}} \in \mathbb{L}_{(2^\kappa)^+,\kappa^+}(\omega \cdot \omega)$

in vocabulary L

(The $\omega \cdot \omega$ refers to quantification of an EF game of length $\omega \cdot \omega$)

2020

Shelah-V.

$\mathcal{K} = \mathrm{Mod}\,(\psi_{\mathcal{K}})$

$\psi_{\mathcal{K}} \in \mathbb{L}_{\beth_2(\kappa)^{+3},\kappa^+}$

in vocabulary L

2021

Leung

$\mathcal{K} = \mathrm{Mod}\,(\psi_{\mathrm{Leung}})$

$\psi_{\mathrm{Leung}} \in \mathbb{L}_{(2^\kappa)^+,\kappa^+}(\omega \cdot \omega)$

in vocabulary L

(The $\omega \cdot \omega$ refers to quantification of an EF game of length $\omega \cdot \omega$)

better logic,

2020

Shelah-V.

$\mathcal{K} = \text{Mod}(\psi_{\mathcal{K}})$

$\psi_{\mathcal{K}} \in \mathbb{L}_{\beth_2(\kappa)^{+3},\kappa^+}$

in vocabulary L

2021

Leung

$\mathcal{K} = \text{Mod}(\psi_{\text{Leung}})$

$\psi_{\text{Leung}} \in \mathbb{L}_{(2^\kappa)^+,\kappa^+}(\omega \cdot \omega)$

in vocabulary L

(The $\omega \cdot \omega$ refers to quantification of an EF game of length $\omega \cdot \omega$)

better logic,

In late 2021, better bound: in $\mathbb{L}_{(2^\kappa)^+,\kappa^+}$

better bound, but use of

$\forall x_0 \exists y_0 \ldots \forall x_i \exists x_i \ldots, \ i < \omega \cdot \omega$

2020

Shelah-V.

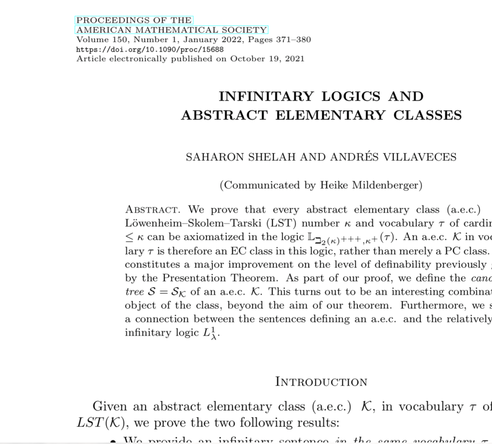
$\mathcal{K} = \text{Mod}(\psi_{\mathcal{K}})$

$\psi_{\mathcal{K}} \in \mathbb{L}_{\beth_2(\kappa)^{+3},\kappa^+}$

in vocabulary L

better logic,

In late 2021, better bound: in $\mathbb{L}_{(2^\kappa)^+,\kappa^+}$


PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 150, Number 1, January 2022, Pages 371–380
https://doi.org/10.1090/proc/15688
Article electronically published on October 19, 2021

INFINITARY LOGICS AND
ABSTRACT ELEMENTARY CLASSES

SAHARON SHELAH AND ANDRÉS VILLAVECES

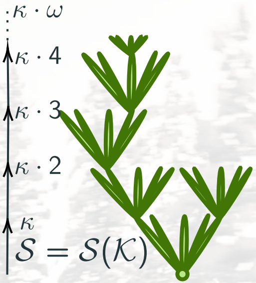
(Communicated by Heike Mildenberger)

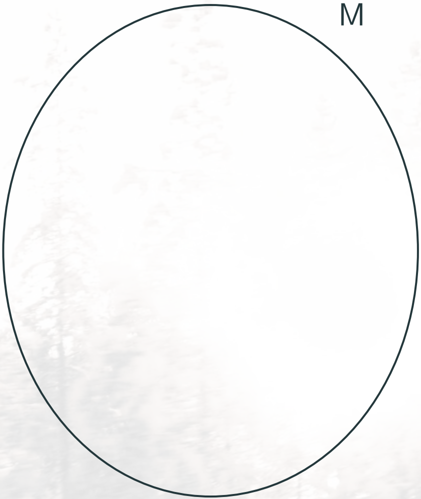
ABSTRACT. We prove that every abstract elementary class (a.e.c.) Löwenheim–Skolem–Tarski (LST) number κ and vocabulary τ of cardin $\leq \kappa$ can be axiomatized in the logic $\mathbb{L}_{\beth_2(\kappa)^{+++},\kappa^+}(\tau)$. An a.e.c. $\mathcal{K}$ in voc lary τ is therefore an EC class in this logic, rather than merely a PC class. constitutes a major improvement on the level of definability previously by the Presentation Theorem. As part of our proof, we define the *cano tree* $\mathcal{S} = \mathcal{S}_{\mathcal{K}}$ of an a.e.c. $\mathcal{K}$. This turns out to be an interesting combinat object of the class, beyond the aim of our theorem. Furthermore, we s a connection between the sentences defining an a.e.c. and the relatively infinitary logic L^1_λ.

INTRODUCTION

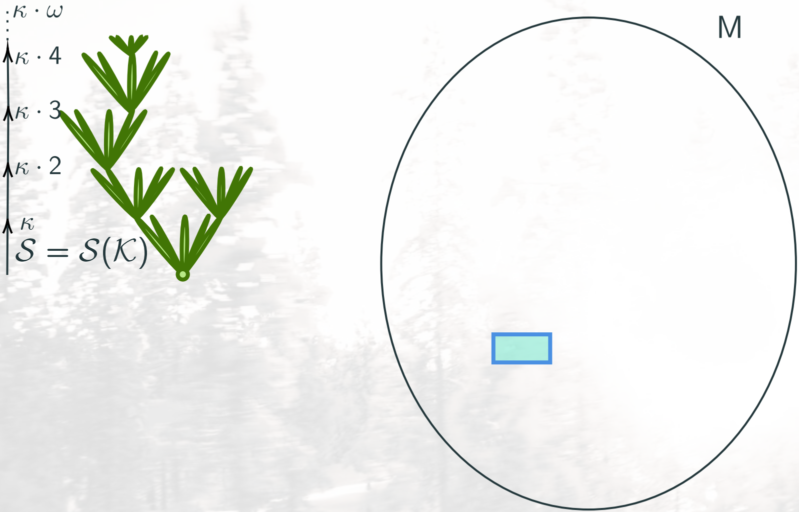
Given an abstract elementary class (a.e.c.) $\mathcal{K}$, in vocabulary τ of $LST(\mathcal{K})$, we prove the two following results:

The mezcal test - Does M $\in \mathcal{K}$?

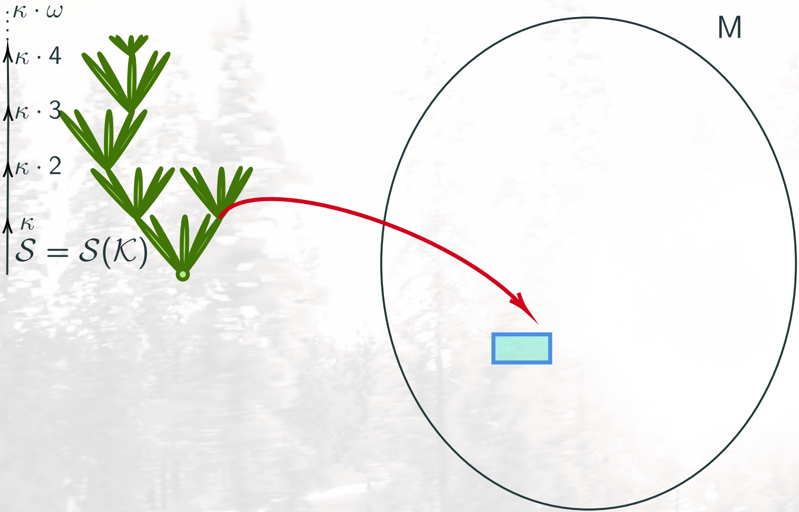




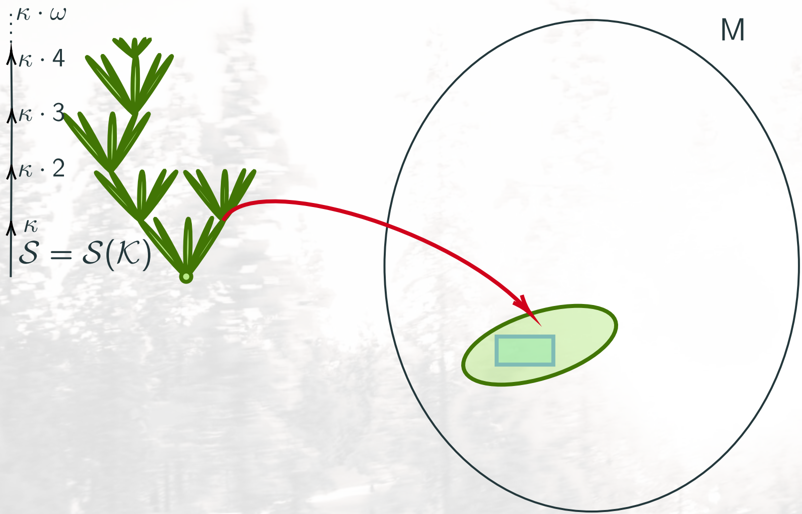
The mezcal test - Does M $\in \mathcal{K}$?



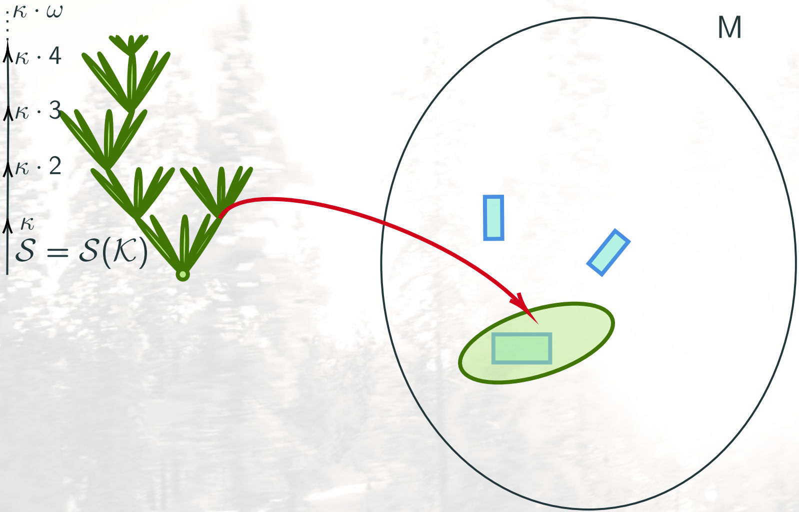
The mezcal test - Does M $\in \mathcal{K}$?



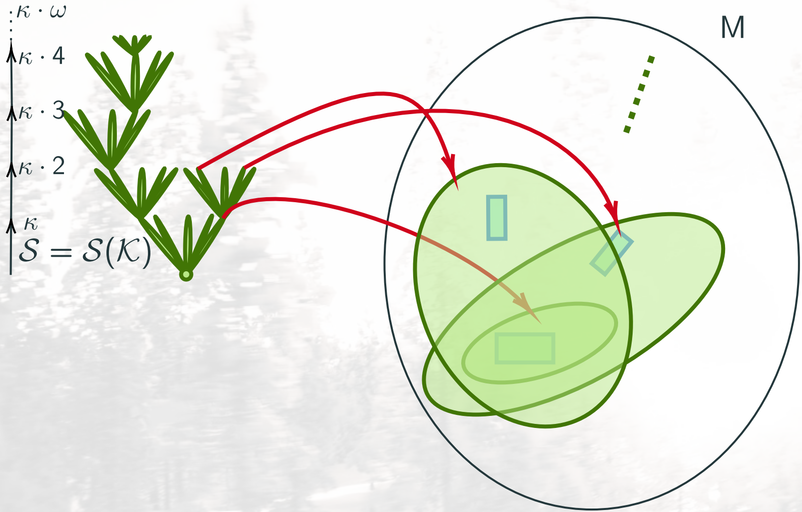
The mezcal test - Does M $\in \mathcal{K}$?

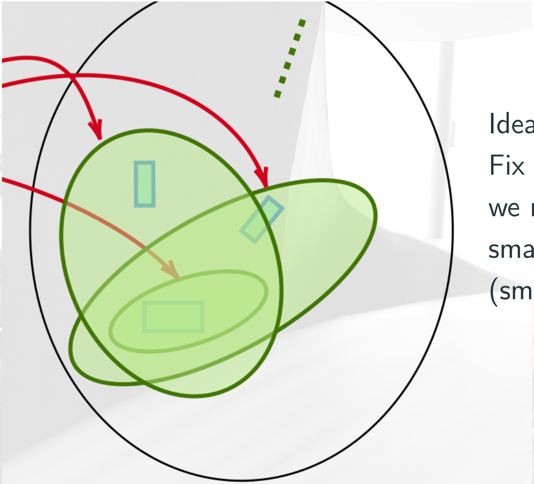


The mezcal test - Does M $\in \mathcal{K}$?



The mezcal test - Does $\mathsf{M} \in \mathcal{K}$?





Idea of our axiomatization:
Fix an L-structure M. How can we realize M as a direct limit of small models $N \in \mathcal{K}$?
(small = size $\kappa = LS(\mathcal{K})$)

Realizing an arbitrary model as a limit

$$M = \lim\{N \subseteq M | N \in \mathcal{K}\}???$$

(Of course, we need a lot of constraints!)

Towards this goal

We use the canonical tree of $\mathcal{K}$: models of size $\kappa = \mathsf{LS}(\mathcal{K})$, with universes

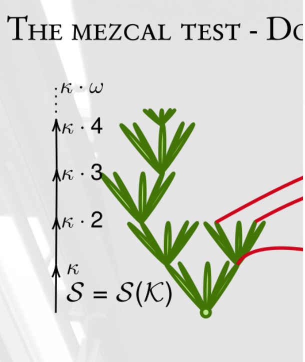
$$\kappa, \kappa + \kappa, \kappa + \kappa + \kappa, \ldots$$

and a whole "system of $\prec_{\mathcal{K}}$-elementary embeddings" between those models:

We use the canonical tree of $\mathcal{K}$: models of size $\kappa = \mathrm{LS}(\mathcal{K})$, with universes

$$\kappa, \kappa + \kappa, \kappa + \kappa + \kappa, \ldots$$

and a whole "system of $\prec_{\mathcal{K}}$-elementary embeddings" between those models:



$\mathcal{S}_{\mathcal{K}}$: the canonical tree of $\mathcal{K}$.

In $\mathcal{S}_{\mathcal{K}}$, $N_1 \triangleleft N_2$ iff $N_1 \prec_{\mathcal{K}} N_2$.

We now use syntax to...

...to "test" the model M **- the test membership in** $\mathcal{K}$

M **must "pass"** $\beth_2(\kappa)^{++} + 2$ **tests (in 2020), or just** $\alpha < (2^\kappa)^+$ **tests (in 2021)**

$I_2(\kappa)^{++}+2$ (2020) / $\alpha < (2^\kappa)^+$ (2021)

sentences, "approximating" $\mathcal{K}$:

$\varphi_{0,0} = \top$

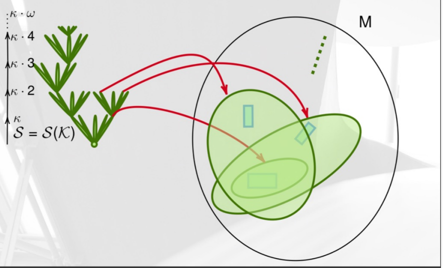
$\varphi_{1,0}$ iterate the "test" against the tree $\mathcal{S}_\alpha$

$\vdots$

$\varphi_{\alpha,0}$

$\vdots$

THE MEZCAL TEST - DOES $M \in \mathcal{K}$?



FORMULAS $\varphi_{M,\gamma,n}(\bar{x}_n)$

For M in the canonical tree $\mathcal{S}$ at level n, a formula with $\kappa \cdot n$ free variables, defined by induction on γ.

- $\gamma = 0$: $\varphi_{0,0} = \top$ ("truth"). If $n > 0$,

$$\varphi_{M,0,n} := \bigwedge \mathrm{Diag}^n_\kappa(M),$$

the atomic diagram of M in $\kappa \cdot n$ variables.

- γ limit: Then

$$\varphi_{M,\gamma,n}(\bar{x}_n) := \bigwedge_{\beta<\gamma} \varphi_{M,\beta,n}(\bar{x}_n).$$

- $\gamma = \beta + 1$: Then $\varphi_{M,\gamma,n}(\bar{x}_n)$ is the $L_{\lambda^+,\kappa^+}(\tau)$ formula

$$\forall \bar{z}_{[\kappa]} \bigvee_{\substack{N \succ_{\mathcal{K}} M \\ N \in \mathcal{S}_{n+1}}} \exists \bar{x}_{=n} \left[\varphi_{N,\beta,n+1}(\bar{x}_{n+1}) \wedge \bigwedge_{\alpha<\alpha_n[N]} \bigvee_{\delta \in S[N]} z_\alpha = x_\delta \right]$$

Formulas $\varphi_{M,\gamma,n}(\bar{x}_n)$

For M in the canonical tree $\mathcal{S}$ at level n, a formula with $\kappa \cdot n$ free variables, defined by induction on γ.

- $\gamma = 0$: $\varphi_{0,0} = \top$ ("truth"). If $n > 0$,

$$\varphi_{M,0,n} := \bigwedge \mathrm{Diag}^{n}_{\kappa}(M),$$

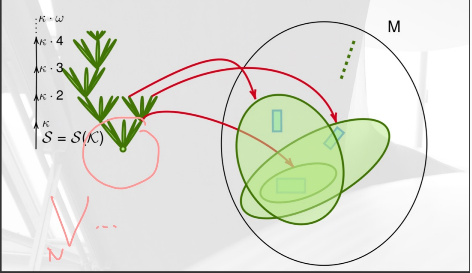
  the atomic diagram of M in $\kappa \cdot n$ variables.
- γ limit: Then

$$\varphi_{M,\gamma,n}(\bar{x}_n) := \bigwedge_{\beta<\gamma} \varphi_{M,\beta,n}(\bar{x}_n).$$

- $\gamma = \beta + 1$: Then $\varphi_{M,\gamma,n}(\bar{x}_n)$ is the $L_{\lambda^+,\kappa^+}(\tau)$ formula

$$\forall \bar{z}_{[\kappa]} \bigvee_{\substack{N \succ_{\mathcal{K}} M \\ N \in \mathcal{S}_{n+1}}} \exists \bar{x}_{=n} \left[\varphi_{N,\beta,n+1}(\bar{x}_{n+1}) \wedge \bigwedge_{\alpha<\alpha_n[N]} \bigvee_{\delta \in S[N]} z_\alpha = x_\delta \right]$$

The mezcal test - Does $M \in \mathcal{K}$?



$$\varphi_{1,0} : \forall z \bigvee_{N \in \mathcal{S}_1} \exists \bar{x}_1 \left[\varphi_{N,0,1}(\bar{x}_1) \wedge \bigwedge_{\alpha<\kappa} \bigvee_{\delta<\kappa} z_\alpha = x_\delta \right]$$

size κ

a copy of N

the copy covers z

Formulas $\varphi_{M,\gamma,n}(\bar{x}_n)$

For M in the canonical tree $\mathcal{S}$ at level n, a formula with $\kappa \cdot n$ free variables, defined by induction on γ.

- $\gamma = 0$: $\varphi_{0,0} = \top$ ("truth"). If $n > 0$,

$$\varphi_{M,0,n} := \bigwedge \mathrm{Diag}^{n}_{\kappa}(M),$$

the atomic diagram of M in $\kappa \cdot n$ variables.

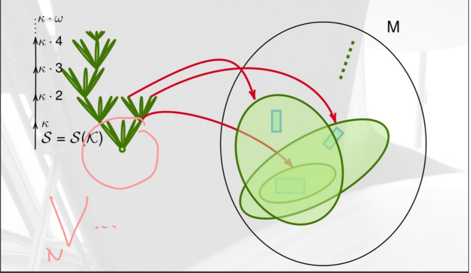
- γ limit: Then

$$\varphi_{M,\gamma,n}(\bar{x}_n) := \bigwedge_{\beta<\gamma} \varphi_{M,\beta,n}(\bar{x}_n).$$

- $\gamma = \beta + 1$: Then $\varphi_{M,\gamma,n}(\bar{x}_n)$ is the $L_{\lambda^+,\kappa^+}(\tau)$ formula

$$\forall \bar{z}_{[\kappa]} \bigvee_{\substack{N \succ_{\mathcal{K}} M \\ N \in \mathcal{S}_{n+1}}} \exists \bar{x}_{=n} \left[\varphi_{N,\beta,n+1}(\bar{x}_{n+1}) \wedge \bigwedge_{\alpha<\alpha_n[N]} \bigvee_{\delta \in S[N]} z_\alpha = x_\delta \right]$$

The mezcal test - Does $M \in \mathcal{K}$?



$$\varphi_{1,0}: \forall z \bigvee_{N \in \mathcal{S}_1} \exists \bar{x}_1 \left[\varphi_{N,0,1}(\bar{x}_1) \wedge \bigwedge_{\alpha<\kappa} \bigvee_{\delta<\kappa} z_\alpha = x_\delta \right]$$

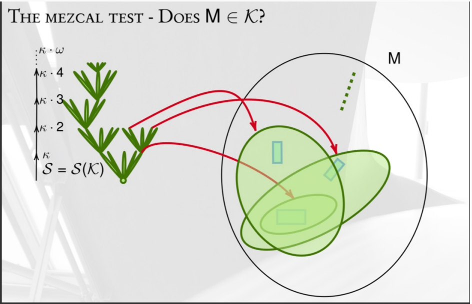
size κ "copy of N" covers z

$M \models \varphi_{1,0}$ if it may "be covered" by levels $N \in \mathcal{K}$, of size κ "

$$\varphi_{1,0}: \forall z \bigvee_{N \in S_1} \exists \bar{x}_1 \left[\varphi_{N,0,1}(\bar{x}_1) \wedge \bigwedge_{\alpha<\kappa} \bigvee_{\delta<\kappa} z_\alpha = x_\delta \right]$$

size κ — version of N — covers z

“ S_1 covers M ”

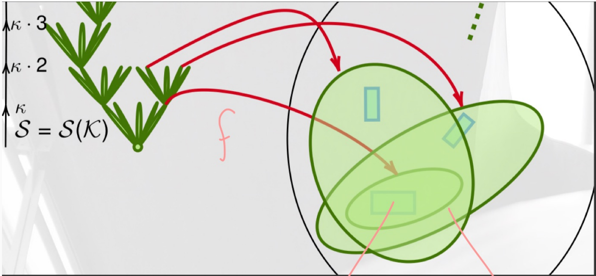


$$M \models \varphi_{2,0} = \forall z \bigvee_{N \in S_1} \exists \bar{x}_2 \left[\varphi_{N,1,2}(\bar{x}_2) \wedge \text{“} z \subseteq \bar{x}_2 \text{”} \right]$$

κ ?

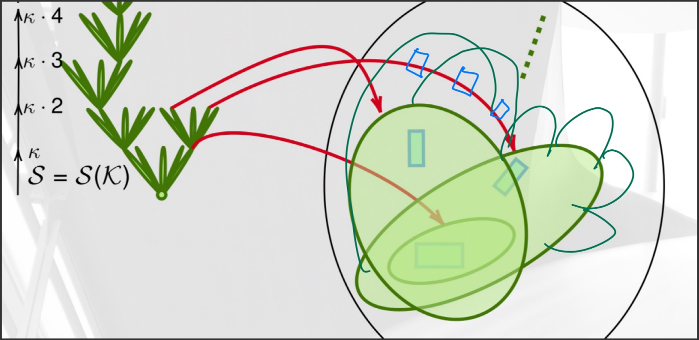
Covers and then covers

$$\forall z \bigvee_{N \in S_1} \exists \bar{x}_2 \left[\forall z' \bigvee_{\substack{N' > N \\ N' \in S_2}} \exists \bar{x}_1 \left((\bar{x}_2 {}^\frown \bar{x}_1) \wedge z' \subset \bar{x}_2 {}^\frown \bar{x}_1 \right) \wedge z \subset \bar{x}_2 \right]$$



Covering notions, refining ...

$$\varphi_{2,0} \begin{cases} \forall z \text{ some } N \in \mathcal{S}_1 \text{ covers } z \text{ (via } f) \\ \forall z' \text{ some } N' \in \mathcal{S}_2 \;\; N' \geq_\kappa N \text{ covers } z' \end{cases}$$



$\varphi_{3,0}$: better cover yet...

Problem: M is big!

As this way of covering may be insufficient, we iterate transfinitely:

$M \models \varphi_{\omega+1,0}$:

$\forall N \bigvee$ covers N BUT $M \models \varphi_{\omega,N,1}(\dots)$ → info of depth ω

$N \in \ell_1$

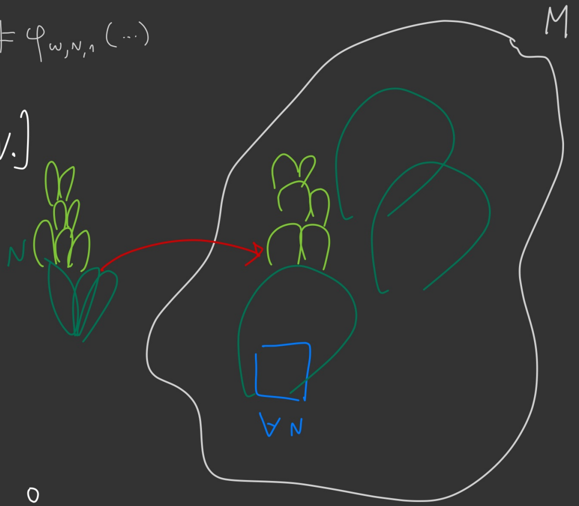
$\forall N \bigvee_{N \in \ell_1}$ cov. N But $M \models \varphi_{\omega,N,1}(\ldots)$

Theorem [Shelah, V.]

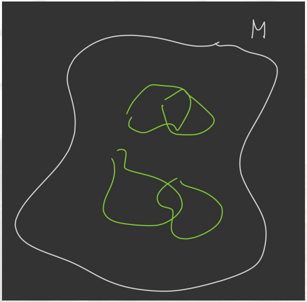
$M \in \mathcal{K}$

$\Updownarrow$

$M \models \varphi_{\beth_2(\kappa)^{++}+2,\,0}$

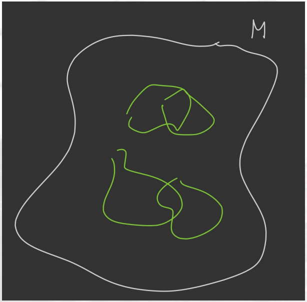


Key Idea



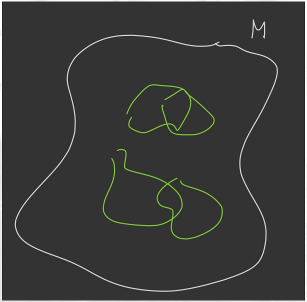
Inside M (because of the sentences $\varphi_{\alpha,0}$ it satisfies), there are “densely” many models of size κ, from the class $\mathcal{K}$.

Key Idea



Inside M (because of the sentences $\varphi_{\alpha,0}$ it satisfies), there are "densely" many models of size κ, from the class $\mathcal{K}$.
These form a $\subseteq$-directed system (again, the sentences. . .).
Now, this per se is extremely weak to guarantee that $M \in \mathcal{K}$.

Key Idea



Inside M (because of the sentences $\varphi_{\alpha,0}$ it satisfies), there are "densely" many models of size κ, from the class $\mathcal{K}$.

These form a $\subseteq$-directed system (again, the sentences. . .).

Now, this per se is extremely weak to guarantee that $M \in \mathcal{K}$.

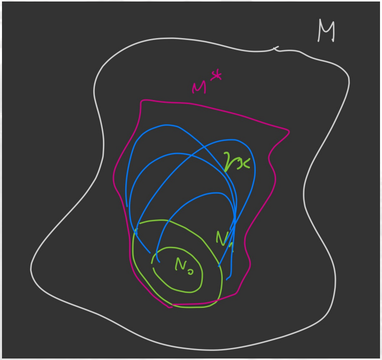
However. . . , as $M \models \varphi_{\beth_2(\kappa)^+ +2,0}$. . . the system will also turn out to be a $\prec_{\mathcal{K}}$-directed system!

Why directed?

However. . . , as $M \models \varphi_{\beth_2(\kappa)^+ + 2, 0}$ . . . the system will also turn out to be a $\prec_{\mathcal{K}}$-directed system!

Why directed?

However..., as $M \models \varphi_{\beth_2(\kappa)^+ + 2, 0} \ldots$ the system will also turn out to be a $\prec_{\mathcal{K}}$-directed system!

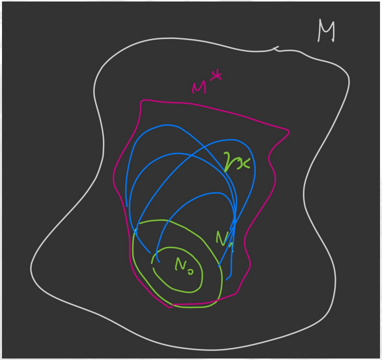


Two combinatorial arguments:

- In 2020, using Komjáth-Shelah's partition relation for well-founded trees.
- In 2021, we reduced complexity

Why directed?

However. . . , as $M \models \varphi_{\beth_2(\kappa)^{+}+2,0}$. . . the system will also turn out to be a $\prec_{\mathcal{K}}$-directed system!

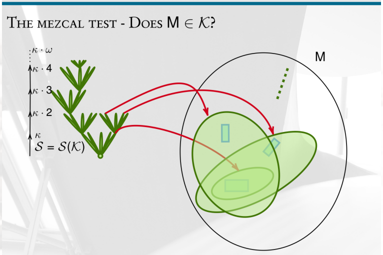


Two combinatorial arguments:

- In 2020, using Komjáth-Shelah's partition relation for well-founded trees.
- In 2021, we reduced complexity

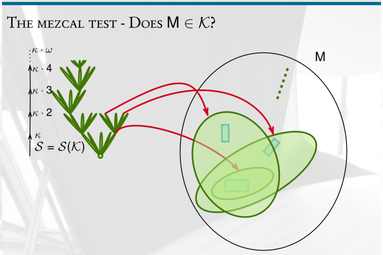
Assuming $N_0 \not\prec_{\mathcal{K}} N_1$, using the tree $\mathcal{S}_{\mathcal{K}}$ and the fact that $M \models \varphi_{\alpha,0}$, we build a **tree of models** converging to the same model - by the axioms of AEC's we may conclude that $N_0 \prec_{\mathcal{K}} N_1$!

Steps:



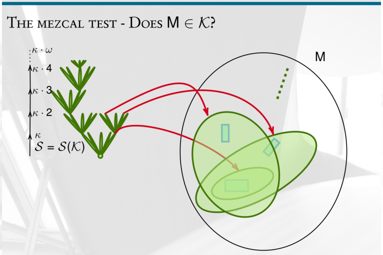
- Build the tree $\mathcal{S}_{\mathcal{K}}$ (ω levels $\kappa \cdot$ n, n $< \omega$)

Steps:



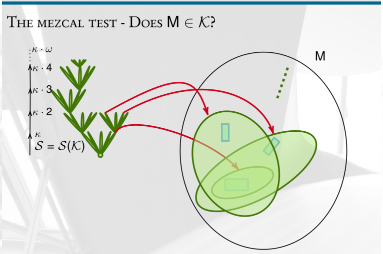
- Build the tree $\mathcal{S}_{\mathcal{K}}$ (ω levels $\kappa \cdot$ n, n $< \omega$)
- Build sentences $\varphi_{0,0}, \varphi_{1,0}, \dots, \varphi_{\alpha,0}, \dots$ capturing ever more "history" of embeddings

Steps:



- Build the tree $\mathcal{S}_{\mathcal{K}}$ (ω levels $\kappa \cdot \mathrm{n}$, $\mathrm{n} < \omega$)
- Build sentences $\varphi_{0,0}, \varphi_{1,0}, \ldots, \varphi_{\alpha,0}, \ldots$ capturing ever more “history” of embeddings
- $\mathrm{M} \models \varphi_{\alpha,0}$ for α “high enough” implies (by very non-trivial combinatorics) that M is a $\prec_{\mathcal{K}}$-direct limit of small models from the class $\mathcal{K}$!

Leung's strategy:



Leung's strategy has similarities, but he replaces the combinatorics by the game quantifier

$$\forall x_0 \exists y_0 \forall x_1 \exists y_1 \ldots \forall x_i \exists y_i \ldots$$

of length $\omega \cdot \omega$.

New Issues:

- The axiomatization shows new aspects of the AEC $\mathcal{K}$, such as:
- Well-tuned complexity of $\mathcal{K}$,
- Connections with categoricity and stability (NIP),

New Issues:

- The axiomatization shows new aspects of the AEC $\mathcal{K}$, such as:
- Well-tuned complexity of $\mathcal{K}$,
- Connections with categoricity and stability (NIP),
- Logical properties controlling $\psi_{\mathcal{K}}$,
- Behaviour of $\prod_{i \in I} \mathfrak{A}_i / \mathcal{U}$ in terms of the logic,
- Bi-interpretability in AECs (Galois theory),
- $\mathcal{K}$'s behaviour in forcing extensions.

Plan

Axiomatizing the un-axiomatized

Why so much stability theory in AECs?

Axiomatizing AECs: attempts old and new

On the Internal Logic of an AEC

The Internal Logic of an AEC

A natural project: finding the internal logic of an AEC. On the face of it, it would seem that an AEC is about a generalized sentence, not about a logic per se. However, the fact they support so many constructions from stability theory (towers of models, structural control by [Galois] types, type omission, minimal pairs, stability spectrum, canonical forking notions for stable AECs, group configuration, etc.) raises the question of finding the natural internal logic of the AEC.

We have now embarked on this large scale project.

Two Internal Logics of an AEC

$$\mathbb{L}_{\mathcal{K}}^{1,\mathsf{aec}} < \mathbb{L}_{\mathcal{K}}^{2,\mathsf{aec}}$$

The two logics

$$\mathbb{L}^{1,\mathrm{aec}}_{\mathcal{K}} < \mathbb{L}^{2,\mathrm{aec}}_{\mathcal{K}}$$

$\psi_{\mathcal{K}} \in \mathbb{L}^{1,\mathrm{aec}}_{\mathcal{K}}$,
fragment of $\mathbb{L}_{(2^\kappa)^+,\kappa^+}$ containing $\psi_{\mathcal{K}}$ (Shelah-V. 2021)

$\psi_{\mathcal{K}} \in \mathbb{L}^{2,\mathrm{aec}}_{\mathcal{K}}$,
second order interpretability of $\mathcal{K}$ (Shelah-V. in progress)

We close $\mathbb{L}_{(2^\kappa)^+,\omega}$ under $\forall x$, $\exists x$, $\bigwedge_{i<2^\kappa} \psi_i$, $\neg$ and $\psi_{\mathcal{K}}$.

This can very easily define well-ordering!

("Non-well orders" form an AEC, of very low "Scott rank", in a natural way!)

For some classes $\mathcal{K}$, the complexity can be extremely high: an AEC may "simulate" Ehrenfeucht-Fraïssé games of arbitrarily high complexity!

For some classes $\mathcal{K}$, the complexity can be extremely high: an AEC may “simulate” Ehrenfeucht-Fraïssé games of arbitrarily high complexity!

Other possibilities:

- Removing $\neg$ from $\mathbb{L}^{1,\mathrm{aec}}_{\mathcal{K}}$?
- Comparing/adapting $\mathbb{L}^1_\kappa$?
- Developing stability theory for $\mathbb{L}^1_\kappa$?
- Transfer stability theory to $\mathbb{L}^{1,\mathrm{aec}}_{\mathcal{K}}$?
- Omitting Types for these logics ?

Thanks! Kiitos paljon!