Around dependent abstract elementary classes

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Valéry - on being in the Ocean

I seem to find myself again when I enter this universal water. I have nothing to do with harvests, with labors; there is nothing for me in the Georgics. . . . I embrace the water with open arms, I love it, possess it, engender with it a thousand strange ideas. Now/In it/, I am the man I want to be. (. . .) I am drunk on my senses.

Paul Valéry - Cahiers (tr. N. Rudavsky-Brody)

Three eras

The Past: A (Rough) Overview of Stability Theory in AECs

The Present: Focus on Dependent AECs; some results

The Now Unfolding Future: new directions

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- Slow start - Terra Incognita
- The past two decades: Acceleration

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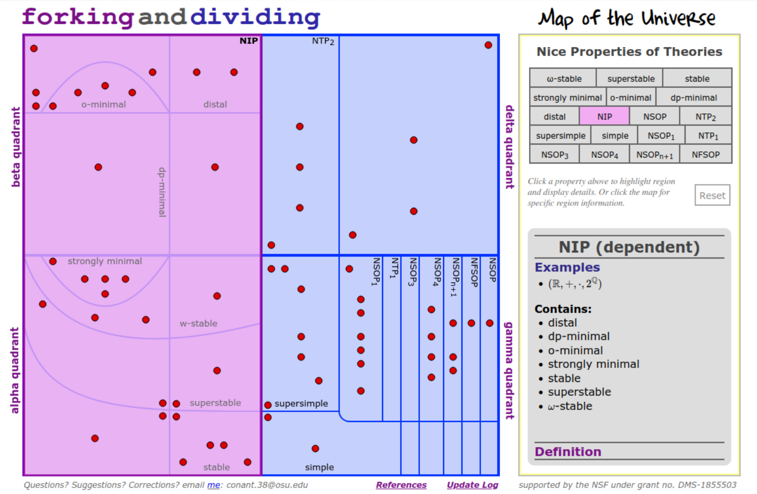
- Shelah-V.
- Nájar-V.

The “outer layer of a well-known map” ?!?



The Conant map (again)

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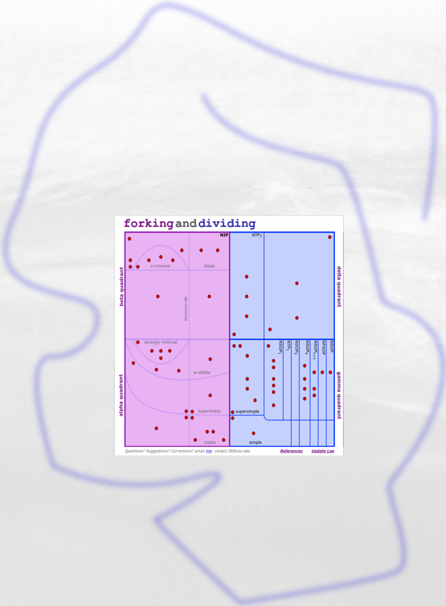

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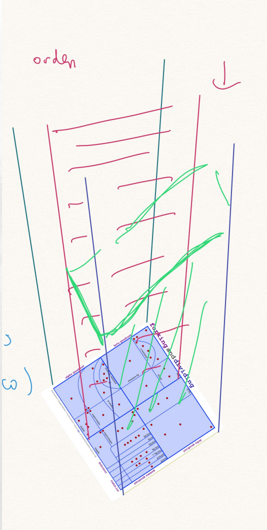

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- Still very pale map outside First Order back then, but beginning of definitions and strong theorems

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- Superstability and Limit Models - the role of splitting (Grossberg-VanDieren-V. 2010, Vasey, Boney, VanDieren, later),
- Simple AECs started (Hirvonen-Hyttinen under strong hypotheses, much later studied under tameness hypotheses by Grossberg and Mazari [2020])

The New Century: new connections

- 2001: the Zilber school: arithmetic geometry studied under the lens of Abstract Elementary Classes (really, axiomatized in $L_{\omega_1\omega}$ or under "quasiminimality hypotheses"):
  (pseudo)-exponential covers, modular forms and functions (the most famous of these analyses for the so-called j-mapping

  $$j : \mathbb{H} \to \mathbb{C},$$

  transferring the group action structure $\mathbb{H} \setminus \Gamma$ (of Fuchsian groups, e.g. $\Gamma = SL_2(\mathbb{Q})$) on (a structure H elementarily equivalent to) $\mathbb{H}$, to the field structure of a model of ACF_0 such as $(\mathbb{C}, +, \cdot, 0, 1)$),
- 2021: Mazari-Armida: model theory of modules: AECs of module structures,

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The dependent area

In FO, three important paradigms:

- Bounded alternation: a formula $\varphi(x, y)$ is dependent if whenever $(b_i)_{i<\omega}$ is an infinite (ind.) sequence, given a, the sequence $\varphi^{\ell}(a, b_i)$ alternates only finitely many times (between φ and $\neg\varphi$).

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- Generic Pair Conjecture: fix λ with $\lambda^+ = 2^\lambda$ and $\lambda^{<\lambda} = \lambda$. T is dependent iff for each saturated $M \models T$ of cardinality λ^+, for each continuous increasing chain $(M_\alpha)_{\alpha<\lambda^+}$ of elementary submodels of M of cardinality λ, there is some club $E \subseteq \lambda^+$ such that for each $\alpha_1 < \beta_1$, $\alpha_2 < \beta_2$ from $E \cap S^{\lambda^+}_{\lambda}$,

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- Recounting types.

First Line: The Generic Pair Conjecture

Studying originally a spectrum of existence question (Under categoricity at three successor cardinals $\kappa, \kappa^+, \kappa^{++}$, get the existence of a model in cardinality κ^{+++}), with Grossberg and VanDieren, we pinned down

- A first definition of "dependent AECs": paradigm 2 recast
  Let $(\mathcal{K}, \prec_{\mathcal{K}})$ be an A.E.C, let $M^* \in \mathcal{K}_{\lambda^+}$ be a universal and homogeneous model, for some λ with $\lambda^+ = 2^\lambda$ and $\lambda^{<\lambda} = \lambda$. $\mathcal{K}$ is gp-dependent at λ iff for each M^* as above, each continuous increasing chain $(M_\alpha)_{\alpha<\lambda^+}$ of $\prec_{\mathcal{K}}$-elementary submodels of M of cardinality λ, there is some club $E \subseteq \lambda^+$ such that for each $\alpha_1 < \beta_1$, $\alpha_2 < \beta_2$ from $E \cap S^{\lambda^+}_\lambda$,

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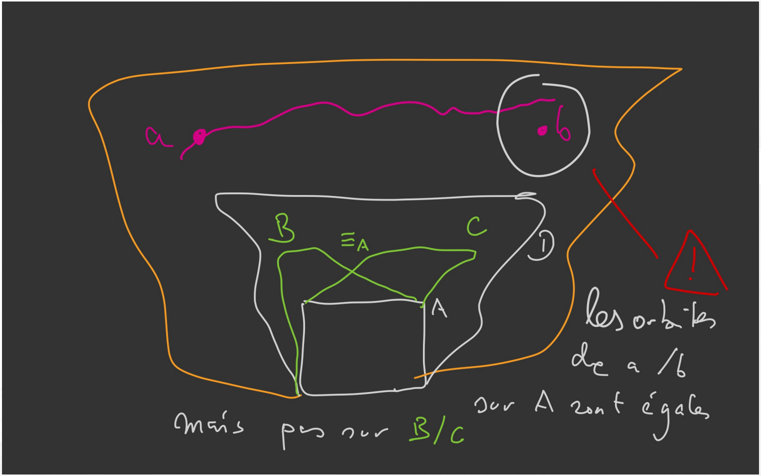
First Line: the GPC - independence notions

- A new notion of independence (Kaplan and Shelah proved later that in First Order this notion corresponds to splitting under stability) - paradigm 3:$p \in S(N)$ splinters over $M \prec N$ iff the notion $\varphi(x, a) \in p$ is a property of the conjugacy class of $\mathrm{tp}(a/M)\ldots$
- This notion captures interesting properties both in FO theories and in AECs!

A picture of splitting/splintering



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Now Happening: New Directions

Paradigm 1, bounded alternation, in many ways the most natural, has been the most difficult to adapt. However, in joint work with Shelah, we have the following results.

- New variants of dependence, not based on paradigm 2, but on the more basic paradigm 1. Examples!
- Extraction of indiscernibles under dependence (improving Kaplan-Lavi-Shelah who have a result for "Homogeneous Diagrams").

Difficult and open:

- Comparison of paradigms 1 and 2.
- Insertion of paradigm 3.
- Genuinely AEC examples of dependence.
- Type decomposition (stable + order/tree).

A related project, with Nájar

Nájar has brought to the center of this work new possibilities, by using my own result with Shelah from 2021 on an originally completely unrelated subject: definability of arbitrary AECs

This gives a much tighter grip on:

- Definability INSIDE an AEC, study of conjugates (recounting types)
- Paradigm 1 revisited.

Thank you for your attention!