Forcing and Independence
(The Turin Lectures)

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Abstract

These *Lectures in Set Theory - Forcing and Independence*, given originally for *Università di Torino*, are a very quick introduction to forcing and its connection to large cardinals. Three additional lectures provide a quick comparison between ZFC foundations and HoTT, virtualization of large cardinals and connections with model theory.

Contents

Introduction

In the Spring of 2021 I had planned a one-semester visit to *Università di Torino*. The invitation was organized by Alessandro Andretta and the Logic Group since 2019 (the length of time depended on a University-wide call for international visiting positions). The pandemic that started in February 2020 in Italy forced most of us to teach remotely. I ended up giving my lectures for a very enthusiastic set of students at the UniTo, remotely! The second part, the research part, of my visit to Turin, was postponed.

Although nothing replaces the chance of meeting colleagues and students personally, I appreciated the chance of giving these lectures. We went in just twelve lecture sessions from the very basics to proper forcing, covering a bit of large cardinals along the way. There were three special lectures, additional to this path: one on a comparison between foundations done through ZFC and foundations done through HoTT. Another one on virtualization of large cardinals, and a third one on a personal view of connections between model theory and set theory.

What this document is not. This is not a full set theory textbook. Rather, this is a collection of lectures that try to favor *synthesis* over detail. There are excellent textbooks where this material is presented: Kunen [4], Jech [3], Halbeisen [2], Džamonja [1], Schindler [5] all present this material, with profusion of details. What I offer here is somewhat different: a very quick road-map "from beginning to forcing", with three detours. I believe there is room for such a synthetic presentation of this material: people from different venues often ask *how to learn the basics of forcing*. Among them, non-specialists in set theory who may have reasons to want to learn forcing for their own mathematical (or philosophical) work, students who are beginning and are considering taking at some point more advanced courses in forcing, physicists or biologists who are intellectually inclined in studying these connections.

I am deeply indebted first and foremost with Ken Kunen, my doctoral thesis advisor in Madison some decades ago, and my first "guide" through forcing. His course at the University of Wisconsin, back in 1993, remains

in my mind and memory an absolute summit of clarity, subdued humor and elegance in presenting the material. Nothing can really equal those courses (and the book; that first edition of Kunen's book that so many of us who ended up doing set theory during part of our lives cherished so much); the inspiration his course provided has led many other endeavours, later in life. Ken passed away a bit less than a year ago; these notes are also a tribute to his memory and the light he brought to the discipline.

I am also thankful to Alessandro Andretta for the invitation to Turin (which I hope will be realized in-person soon!), to the rest of the Turin Logic group, and to a very enthusiastic group of students who followed these lectures.

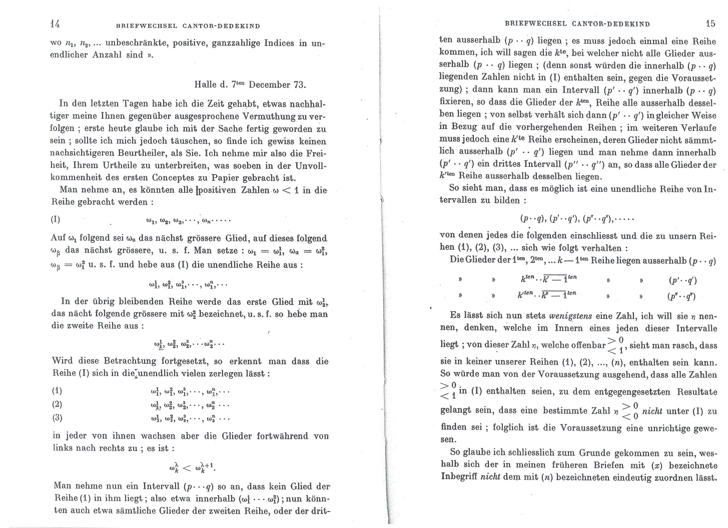
I am especially indebted to Mirna Džamonja, Vika Gitman and Fernando Zalamea for their special guest appearances during the special lectures, the detours from the main road. Their accompaniment was fundamental during those three special sessions.

And I am thankful to Chiara Romano, Miriam Marzaioli and Beatrice Degasperi for their corrections of my Italian and mathematical questions and suggestions: one of the special lectures was in that language - they were very helpful in proof-reading my written Italian beforehand. Gli errori scritti che ci sono forse rimasti e le volte che mi son' sbagliato mentre parlavo sono soltanto colpa mia! La lezione speciale sarebbe stata tutt'altra senza il loro aiuto!

1 Why Set Theory? Why Independence?

1.1 Classical problems

On September 7, 1873: **Cantor** proved the reals are not countable. He was studying the structure of convergence sets of trigonometric series in $\mathbb{R}$ and ended up *discovering/inventing* transfinite induction, the structure of cardinals and ordinals.



wo $n_1, n_2, \ldots$ unbeschränkte, positive, ganzzahlige Indices in unendlicher Anzahl sind ».

Halle d. 7ten December 73.

In den letzten Tagen habe ich die Zeit gehabt, etwas nachhaltiger meine Ihnen gegenüber ausgesprochene Vermuthung zu verfolgen ; erste heute glaube ich mit der Sache fertig geworden zu sein ; sollte ich mich jedoch täuschen, so finde ich gewiss keinen nachsichtigeren Beurtheiler, als Sie. Ich nehme mir also die Freiheit, Ihrem Urtheile zu unterbreiten, was soeben in der Unvollkommenheit des ersten Conceptes zu Papier gebracht ist.

Man nehme an, es könnten alle positiven Zahlen $\omega < 1$ in die Reihe gebracht werden :

(I) $$\omega_1, \omega_2, \omega_3, \cdots, \omega_n \cdots\cdots$$

Auf ω_1 folgend sei ω_α das nächst grössere Glied, auf dieses folgend ω_β das nächst grössere, u. s. f. Man setze : $\omega_1 = \omega_1^1$, $\omega_\alpha = \omega_1^2$, $\omega_\beta = \omega_1^3$ u. s. f. und hebe aus (I) die unendliche Reihe aus :

$$\omega_1^1, \omega_1^2, \omega_1^3, \cdots, \omega_1^n, \cdots$$

In der übrig bleibenden Reihe werde das erste Glied mit ω_2^1, das nächt folgende grössere mit ω_2^2 bezeichnet, u. s. f. so hebe man die zweite Reihe aus :

$$\omega_2^1, \omega_2^2, \omega_2^3, \cdots \omega_2^n \cdots$$

Wird diese Betrachtung fortgesetzt, so erkennt man dass die Reihe (I) sich in die unendlich vielen zerlegen lässt :

(1) $$\omega_1^1, \omega_1^2, \omega_1^3, \cdots, \omega_1^n, \cdots$$

(2) $$\omega_2^1, \omega_2^2, \omega_2^3, \cdots, \omega_2^n \cdots$$

(3) $$\omega_3^1, \omega_3^2, \omega_3^3, \cdots, \omega_3^n \cdots$$

in jeder von ihnen wachsen aber die Glieder fortwährend von links nach rechts zu ; es ist :

$$\omega_k^\lambda < \omega_k^{\lambda+1}.$$

Man nehme nun ein Intervall $(p \cdots q)$ so an, dass kein Glied der Reihe (1) in ihm liegt ; also etwa innerhalb $(\omega_1^1 \cdots \omega_1^2)$; nun könnten auch etwa sämtliche Glieder der zweiten Reihe, oder der dritten ausserhalb $(p \cdot\cdot q)$ liegen ; es muss jedoch einmal eine Reihe kommen, ich will sagen die k^{te}, bei welcher nicht alle Glieder ausserhalb $(p \cdot\cdot q)$ liegen ; (denn sonst würden die innerhalb $(p \cdot\cdot q)$ liegenden Zahlen nicht in (I) enthalten sein, gegen die Voraussetzung) ; dann kann man ein Intervall $(p' \cdot\cdot q')$ innerhalb $(p \cdot\cdot q)$ fixieren, so dass die Glieder der k^{ten}, Reihe alle ausserhalb desselben liegen ; von selbst verhält sich dann $(p' \cdot\cdot q')$ in gleicher Weise in Bezug auf die vorhergehenden Reihen ; im weiteren Verlaufe muss jedoch eine k'^{te} Reihe erscheinen, deren Glieder nicht sämmtlich ausserhalb $(p' \cdot\cdot q')$ liegen und man nehme dann innerhalb $(p' \cdot\cdot q')$ ein drittes Intervall $(p'' \cdot\cdot q'')$ an, so dass alle Glieder der k'^{ten} Reihe ausserhalb desselben liegen.

So sieht man, dass es möglich ist eine unendliche Reihe von Intervallen zu bilden :

$$(p \cdot\cdot q), (p' \cdot\cdot q'), (p'' \cdot\cdot q''), \cdots\cdots$$

von denen jedes die folgenden einschliesst und die zu unsern Reihen (1), (2), (3), ... sich wie folgt verhalten :

Die Glieder der $1^{\text{ten}}, 2^{\text{ten}}, \ldots k - 1^{\text{ten}}$ Reihe liegen ausserhalb $(p \cdot\cdot q)$

» » $k^{ten} \cdot\cdot \overline{k' - 1}^{ten}$ » » $(p' \cdot\cdot q')$

» » $k'^{ten} \cdot\cdot \overline{k'' - 1}^{ten}$ » » $(p'' \cdot\cdot q'')$

Es lässt sich nun stets *wenigstens* eine Zahl, ich will sie η nennen, denken, welche im Innern eines jeden dieser Intervalle liegt ; von dieser Zahl η, welche offenbar $\substack{> 0 \\ < 1}$, sieht man rasch, dass sie in keiner unserer Reihen (1), (2), ..., (n), enthalten sein kann. So würde man von der Voraussetzung ausgehend, dass alle Zahlen $\substack{> 0 \\ < 1}$ in (I) enthalten seien, zu dem entgegengesetzten Resultate gelangt sein, dass eine bestimmte Zahl $\eta \substack{> 0 \\ < 0}$ *nicht* unter (I) zu finden sei ; folglich ist die Voraussetzung eine unrichtige gewesen.

So glaube ich schliesslich zum Grunde gekommen zu sein, weshalb sich der in meinen früheren Briefen mit (x) bezeichnete Inbegriff *nicht* dem mit (n) bezeichneten eindeutig zuordnen lässt.

The question of existence of subsets of the reals with size **strictly** between $\aleph_0$ and $2^{\aleph_0}$ was left open by Cantor's result:

$$2^{\aleph_0} \stackrel{?}{=} \aleph_1.$$

The whole 20th century was marked by attempts to tackle this question (Hilbert '00, Gödel '40, Cohen '62) and opened extremely rich interaction between Logic, Topology, Analysis and. . . Set Theory. Later on, with Model Theory.

In Set Theory, the most studied (although certainly not the only) property of the set of reals, $\mathbb{R}$, is its *cardinality*, $2^{\aleph_0}$. The only ZFC-restriction (besides being uncountable) is due to König: $\operatorname{cf} 2^{\aleph_0} \neq \omega$.

Suslin asked around 100 years ago how to characterize $(\mathbb{R}, <)$.

A first attempt of a characterization was: *total dense linear order with no endpoints + Dedekind complete*.

But this was not sufficient!

Cantor in the 1890s already had another kind of answer: add to the "first attempt" the property "having a countable dense subset" (separability).

Suslin asked whether this could be weakened to the "ccc". Suslin Hypothesis (SH): YES. Later, however… SH is *independent*.

A word about Set Theory under $\neg CH$: The really interesting question is then *how different* from $\aleph_0$ are cardinals such as $\aleph_1$, or in general, cardinals $\kappa < 2^{\aleph_0}$, if CH fails. We may put these questions more precisely:

(a) $\forall X \subset \mathbb{R}(|X| = \aleph_1 \Rightarrow X$ has Lebesgue measure $0)$?

(b) $\forall X \subset \mathbb{R}(|X| = \aleph_1 \Rightarrow X$ is of first category$)$?

As it turns out, the four theories

$$\neg CH \pm (a) \pm (b)$$

are all consistent!

Another important step, connecting classical problems and forcing, was Martin's Axiom (MA). A limitation of $\neg CH$ is it doesn't construct anything. In this sense MA will be our crucial example of a strong existence axiom beyond ZFC.

There are, loosely speaking, three main families of axioms strengthening ZFC:

Family	Examples	Advantages
Large Cardinal Axioms	Compact, Measurables	Linear Hierarchy
Forcing Axioms	MA, MM, PFA,	Topology, $\neg$CH
"Determinacy" Axioms	AD, PD, $\text{AD}^{L(\mathbb{R})}$,…	$\forall X$ X is measurable

Although the proof of the following theorem requires *iterated forcing* (we do study forcing but not iterated forcing here), we can state it here in connection with the classical questions above.

Theorem 1.1. MA is consistent with $ZFC + \neg CH$ (and in that case $2^{\aleph_0}$ is regular).

So, $MA + \neg CH$ decides many things (e.g. $MA + \neg CH \Rightarrow$ Yes to (a) and (b) above $\Rightarrow SH$)

Yet many other questions are not decided by $MA + \neg CH$. For instance, the claim

all $\aleph_1$-dense subsets of $\mathbb{R}$ are isomorphic

is independent of $MA + \neg CH$!

Here is the statement of MA, which we study later in these notes, in full detail: given X a compact Hausdorff ccc space, X is not a union of $< 2^{\aleph_0}$ many nowhere dense subsets.

1.2 On ZFC axioms

We recall the ZFC axioms.

- **Extensionality:** $\forall x \forall y \left(\forall z \left(z \in x \leftrightarrow z \in y \right) \rightarrow x = y \right)$,
- **Pairing:** $\forall x \forall y \exists z \left(x \in z \wedge y \in z \right)$,
- **Unions:** $\forall x \exists y \forall z \left(\exists w \left(z \in w \wedge w \in x \right) \rightarrow z \in y \right)$,
- **Powerset:** $\forall x \exists y \forall z \left(\forall w \left(w \in z \rightarrow w \in x \right) \rightarrow z \in y \right)$,
- **Comprehension$_\varphi$:** $\forall x \forall \vec{y} \exists z \forall w \left(w \in z \leftrightarrow w \in x \wedge \varphi \left(x, \vec{y}, w \right) \right)$, where φ is a formula of $\mathcal{L}_\in$,
- **Infinity:** $\exists x \left(\emptyset \in x \wedge \forall y \left(y \in x \rightarrow S(y) \in x \right) \right)$, where $S(y) = y \cup \{y\}$, and enough other axioms are in place so that these definitions make sense,
- **Replacement$_\varphi$:** $\forall x \forall \vec{y} \exists z \forall w \left(w \in z \leftrightarrow \exists t \left(t \in x \wedge \varphi \left(t, w, x, \vec{y} \right) \right) \right)$, where $\varphi \left(w, z, \ldots \right)$ is a formula of $\mathcal{L}_\in$ that behaves *functionally* in t for w,
- **Foundation:** $\forall x \left(x \neq \emptyset \rightarrow \exists y \left(y \in x \wedge y \cap x = \emptyset \right) \right)$.

The previous are the ZF axioms. In addition, the Axiom of Choice AC gives the system ZFC.

Some important **basic facts** about the axioms include: Foundation is in a way unavoidable: in $ZFC-$**Foundation** one may build a model (sometimes called WF) consisting of all well-founded sets, and all sets may be mapped into that model. **Comprehension** and **Replacement** are axiomatic *schemes*. **Replacement** is necessary for the existence of the ordinal $\omega + \omega$ (and many other constructs).

An *ordinal* is a transitive set (i.e., a set x such that $z \in y \in x \rightarrow z \in x$) that is well-ordered by the relation $\in$. Under **Foundation**, one may adapt this definition: in that case, x is an ordinal iff x is transitive and *totally ordered* by $\in$.

A useful theorem:

Theorem 1.2. Every increasing continuous function from ordinals to ordinals has a fixed point.

Examples of such functions include: $\aleph$, $\beth$, $+, \cdot$ on ordinals. They all have arbitrarily high fixed points. The most famous are (perhaps) fixed points of $\beth$: cardinalities κ such that $\beth_\kappa = \kappa$.

Finally, the König theorem: under the Axiom of Choice, given $I \neq \emptyset$ and given cardinals $\lambda_i < \kappa_i$ for $i \in I$,

$$\sum_{i \in I} \lambda_i < \prod_{i \in I} \kappa_i.$$

Corollaries of this are the Cantor theorem ($\kappa < 2^\kappa$, for all κ) and the most important ZFC limitation in cardinal arithmetic: $\kappa < \kappa^{\operatorname{cf}\kappa}$, for all infinite κ (so, for example, $2^{\aleph_0} \neq \aleph_\omega$, etc.).

2 The Transfinite Recursion Theorem

2.1 Transfinite Inductive Definitions

x is an *ordinal* iff x is transitive and well-ordered by $\in$

$$\text{Foundation} \Longrightarrow \begin{cases} x \text{ totally ordered by } \in \\ \Updownarrow \\ x \text{ well-ordered by } \in \end{cases}$$

Connection with Fraenkel-Mostowski: see Exercise 3.
$ZC^- = ZFC \setminus \{\text{Foundation, Replacement}\}$

$ZC^- \vdash 90\%$ of mathematics

Replacement used for two things:

- iterate $\wp$ to infinity,
- get von Neumann ordinals

In ZC^- one can build $\mathbb{R}$…but one cannot build $\omega + \omega$…

2.2 Metamathematics and logic in ZF^-

Theorem 2.1 (Transfinite Recursion). $\forall A \forall B \forall R \forall G$ (if R is a well-founded and set-like binary relation, and

$$G : (\text{ set of partial functions } A \to B) \longrightarrow B.$$

$$\text{then } \exists! f : A \to B\,(\forall x \in A\,[f(x) = G(f \restriction \operatorname{pred}_{A,R}(x))]))\,.$$

Tricky part: do the previous in *proper classes*, such as $\aleph$ or $\beth$, or $\text{ ran } : WF \to On$.

Abbreviations? What does the following expression abbreviate?

$$\varphi : \forall n \in \omega \left(2^{\aleph(n)} = \aleph(n+3)\right)?$$

Two ways:

- φ abbreviates a sentence of $\mathcal{L}_\in$, or
- (easier): augment the language $\mathcal{L}_\in$, adding new predicate symbols such as On, $\aleph$, etc.

This can be done in such a way that we get (formal) definitions, and a chain of theories $T_0 \subseteq T_1 \subseteq \cdots \subseteq T_k$ such that

- T_k is a conservative extension of T_0 (the theory is not strengthened), and
- If $\varphi(x_1, \ldots, x_n) \in \mathcal{L}_k$, there is $\psi(x_1, \ldots, x_n) \in \mathcal{L}_0$ such that $T_k \vdash \forall \vec{x}(\varphi(\vec{x}) \leftrightarrow \psi(\vec{x}))$.

3 Small models of set theory

3.1 Structure of the V_α's and $H(\kappa)$'s

Basic lemmas: $x \in y \in WF \Rightarrow \text{ ran } (x) < \text{ ran } (y)$. V_α is transitive, $|V_\alpha| = \beth_\alpha$ if $\alpha > \omega^2$; in general, $|V_{\omega+\alpha}| = \beth_\alpha$.

$H(\kappa) := \{x \mid |\text{ trcl } (x)| < \kappa\}$. $H(\omega) = V_\omega$ (HF, the hereditarily finite sets). $H(\kappa)$ is transitive.

For every infinite cardinal κ, $H(\kappa) \subseteq V_\kappa$. Moreover, under AC, for $\kappa > \aleph_0$,
$H(\kappa) = V_\kappa \Leftrightarrow \kappa = \beth_\kappa$.

(Lemma: $t \in WF$ implies $(\text{ ran } \restriction t) : t \to \text{ ran } (t)$ is onto.)

$H(\omega_1) \subseteq V_{\omega_1}$…but $|V_{\omega_1}| = \beth_{\omega_1}$ whereas $|H(\omega_1)| = \beth_1$! (Usually, the $H(\kappa)$'s are "tall but thin" inside the V_κ's)

$$\text{Notice: } ZFC^- \vdash \begin{cases} V_{\omega+\omega} \models ZC = ZFC - \textbf{Replacement} \\ V_\omega \models ZFC - \textbf{Infinity} \\ H(\omega_1) \models ZFC - \textbf{Powerset} \end{cases}$$

yet

$$ZFC^- \nvdash \exists \text{ a set model of ZFC},$$

by Gödel's Second Incompleteness Theorem.

3.2 Small Models of ZFC. Small Large Cardinals.

Which formulas hold in our small models V_κ, $H(\theta)$?

Definition 3.1. (AC) κ is strongly inaccessible if

1. $\kappa > \omega$,
2. κ is regular,
3. $\forall \lambda < \kappa (2^\lambda < \kappa)$.

If κ is strongly inaccessible, then $V_\kappa = H(\kappa) \models ZFC$.

If κ is strongly inaccessible then $\beth_\kappa = \kappa$. The opposite is far from true! (Consider $\kappa = \sup(\omega, \beth_\omega, \beth_{\beth_\omega}, \beth_{\beth_{\beth_\omega}}, \dots)$, then $\beth_\kappa = \kappa$ but $\text{cf}(\kappa) = \omega$...! For this κ, $V_\kappa \not\models$ **Replacement**,

If $V_\gamma \models ZFC$ then $\gamma = \beth_\gamma$ and γ is the γ-th cardinal in $\{\kappa | \kappa = \beth_\kappa\}$.

1. If M transitive then $M \models$ **Extensionality**
2. If $M \subseteq WF$ then $M \models$ **Foundation**
3. If M is transitive and $\forall z \in M \forall y \subseteq z (y \in M)$ then $M \models$ **Comprehension**
4. $V_\alpha \models$ **Pairing** iff α is a limit ordinal
5. All the V_α's and all the $H(\kappa)$'s satisfy **Unions**
6. $V_\omega = H(\omega) \models$ **Replacement**, (AC) $H(\kappa) \models$ **Replacement** if κ is regular
7. $ZFC \nvdash \exists \kappa (\kappa)$ inaccessible
8. If M is transitive and $\forall A \in M \forall B (|B| \leq |A| \rightarrow B \in M)$ then $M \models$ **Replacement**

9. If M is transitive, $\omega \in M$ and $M \models$ **Comprehension**, **Unions**, **Pairing**…then $M \models$ **Infinity**
10. If M is transitive and closed under $\wp$ then $M \models$ **Powerset**
11. (ZF^-) $H(\kappa) \models AC$; under AC, $V_\gamma \models AC$ if γ is a limit.

4 Constructibility and Gödel's L

4.1 Comprehension, Replacement and Choice

- M transitive and closed under subsets of its elements implies $M \models$ **Comprehension**
- If M transitive and $\forall A \in M \forall B\, (|B| \leq |A| \rightarrow B \in M)$ then M models **Replacement**
- $V_{\omega+\omega}$ does not satisfy the hypothesis. But $H(\kappa)$ does, when κ is regular!
- In detail (in ZF^-): why $H(\kappa) \models AC$ (for all κ)and (under AC) $V_\gamma \models AC$ if γ is limit.

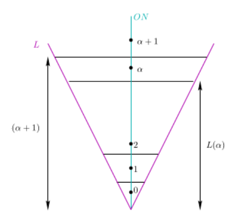
4.2 Gödel's L, basic properties

Goal: To show that $Con(ZF^-) \rightarrow Con(ZFC + GCH)$.

Working in ZF (or in ZF^-) we **define** the class L and show that "$L \models ZFC + GCH$".

But we express the previous statement using *relativizations*!

- $L_0 = L(0) = \emptyset$,
- $L_{\alpha+1} = L(\alpha+1) = \mathcal{D}(L(\alpha))$,
- $L_\gamma = L(\gamma) = \bigcup_{\alpha<\gamma} L(\alpha)$,
- $L = \bigcup_{\alpha \in On} L(\alpha)$.



The crucial difference is the use of $\mathcal{D}$ in the successor stage. $\mathcal{D}(x)$ is the class of **definable** subsets of x.

Notice $L(n) = V_n$ for all finite n, and $L_\omega = V_\omega$. However, $L_{\omega+1}$ is quite different from $V_{\omega+1}$! BUT since we want L to satisfy **Comprehension,** we need to put in L sets like $\{n < \omega \mid n \text{ is even}\}$.

In general, for $\mathfrak{A}$ a τ-structure, $\mathcal{D}(\mathfrak{A}) = \{S \subseteq A \mid S \text{ is FOD in } \mathfrak{A}\}$. A set $S \subseteq A$ is FOD (First Order Definable) in $\mathfrak{A}$ if it is the "solution set" of some formula: for some τ-formula $\varphi(x, \vec{y})$, and some parameters $\vec{b}$ from A, we have

$$S = \varphi(\mathfrak{A}, \vec{b}) = \left\{a \in A \mid \mathfrak{A} \models \varphi(a, \vec{b})\right\}.$$

We'll prove: $\mathcal{D}(\mathfrak{A})$ contains all finite and cofinite subsets of A. If A is well-orderable and infinite then $|\mathcal{D}(\mathfrak{A})| = |A|$.

If $\triangleleft$ well orders A, then we can define $E(\triangleleft)$ a well-order of $\mathcal{D}(A)$.

- L_α is transitive,
- $\alpha \leq \beta \to L_\alpha \subseteq L_\beta$,
- $L_\alpha \subseteq V_\alpha$ for all α,
- $L_\alpha \in L_{\alpha+1} \setminus L_\alpha$,
- $\alpha \in L_{\alpha+1} \setminus L_\alpha$,

4.3 Discussion: Why First Order?

The same idea *could* work outside FO. In fact $HOD = L^{(2)}$ (second order definability) has many similar properties... but we cannot GCH holds in HOD. Other logics have been studied recently [Kennedy-Magidor-Väänänen(2020)].

Although "well-ordered" is second order, if A is transitive and $A \subset WF$ then $On \cap A \in \mathcal{D}(A)$.

5 Gödel: $Con(ZF^-) \to Con(ZFC + GCH)$

An important function: $\rho(x)$, the L-rank of x: the minimum α such that $x \in L_{\alpha+1} \setminus L_\alpha$. Notice $\rho(x) < \rho(y) \to x <_L y$. Also, if α is limit, $<_L \restriction L_\alpha \in \mathcal{D}(L_\alpha)$, and $\rho \restriction L_\alpha \in \mathcal{D}(L_\alpha)$.

5.1 The idea of Gödel's proof

Goal: to prove $L \models ZF + AC + GCH$. We break this in *two stages:*

- $L \models ZF + V = L$,
- $ZF + V = L \vdash AC + GCH$.

$$V = L: \qquad \forall x \exists \alpha \, (x \in L_\alpha)$$

The sentence "$L \models V = L$" should be trivial…but it isn't: what's trivial is $\forall x \in L \exists \alpha (x \in L_\alpha)$…but $L \models V = L$ really means

$$\forall x \in L \exists \alpha \left(x \in L^L(\alpha) \right)!$$

L^L means carrying the construction of L…in L. We will need some **absoluteness** in the form $L^L(\alpha) = L_\alpha$.

We already know: $ZF + V = L \vdash AC$ (using $<_L$, the global well-order of L). We also know that $ZF + V = L \vdash \forall \alpha \geq \omega \, (|L_\alpha| = |\alpha|)$ (the conclusion uses the axiom of choice).

Later, with downward Löwenheim-Skolem (and Condensation, and Mostowski Collapse), we will get that $\wp(\kappa) \subseteq L(\kappa^+)$ and we will be able to conclude that $2^\kappa = \kappa^+$.

Notice in ZF we prove: $L \models ZFC$, $L \models |L_\alpha| = |\alpha|$. So, in L, there is

$$f : L_\alpha \xrightarrow[onto]{1-1} \alpha.$$

But f then also "lives" in V…and "f is a bijection" is absolute.

The first step (showing that $L \models ZF + V = L$) is proved similar to WF with two exceptions:

- $L \models V = L$ [this uses absoluteness],
- $L \models$ **Comprehension** [this uses Downward Löwenheim-Skolem]

The other axioms are easy (e.g., to see that $L \models$ **Powerset**, we do not need the "real powerset": if $x \in L$, let $\alpha = \sup \left\{ \rho(z) + 1 \mid z \in \wp(x) \cap L \right\}$. This ordinal α exists, by **Replacement** (in V). Then $\wp(x) \cap L \subseteq L_\alpha$. Let $y = L_\alpha \in L_{\alpha+1} \subset L$….

5.2 Löwenheim-Skolem, Reflection Principle. Relativizations

Why is the proof of $L \models$ **Comprehension** not trivial?

- Comparing "truth in L" and "truth in some L_α"
- Löwenheim-Skolem and Reflection
- 4 versions ($2 \cdot 2$: classes vs sets / sets vs hierarchy)

5.3 Absoluteness

$\varphi(\vec{x})$ is *absolute* for M iff $\forall \vec{x} \in M^n \left(\varphi(\vec{x})^M \Leftrightarrow \varphi(\vec{x})\right)$.

This is the same as $M \prec_\varphi V$, but in a correct way.

Notions such as "ordinal", "function" are absolute. We plan to see that $(\xi \mapsto L_\xi)$ is absolute.

"Second order" notions such as "cardinality" are usually not absolute. But ω is. Some questions on absoluteness are difficult. For example, if GC denotes the Goldbach Conjecture, then is

$$\varphi(x) = (GC \wedge x = \omega_1) \wedge (\neg GC \wedge x = \emptyset)$$

absolute for ctm (countable transitive models) of ZFC?

Remember $o(M) = On \cap M$. When M is transitive, $o(M)$ is an ordinal - the first ordinal that does not belong to M. When we'll have that "ordinal" is absolute, we will know that $o(M) = On^M$.

The 8 absoluteness lemmas

- **Lemma 1:** Absolute formulas are closed under propositional connectives and *bounded* quantifiers.

  Examples: All Δ_0 formulas are absolute for transitive models M. $x \subseteq y$, $\emptyset$, etc.
- **Lemma 2:** If φ and ψ are equivalent in M and in V, and ψ is absolute for M, then φ also is.

  Examples: x is an ordinal. The official definition is not Δ_0 but under Foundation we can use total order instead of well-order. So, being an ordinal is absolute for models of ZF.

- **Lemma 3:** If f is defined using $\varphi(\vec{x}, y)$, and $\varphi(\vec{x}, y)$ is absolute for M, and f^M is defined, then f is absolute for M.

  Examples: to check f^M is defined, either $M \models T, T \vdash \forall\vec{x}\exists! y\varphi(\vec{x}, y)$. For example, if $T = Z - P - Inf$, for transitive models of T, "Singleton" is absolute. Or else, one verifies directly that M is closed under f. If M is closed under $\{x\}$, $\{x, y\}$, $\bigcup x$, $\ldots$, then those notions are absolute.

- **Lemma 4:** If M is a transitive model of $Z - P - Inf$ then M is closed under finitary operations, i.e. $[M]^{<\omega} \subseteq M$, $M^{<\omega} \subseteq M$ and $\omega \subseteq M$.

  Examples: $f : x \to y$ is a bijection, x is a natural number, x is a finite set, R is an acyclic relation on A, x is an ordinal... are absolute

- **Lemma 5:** "R is well founded" is absolute for transitive M satisfying $ZF - P$.

  This one requires more detail!

- **Lemma 6:** The constants ω and HF are absolute for transitive $M \models ZF - P$.

  Similarly, "x is a ZF axiom", "p is a proof of $0 \neq 0$ from ZF" and "Con(ZF)" are absolute!

  Working in ZF, we can**not** prove $Con(ZF)$, but we can prove that **if** there exists a transitive $M \models ZF$ **then** $Con(ZF)$ (by soundness of $\vdash$), and therefore $(Con(ZF))^M$. By Gödel, there exists a model $\mathfrak{A} \models (ZF + \neg Con(ZF))$, of the form $\mathfrak{A} = (A, E)$; then E cannot be well-founded (Mostowski). In fact, the length of the proof of $\neg Con(ZF)$ is a non-standard *integer*.

  All Π^1_1 formulas are absolute.

  Most of topology and analysis are absolute. For example,

  $$\forall x \in \mathbb{R} \left(\sum_n a_n \cos(nx) \text{ converges} \right)$$

  is absolute (as it is Π^1_1).

- **Lemma 7:** If M is a transitive model of $ZF - P$ then all logical syntax is absolute for M.

- **Lemma 8:** $\models$ is absolute (for transitive models...)

5.4 Putting together Gödel's Proof. Condensation

Proof of $ZF + V = L \vdash GCH$: assume $ZF + V = L$ and fix θ an infinite cardinal. We want to see that $2^\theta = \theta^+$. We check that $\wp(\theta) \subseteq L_{\theta^+}$; we already know that under AC, $|L_{\theta^+}| = \theta^+$.
Let $x \subseteq \theta$. We need to find $\alpha < \theta^+$ such that $x \in L_\alpha$. We need the following two lemmas.

- **Lemma 1:** if $\kappa > \omega$ is regular then $L_\kappa \models ZF - P + V = L$. (Proof: the most interesting cases are **Comprehension** and $V = L$. But $L_\kappa \models V = L$ since L_α is absolute and $L_\kappa \cap On = \kappa$. To see that Comprehension holds we use Reflection for set hierarchies and the regularity of κ: if $B \in L_\kappa, a_1, \dots, a_n \in L_\kappa$, to see that $E = \{b \in B \mid \varphi(b, a_1, \dots, a_n)\} \in L_\kappa$, let $\alpha < \kappa$ such that $L_\alpha \prec L_\kappa$ and $B, a_1, \dots, a_n \in L_\alpha$. Then $E = \{b \in B \mid L_\kappa \models \varphi(\dots)\} \in \mathcal{D}(L_\alpha) = L_{\alpha+1} \subset L_\kappa \dots$ thus $E \in L_\kappa$.)
- The **Condensation** lemma:

> If M is transitive and $M \models ZF - P + V = L$ then $M = L_\alpha$ for some α.

(Proof: Let $\alpha = o(M)$. This is the minimum ordinal outside of M. α is limit, since M is closed under successor. So, $L_\alpha = \bigcup_{\xi<\alpha} L_\xi$. Since $M \models V = L$, if $x \in M$ there is $\xi < \alpha$ such that $x \in L_\xi$ [absoluteness of L_ξ] and thus $M \subseteq L_\alpha$. But if $\xi < \alpha$ then $L_\xi = (L_\xi)^M \in M$; as M is transitive, $L_\xi \subseteq M$. Thus, $L_\alpha \subseteq M$.)

By the assumption $V = L$, we know that $x \in L_\kappa$ for some κ. Wlog, κ is a regular cardinal. Let $S := \{x\} \cup \theta$. Clearly, S is transitive and has size θ. Let $A \prec L_\kappa$ with $S \subseteq A$ and $|A| = \theta$. Now use the Mostowski collapse to find T transitive, $\pi : A \xrightarrow{\approx} T$ the Mostowski isomorphism. Since S is transitive, $\pi \restriction S = id$ and therefore $x = \pi(x) \in T$. By Lemma 1,

$$T \approx A \prec L_\kappa \models ZFC - P + V = L.$$

By Condensation, $T = L_\alpha$ for some α. Since $|T| = \theta$, $\alpha < \theta^+$. So $x \in T = L_\alpha \subseteq L_{\theta^+}$, $\wp(\theta) \subseteq L_{\theta^+}$ and $2^\theta = \theta^+$. □

6 Martin's Axiom

Martin and Silver at the end of the 1960's *amalgamated* many constructions of iterated forcing about the continuum into **one unique axiom** called Martin's Axiom (MA).

$$Con(ZFC + c \text{ is big } + \text{ other desirable properties}).$$

They just proved with one iteration

$$Con(ZFC + c \text{ is big } + MA),$$

and showed that the good properties (SH, U1C, U0M) hold.

$MA : MA_\kappa, \forall \kappa < c$

MA_κ: given a ccc partial order $\mathbb{P}$ and a family $\mathcal{D}$ of $\leq \kappa$ dense subsets of $\mathbb{P}$, there exists a filter $G \subseteq \mathbb{P}$ such that $D \cap G \neq \emptyset$, for each $D \in \mathcal{D}$.

Last week: End of Gödel's Proof. Beginning of description of MA: ccc, dense orders, filters generic for a family $\mathcal{D}$.

6.1 The connection with Baire Category

The statement of Martin's Axiom (MA): $MA : MA_\kappa, \forall \kappa < c$

MA_κ: given a ccc partial order $\mathbb{P}$ and a family $\mathcal{D}$ of $\leq \kappa$ dense subsets of $\mathbb{P}$, there exists a filter $G \subseteq \mathbb{P}$ such that $D \cap G \neq \emptyset$, for each $D \in \mathcal{D}$.

Fix X a locally compact Hausdorff space.

Let $\mathbb{P} = \{U \subset X \mid U \text{ open}, U \neq \emptyset, \bar{U} \text{ compact}\}$, ordered by $\subseteq$.

We will see that $MA(\kappa) \longrightarrow \mathbb{R}$ is not a union of κ many closed nowhere dense (nwd) sets.

When $\kappa = \omega$, this is a version of Baire's Category Theorem: a locally dense compact Hausdorff space can never be the union of ω many closed nwd sets. Proof in class.

Generic Existence Theorem = $MA(\omega)$ = Baire Category Theorem: given $\mathbb{P}$ a partial order and $\mathcal{D}$ a **countable** family of dense subsets of $\mathbb{P}$, there exists $G \subset \mathbb{P}$ such that for each $D \in \mathcal{D}$, $G \cap D \neq \emptyset$.

6.2 Why below continuum. Basic properties

Why ccc? Consider "$SMA(\kappa)$" (Super-Martin Axiom - remove the ccc restriction from the statement of MA).

$ZFC \vdash SMA(\omega)$.

But $SMA(\kappa)$ implies that given X locally compact Hausdorff, X is not a union of $\leq \kappa$ nwd sets.

But then $SMA(\omega_1)$ is inconsistent: the space $X = (\omega_1+1)^\omega$ is compact and Hausdorff... but it is the union of the closed nowhere dense sets $C_\alpha := ((\alpha+1) \cup \{\omega_1\})^\omega$.

So, for κ beyond ω, $SMA(\kappa)$ is just false!

7 The amoeba partial order $\mathbb{A}^\varepsilon$

Theorem 7.1. $MA(\kappa)$ implies that in $\mathbb{R}$, a union of $\leq \kappa$ Lebesgue-null sets is also Lebesgue-null.

We use the "amoeba" partial order $\mathbb{A}^\varepsilon$ for ε

$$\mathbb{A}^\varepsilon := \{p \subset \mathbb{R} \mid p \text{ open}, \mu(p) < \varepsilon\}, \quad \leq = \supseteq .$$

Fix $\varepsilon > 0$ and let $(N_\alpha)_\kappa$ be a family of κ many null sets. $\mathbb{A}^\varepsilon$ is ccc (we'll see why) and **density argument** will allow us to conclude the theorem:

For $\alpha < \kappa$, let $D_\alpha = \{p \in \mathbb{A}^\varepsilon \mid N_\alpha \subset p\}$. This is a dense subset (pseudopod argument). Then, by $MA(\kappa)$, there is a filter G generic for all the D_α's. Let $U = \bigcup G = \bigcup_{p\in G} p$. Then U has measure $\leq \varepsilon$ and covers all the N_α's.

8 Cohen forcing

So far we have: SMA_{ω_1} implies that a compact Hausdorff space cannot be a union of $\aleph_1$ many closed nwd sets, but $X = (\omega_1 + 1)^\omega$ is compact Hausdorff, and a union of $\aleph_1$-many closed nwd sets. The amoeba forcing used to show that MA_κ implies that a union of κ-many null sets of reals is null, and the corollary: MA implies c is regular.

8.1 $Fn(I, J)$. Cohen's partial order $Fn(I, 2)$

The partial order $\mathbb{P} = Fn(I, J) = \{p \subset I \times J \mid p$ is a partial function and $|p| < \omega\}$, with $\leq = \supseteq$, is crucial from now on.

G prefilter implies it is a compatible family of functions, therefore $\bigcup G = \bigcup_{p \in G} p$ is a partial (maybe total) function from I to J. We can view elements $p \in \mathbb{P}$ as **finite approximations of** $\bigcup G$... and later, to adding (many) new reals.

Some dense sets:

- $(i \in I)$ $D_i = \{p \mid i \in \text{ dom } (p)\}$ is dense (if $J \neq \emptyset$),
- $(j \in J)$ $E_j = \{p \mid j \in \text{ ran } (p)\}$ is dense (if I is infinite).

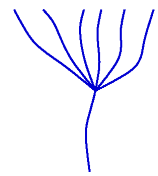
Meeting all the D_i's and all the E_j's would give a function $f_G = \bigcup G : I \to J$ that would be onto. This is impossible if (for example) $I = \omega$, $J = \omega_1$. [This does not contradict MA_{ω_1}, as $Fn(\omega, \omega_1)$ is not ccc]. However, **if J is finite or countable, then $Fn(I, J)$ is ccc.**

8.1.1 Δ-systems and the ccc. Productivity

To prove the previous, we use the Δ-system lemma:

Given finite sets a_ξ, for $\xi < \omega_1$, there exists uncountable $S \subseteq \omega_1$ such that $\{a_\xi \mid \xi \in S\}$ is a Δ-system. That is, there is some r such that if $\xi_1 \neq \xi_2 \in S$, $a_{\xi_1} \cap a_{\xi_2} = r$.

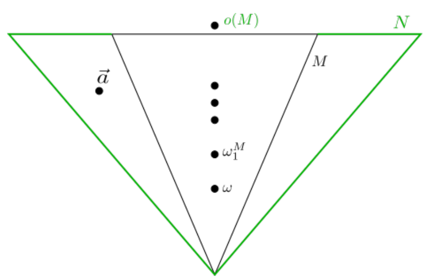
We prove the Δ-system lemma and a famous application: **under MA_{ω_1}, every product of ccc spaces is also ccc.**



Δ-system lemma: $\omega_1 \xrightarrow{\Delta} (\omega_1)^{<\omega}$. Similarly, if κ regular, then $\kappa \xrightarrow{\Delta} (\kappa)^{<\omega}$. This fails for singular κ!

Even stronger: $c^+ \xrightarrow{\Delta} (c^+)^\omega$, but this fails for c: for $x \in \mathbb{R}$, let a_x be countable, $a_x \subset \mathbb{Q}$, $a_x \to x$. So, if $x \neq y$, $a_x \cap a_y$ must be finite. Suppose that $\{a_x | x \in D\}$ forms a Δ-system ($|D| = \omega_1$). Then, considering $\{a_x \setminus r | x \in D\}$, we would have ω_1-many disjoint non-empty subsets of $\mathbb{Q}$... something impossible.

8.2 The generic extension (I)



The *very rough* idea to get $N \models ZFC + \neg CH$ is to start with a countable transitive model (ctm) of ZFC. $\omega_1^M, \omega_2^M, \ldots$ are then countable ordinals in V.

In V, we can fix $\vec{a} = \langle a_\xi \mid \xi < \omega_2^M \rangle$ *outside* of M. Then adjoin $\vec{a}$ to M and get $N = M[\vec{a}] \models c \geq \omega_2$.

But there are problems with this:

- How do we build N so as to get $N \models ZFC$?
- How do we guarantee $\omega_2^N = \omega_2^M$?

We will build N in such a way that

- We get the same ordinals: $M \cap On = N \cap On$,
- N is transitive,
- $N = M[G]$ for G a generic filter over $\mathbb{P}$ ($G \cap D \neq \emptyset$ for all dense $D \subseteq \mathbb{P}$ such that $D \in M$),
- We will prove that if ($\mathbb{P}$ is ccc)^M then M and $M[G]$ have the same *cardinals*.
- We will use dense sets of $\mathbb{P} = Fn(\omega_2^M, 2)$ to get $(2^{\aleph_0} = \aleph_2)^N$.

9 Construction of $V[G]$. $\mathbb{P}$-names

Fix M a ctm of ZFC. We want $N \supseteq M$ with $o(N) = o(M)$, N transitive, $N \models ZFC$…but also there are elements in $N \setminus M$, and (at least for the sake of Cohen's forcing where we add new reals) the notion of cardinality to be preserved between M and N.

Notice we do not always want that last point: $Fn(\omega, \omega_1)$ *collapses* ω_1.

We already know that if $\mathbb{P} \in M$ and $\mathbb{P}$ is *separative* (or "non-atomic"), then if G is $\mathbb{P}$-*generic* over M, $G \notin M$ (as in that case, G cannot meet the dense set $D = \mathbb{P} \setminus G$, but $D \in M$ if $\mathbb{P} \in M$).

Before using G to build N, we construct the universe of "$\mathbb{P}$-names", $V^{\mathbb{P}}$.

σ is a $\mathbb{P}$-name iff all elements of σ are of the form $\langle \tau, p \rangle$, where τ is a $\mathbb{P}$-name and $p \in \mathbb{P}$.

Examples (in class), and a construction of $V^{\mathbb{P}}$ as a hierarchy.

$M^{\mathbb{P}} = M \cap V^{\mathbb{P}}$. Now, the **values** of $\mathbb{P}$-names in M (computed using G) are the elements of $M[G]$:

For σ a $\mathbb{P}$-name, $\mathrm{val}(\sigma, G) = \sigma_G := \{\tau_G \mid \text{for some } p \in G(\langle \tau, p \rangle \in \sigma)\}$

$$M[G] = \left\{\sigma_G \mid \sigma \in M^{\mathbb{P}}\right\}$$

Canonical names: for $a \in M$, the name $\check{a} = \left\{\langle \check{b}, \mathbb{1} \rangle \mid b \in a\right\}$ is such that $(\check{a})_G = a$. Therefore, $M \subset N$.

Now, G has a name: $\Gamma = \{\langle \check{p}, p \rangle \mid p \in \mathbb{P}\}$.

$\Gamma_G = \{\sigma_G \mid \exists p \in G\,(\langle \sigma, p \rangle \in \Gamma)\} = \{(\check{p})_G \mid \exists p \in G\,(\langle \check{p}, p \rangle \in \Gamma)\} = \{p \mid p \in G\} = G$.

10 Large Cardinals

10.1 Measurable Cardinals

κ is **measurable** iff there exists $j : V \to M$ a non-trivial elementary embedding with critical point $crit(j) = \kappa$.

j must move an ordinal; the minimal one moved is the *critical point* κ. κ must be a cardinal, and $j(\kappa) > \kappa$.

> κ is measurable iff there exists a non-principal, κ-complete ultrafilter on κ.

We proved in class one direction: if κ is measurable, let $j : V \to M$ have critical point κ. Then $\mathcal{U}_\kappa = \{X \subset \kappa \mid \kappa \in j(X)\}$ is a non-principal, κ-complete ultrafilter on κ. The other direction is proved by taking the ultrapower of V with respect to the ultrafilter $\mathcal{U}$. The κ-completeness (really, the ω_1-completeness) of the ultrafilter gives a *well-founded* ultrapower $i : V \to V^\kappa/\mathcal{U}$. We can therefore apply the Mostowski collapse π and get a transitive model M. Then $j = \pi \circ i : V \to M$ is a non-trivial elementary embedding with critical point κ.

10.2 Other Large cardinals

Given $\lambda > \kappa$, κ is λ-**supercompact** iff there exists $j : V \to M$ with critical point κ such that ${}^\lambda M \subset M$ and $j(\kappa) > \lambda$. κ is **supercompact** if it is λ-supercompact for all $\lambda > \kappa$.

Given $\lambda > \kappa$, κ is λ-**strong** iff there exists $j : V \to M$ with critical point κ such that $V_\lambda \subset M$ and $j(\kappa) > \lambda$. κ is **strong** if it is λ-strong for all $\lambda > \kappa$.

The *key point* is that

- They are characterized by *elementary embeddings* from V to some inner model M,
- This inner model M can never be equal to V (under AC, Kunen's bound) and making M more and more similar to V provides larger and larger notions,
- Most of these notions can also be characterized by *set* objects (like $\mathcal{U}_j$ - some of them quite complex ("extender-systems", etc.)

10.3 Collapse forcing

Fix M a ctm of ZFC and let $\mathbb{P} \in M$ and G $\mathbb{P}$-generic over M. When $\mathbb{P}$ is separative, $G \notin M$. Then we get $N[G] = \{\sigma_G \mid \sigma \in M^{\mathbb{P}}\}$: the generic extension consists of values of $\mathbb{P}$-names in M, "computed" using the generic filter G.

We have then $M[G] \supset M$, with $o(M[G]) = o(M)$, $M[G]$ transitive, $M[G] \models ZFC$.

Remember that $\mathbb{P} = Fn(\omega, \omega_1)$ *collapses* ω_1: if G is $\mathbb{P}$-generic, then G meets the dense sets $D_i = \{p \in \mathbb{P} \mid i \in \operatorname{dom}(p)\}$ and $E_j = \{p \in \mathbb{P} \mid j \in \operatorname{ran}(p)\}$, and thus $f_G = \bigcup G : \omega \to \omega_1$ that is **surjective**. So, ω_1^M has *collapsed* (i.e., become a countable ordinal) in $M[G]$.

$Fn^{<\lambda}(\kappa, \lambda) = \{p \subset \kappa \times \lambda \mid p \text{ is a partial function and } |p| < \lambda\}$ collapses likewise κ to λ.

Lévy collapse: If κ is regular and λ is strongly inaccessible, then the Lévy collapse $\mathbb{L}(\lambda, \kappa)$ is the set of functions p on subsets of $\lambda \times \kappa$ with domain of size , κ and with $p(\alpha, \xi) < \alpha$ for every (α, ξ) in $\operatorname{dom}(p)$. This poset collapses all cardinals less than λ onto κ, but keeps λ as the successor of κ.

11 Controlling the forcing relation $\Vdash$

11.1 The forcing relation $\Vdash$

Also, $o(M[G]) = o(M)$: clearly, since $M \subset M[G]$, $o(M) \leq o(M[G])$. Since $\forall \tau \in M^{\mathbb{P}} (\operatorname{ran}(\tau_G) \leq \operatorname{ran}(\tau))$, we also have $o(M[G]) \leq o(M)$.

Pairs and Unions: given $\sigma, \tau \in V^{\mathbb{P}}$, let $\operatorname{pair}(\sigma, \tau) = \{\langle \sigma, \mathbb{1} \rangle, \langle \tau, \mathbb{1} \rangle\}$. Then $\operatorname{pair}(\sigma, \tau)_G = \{\sigma_G, \tau_G\}$. Let $\operatorname{ordpair}(\sigma, \tau) = \operatorname{pair}(\operatorname{pair}(\sigma, \sigma), \operatorname{pair}(\sigma, \tau))$. Then $\operatorname{ordpair}(\sigma, \tau)_G = \langle \sigma_G, \tau_G \rangle$. Therefore, since $M[G]$ is transitive and we have these constructions, $M[G] \models$ **Foundation**, **Extensionality**, **Pairing**. Recall $\tau_1 \cup \tau_2$ names the union: $(\tau_1 \cup \tau_2)_G = (\tau_1)_G \cup (\tau_2)_G$. For intersections, *the same idea does not work:* maybe $\tau_1 = \{\langle \sigma_1, \mathbb{1} \rangle\}$, $\tau_2 = \{\langle \sigma_2, \mathbb{1} \rangle\}$; it may happen that $\sigma_1 \neq \sigma_2$ but $(\sigma_1)_G = (\sigma_2)_G$; in this case $\tau_1 \cap \tau_2 = \emptyset$ yet $(\tau_1)_G \cap (\tau_2)_G \neq \emptyset \ldots$

"Living in M", we may still reason about objects in $M[G]$, using forcing:

> $p \Vdash \varphi$ iff $\forall G$ $\mathbb{P}$-generic over M, if $p \in G$ then $M[G] \models \varphi$.

If φ is logically valid, then $p \Vdash \varphi$ for all φ, and there exists no p such that $p \Vdash \neg\varphi$, by the existence of generic sets.

If $p \Vdash \varphi$ and $q \leq p$ then $q \Vdash \varphi$, as filters are upward closed. The strongest assertion is then $\mathbb{1} \Vdash \varphi$ (equivalent to all $p \Vdash \varphi$).

11.2 Truth Lemma, Definability Lemma. $M[G] \models ZFC$

The **two fundamental tools** are the **Definability Lemma (DL)** and the **Truth Lemma (TL)**:

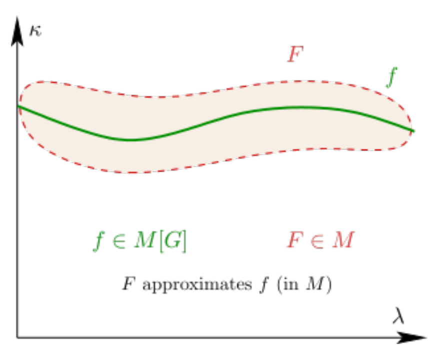
DL: $\{\langle p, \vec{\tau}\rangle \mid p \Vdash \varphi(\vec{\tau})\}$ is definable in M, for each formula $\varphi(\vec{x})$ in $\mathcal{L}_\in$.
TL: For each sentence in $\mathcal{L}_\in$, $\quad M[G] \models \varphi \Leftrightarrow \exists p \in G\,(p \Vdash \varphi)$.

With this, we prove that $M[G] \models$ **Comprehension** (in class). We also prove that $M[G] \models AC$.

11.3 Approximation Lemma (for *ccc* forcings). *ccc* forcings preserve all cardinals

A crucial point of Cohen's construction of a model where CH fails is that his forcing *preserves cardinals*. More precisely, if $(\mathbb{P} \text{ is } ccc)^M$ then $\forall \alpha < o(M)\left(\omega_\alpha^M = \omega_\alpha^{M[G]}\right)$.

We prove much more (using the following Approximation Lemma): *ccc* forcings preserve cardinals, regularity, weak inaccessibiity, weak Mahloness, etc.



Lemma 11.1 (Approximation Lemma for *ccc* forcings). Let $\mathbb{P}$ be *ccc* (in M) and let (in $M[G]$) $f : \lambda \to \kappa$ where λ, κ are ordinals. Then there exists $F \in M$, $F : \lambda \to [\kappa]^{\leq\omega}$ such that $\forall \alpha < \lambda\,(f(\alpha) \in F(\alpha))$.

12 Cohen and Random forcing

12.1 Bounds on cardinalities in $M[G]$

Counting subsets requires putting a bound on the number of names. Let $\tau \in M^{\mathbb{P}}$. A **nice name** of a subset of τ is a name of the form

$$\bigcup_{\pi \in \operatorname{dom} \tau} \{\pi\} \times A_\pi$$

where A_π is an antichain of $\mathbb{P}$.

Crucial Lemma:

If, in $M[G]$, $A \subseteq \tau_G$, then $A = \sigma_G$ for some nice name σ of a subset of τ.

12.1.1 $M[G] \models$ Powerset. Nice names

A Corollary is that $M[G] \models$ **Powerset**. To prove the crucial lemma, we show that if $A \subseteq \mathbb{P}$ is an antichain, $A \in M$, G is $\mathbb{P}$-generic over M, then **either** $G \cap A \neq \emptyset$ **or** $\exists q \in G(q \perp A)$. So, if A is a maximal antichain, $A \cap G \neq \emptyset$.

With this, we can prove that if in M,$\mathbb{P}$ is *ccc*, $|\mathbb{P}| = \kappa \geq \aleph_0$ and $\theta = \kappa^{\aleph_0}$ then $M[G] \models 2^{\aleph_0} \leq \theta$. So, if $\mathbb{P} = Fn(\kappa, 2)$, we have $M[G] \models 2^{\aleph_0} = \kappa$.

12.2 Around the proof of DL and TL

We define in M a relation $p \Vdash^* \varphi$ by recursion on φ and then we verify that $\Vdash = \Vdash^*$ and TL, recursively.

The definition of $\Vdash^*$ on atomic formulas is well-founded (see Kunen, ex. VII, B3)[1]. We define $p \Vdash^* \varphi \wedge \psi \Leftrightarrow p \Vdash^* \varphi$ and $p \Vdash^* \psi$, $p \Vdash^* \forall x \varphi(x) \Leftrightarrow$ for all names τ, $p \Vdash^* \varphi(\tau)$, $p \Vdash^* \neg\varphi \Leftrightarrow \neg\exists q \leq p\,(q \Vdash^* \varphi)$.

For each sentence $\varphi \in \mathcal{L}(M^{\mathbb{P}})$, for each $p \in \mathbb{P}$,

- $p \Vdash \varphi$ iff $(p \Vdash^* \varphi)^M$,
- (LV) for each G generic, $M[G] \models \varphi \Leftrightarrow \exists p \in G\,(p \Vdash \varphi)$.

12.3 Cohen and Random Forcing

${}^{\kappa}2$ is a compact Hausdorff topological space (product space of κ copies of 2). Basic clopens have finite support: $s \in Fn(\kappa, 2) \to K_s = \{f \in {}^{\kappa}2 \mid f \supset s\}$. Let $\mathcal{B}({}^{\kappa}2)$ be the *Baire* subsets: the minimal σ-algebra containing the clopens. **Forcing with $Fn(\kappa, 2)$ is equivalent to forcing with the partial order** $\mathbb{P}_c^{\kappa} = \{p \in \mathcal{B}({}^{\kappa}2) \mid p \text{ is not meager}\}, \leq = \subset$.

Random forcing: $\mathbb{P}_r^{\kappa} = \{p \in \mathcal{B}({}^{\kappa}2) \mid \mu(p) > 0\}, \leq = \subset$, where μ is the product measure of the "coin throwing" measure.

In M, let κ be uncountable. Extending using $\mathbb{P}_c^{\kappa}$ or $\mathbb{P}_r^{\kappa}$ yields $2^{\aleph_0} = \kappa$. However, mathematical properties will be different:

- **Random reals:**
  - $\forall X \subseteq \mathbb{R}\,(|X| < \kappa \to X \text{ is meager})$,
  - $\exists X \subseteq \mathbb{R}\,(|X| = \omega_1 \wedge X \text{ is not null})$.
- **Cohen reals:**
  - $\forall X \subseteq \mathbb{R}\,(|X| < \kappa \to X \text{ is null})$,
  - $\exists X \subseteq \mathbb{R}\,(|X| = \omega_1 \wedge X \text{ is not meager})$.

A great source for this measure/topology duality is Kunen's *Random and Cohen reals*, in the **Handbook of Set Theoretic Topology.**

[1] $p \Vdash^* \tau_1 = \tau_2 \Leftrightarrow \forall q \leq p \forall \sigma \in \operatorname{dom} \tau_1 \cup \tau_2\,(q \Vdash^* \sigma \in \tau_1 \Leftrightarrow \sigma \in \tau_2)$
$p \Vdash^* \sigma \in \tau \Leftrightarrow \forall q \leq p \exists r \leq q \exists \langle \pi, s\rangle \in \tau\,(r \leq s \wedge r \Vdash^* \pi = \sigma)$

13 Proper forcing

Last week: We defined **nice names** as a way to control upper bounds on cardinalities. We proved every subset of a given set in $M[G]$ has a nice name. We also discussed the (braided) induction in the proof of DL and TL (in detail: the $\neg$ step). Then, on Thursday, the lecture **Tra teoria dei modelli e teoria degli insiemi**.

13.1 Proper Forcing (the Second Forcing Revolution)

The Approximation Lemma, restated: if $\mathbb{P}$ is *ccc*, $A \subset On$ is countable (or of cardinality $\leq \kappa$) in $M[G]$ **then** there is a set $B \subset On$, in M, countable (or of cardinality $\leq \kappa$) in M, such that $\mathbb{1} \Vdash \tau \subseteq \check{B}$, for τ a name of A.

This implies that $\Vdash_{\mathbb{P}} \aleph_1^V$ is uncountable and hence $\aleph_1^V = \aleph_1^{V^{\mathbb{P}}}$.

Properness is **more general** than *ccc*, also more general than σ-closed, but still sufficient to preserve ω_1.

The key property of a proper forcing $\mathbb{P}$ is:

- if $p \Vdash \tau \subset On$ is countable, then there is $q \leq p$ and a countable set B such that $q \Vdash \tau \subseteq \check{B}$.
- if $\mathbb{P}$ is also $\aleph_2 - cc$, then $\mathbb{P}$ preserves all cardinalities and cofinalities.

Given $\mathbb{P}$, we say κ is *large enough for* $\mathbb{P}$ if the family of dense sets of $\mathbb{P}$ belongs to H_κ.

1. Given $\mathbb{P}$ and κ large enough for $\mathbb{P}$, given $N \prec H_\kappa$, the condition $p \in \mathbb{P}$ is N**-generic** if for each dense $D \in N$, for each $q \leq p$ there exists $r \in D \cap N$ such that $r \not\perp q$.
2. A forcing $\mathbb{P}$ is **proper** if for every κ large enough, for each countable $N \prec H_\kappa$ such that $\mathbb{P} \in N$, every condition $p \in \mathbb{P} \cap N$ has an N-generic extension.

- The antichain game $G_{ac}(\mathbb{P}, p)$: Player I plays a maximal antichain A_0 below p. Player II responds with a countable subset B_0^0. Then I plays again a maximal antichain A_1 above p and II responds with two countable sets $B_0^1 \subseteq A_0, B_1^1 \subseteq A_0$. In the n-th move, I plays a maximal antichain A_n above p and II responds with countable sets $B_0^n \subseteq A_0, \ldots, B_n^n \subset A_n$. After ω moves, II wins if there is a condition $q \leq p$ such that (for $B_n := \bigcup_{k \leq n} B_n^k$) for each n, the conditions in A_n compatible with q are all included in B_n.

- $\mathbb{P}$ is proper iff for all $p \in \mathbb{P}$ player II has a winning strategy in $G_{ac}(\mathbb{P}, p)$. So *ccc* forcings are proper!
- Why antichains? Not collapsing ω_1 makes us look at sequences $(\check{\alpha}_n \mid n \in \omega)$ of ordinals in $M[G]$. This is coded by antichains, this way: if $A \leq \mathbb{P}$ is a maximal antichain below p and $f : A \to On$ is a function, $\dot{\alpha} = \dot{\alpha}_{A,f} := \{(f(q), q) \mid q \in A\}$ is a name for an ordinal "below p" ($p \Vdash \dot{\alpha}_{A,f} \in On$) and for each $q \in A$, $q \Vdash \alpha = \dot{f}(q)$.

Pure combinatorics is replaced by better understanding of models of set theory; genericity is linked to them.

(Countable support) iterations of proper forcings are proper.

Iterated forcing: let $\mathbb{Q} \in M[G]$, where G is $\mathbb{P}$-generic. Then $\mathbb{P} * \mathbb{Q} = \left\{(p, \dot{q}) \mid p \in \mathbb{P} \wedge p \Vdash \dot{q} \in \dot{\mathbb{Q}}\right\}$.

Forcing with $\mathbb{P}$ to get $M[G]$, and then (in $M[G]$) with $\mathbb{Q}$ to get $M[G][H]$ can be seen from M as forcing with $\mathbb{P} * \mathbb{Q}$. The consistency of MA with ZFC requires *iterating* this idea continuum many times.

At some point in the lectures last week, I said (but did not provide detail):

> Properness is **more general** than *ccc*, also more general than σ-closed, but still sufficient to preserve ω_1.

The fact that properness is more general than *ccc* was clear from the definition and descriptions; I want to develop a bit more on σ-closedness and more in general on κ-closedness. Especially since in Miriam's presentation today, the σ-closure (also called $\aleph_0$-closure) of the forcing consisting of regular α-trees, for $\alpha < \omega_1$, was used (and was important to see that $\aleph_1$ was preserved).

A forcing $\mathbb{P}$ is κ-closed iff for every *descending* sequence of conditions $(p_i)_{i<\kappa}$ there exists a condition q such that $q < p_i$ for all $i < \kappa$. An important case is $\kappa = \aleph_0$; in this case we sometimes say σ-closed (for historical reasons).

The crucial fact is

κ-closed forcing preserves all cardinalities $\leq \kappa^+$.

This should be contrasted with **all κcc forcing preserves all cardinalities $\geq \kappa$** (exercise: adapt the proof of the Approximation Lemma for the weaker notion κcc and use it to show that cardinal arithmetic as witnessed by bijections between sets of size $\geq \kappa$ is not affected by the forcing).

I now prove that $\aleph_0$-closed forcing preserves $\aleph_1$. The crucial point is to notice that

if $A \in M[G]$ is a countable set (in $M[G]$) consisting of elements from M, then $A \in M$.

This clearly implies the preservation of $\aleph_1$, since **no new countable sets consisting of elements of M are added**; if $\aleph_1$ was collapsed, a bijection witnessing this would be a countable set in $M[G]$ consisting of elements of M, and this is impossible.

So let $A \in M[G]$ be countable in $M[G]$, and let $f \in M[G]$ enumerate A: $A = \{f(0), f(1), \ldots, f(n)\}$. Since $M[G] \models f : \omega \to A$, the Truth Lemma provides a condition p such that $p \Vdash \dot{f} : \check{\omega} \to \dot{A}$. This still is far away from allowing us to conclude $f \in M$. But in $M[G]$ we have $f(0) = a_0$ (where a_0 is the first element enumerated). Then there is (again, by the Truth Lemma) some $p_0 \leq p$ such that $p_0 \Vdash \dot{f}(\check{0}) = \check{a_0}$. We can continue this way by induction and get

$$\cdots \leq p_{n+1} \leq p_n \leq \cdots \leq p_1 \leq p_0 \leq p$$

such that $p_n \Vdash \dot{f}(\check{n}) = \check{a_n}$ (we usually say "p_n decides the correct value of f at n").

Now, by the σ-closure of $\mathbb{P}$, there is q such that $q \leq p_n$ for all $n < \omega$.

By the monotonicity of forcing, $q \Vdash \dot{f}(\check{n}) = \check{a_n}$ for all $n < \omega$. But then, by the definability lemma, using q in M we may decode the whole enumeration and therefore $f \in M$... and thus $A \in M$.

This implies that **no new countable subsets** (consisting of old elements) are added when forcing with $\mathbb{P}$ and therefore $\aleph_1$ is preserved.

The exercise for you is to generalize this proof to κ-closed. If now $\mathbb{P}$ is κ-closed, **no new subsets of size $\leq \kappa$ consisting of old elements are added**, and therefore all cardinalities up to κ are preserved!

Exercises

1. (Rothberger) Assume that $\mathbb{R}$ is not a union of κ many null sets. Prove that every subset of $\mathbb{R}$ of cardinality κ is meager. *Hint:* the exact definitions of "meager" and "null" are not essential to this problem. It is enough to use their invariance under translation and that $\mathbb{R} = M \cup N$ where M is meager and N is null.
2. Describe which ZFC axioms hold in the following models (concentrate on what you consider is most interesting in each case):
   a. $\langle 1, \in \rangle$,
   b. $\langle \omega, \in \rangle$,
   c. $\langle \{a\}, E \rangle$, where aEa,
   d. $\langle \omega, F \rangle$, where nFm iff the n-th position in m's binary expansion is 1.
3. This exercise (adapted from Kunen) shows what can be and what cannot be done with recursion.
   We begin by building a formal language with a constant symbol for each set. One way of achieving this is the following. Let
   a. *constant symbols:* $\langle diamond, x \rangle$ for each set x. One can abbreviate $\langle diamond, x \rangle$ as $^{\diamond}x$.
   b. *variables:* $\langle var, n \rangle$, where n is a natural number. We use v_n as a synonym of $\langle var, n \rangle$.
   c. *terms:* variable or constant symbols.
   d. *atomic formulas:* any object of the form $\langle \in, x, y \rangle$ o $\langle =, x, y \rangle$, with x and y terms.
   e. *formulas $fmla$:* atomic formulas, or any object of the form $\langle \vee, \varphi, \psi \rangle$, or $\langle \neg, \varphi \rangle$, or $\langle \exists, x, \varphi \rangle$, where φ, ψ are formulas and x is a variable.

You may treat $diamond$, $\in$, $=$, $\vee$, $\neg$, $\exists$, var as if they were other names of $0, 1, 2, 3, 4, 5, 6$ respectively. $\langle x, y, z\rangle$ may be defined as $\langle x, \langle y, z\rangle\rangle$.

1. The first four notions were defined explicitely; however, the notion $fmla$ is defined recursively. Show how to formalize this as a definition by recursion on a well founded relation. **Which well-founded relation is used in the definition of syntactic notions such as $complexity(\varphi)$ and $freevar(\varphi)$?**
2. If M is a set, one may define $truth(\varphi, M) \in \{T, F\}$ for sentences φ that only use ${}^{\diamond}z$ for $z \in M$. This is just an instance of Tarski's definition of truth in model theory. **Which well-founded relation is used here?**
3. One would also like to define truth in the universe V. That is, define $truth(\varphi)$ recursively on φ. For instance,

   $$truth(\langle\exists, x, \varphi\rangle) = T \text{ iff } \exists z[truth(subst(\varphi, x, {}^{\diamond}z)) = T].$$

   Therefore, working in ZF, one may show $Con(ZF)$ by showing that $truth(\varphi) = T$ for every $\varphi \in ZF$, and conclude by Gödel that ZF is inconsistent. **What is wrong with the previous argument? Why does the definition of truth work for set models but not for proper class models?**
4. Show that $rank(\bigcup x)$ es $rank(x) - 1$ if $rank(x)$ is a successor, and $rank(x)$ if it is a limit.
5. Here is the Fraenkel-Mostowski method to obtain models where Foundation fails. Work in ZF, and assume that F is some (definable class) function that permutes V. Consider the model whose domain is V, with membership interpreted as E, where $xEy \Leftrightarrow x \in F(y)$. Prove that all axioms of ZF^- hold in (V, E). Now, choosing an appropriate F, make the model satisfy $\exists x(x = \{x\})$ (try e.g. $F(0) = 1$, $F(1) = 0$, and $F(x) = x$ for $x \notin 2$). **With some other F, get a model with a set a that is transitive and totally ordered by membership but is not an ordinal.**

4. Let $M_0 = On$ and $M_{n+1} = \mathcal{P}(M_n)$; let $M = \bigcup_{n\in\omega} M_n$. Prove that this definition makes sense even though all the M_n's are proper classes. Show that M is a transitive model of Z + "every well-order is isomorphic to a von Neumann ordinal" and that some instance of Replacement fails in M.

5. (AC) Show that for every infinite cardinal κ, $|H(\kappa)| = \sup\{2^\lambda | \lambda < \kappa\}$. *Hint: notice that every x may be recovered from the isomorphism type of $\in$ on $\in$ sobre $\{x\} \cup \operatorname{trcl}(x)$. If $|\operatorname{trcl}(x)| = \lambda$, then this isomorphism type may be represented by a relation on λ.*
6. (AC) Assume that $V_\alpha \prec V_\beta$, α is regular and $\alpha < \beta$. Prove that α is strongly inaccessible, and it is the αth strongly inaccessible.
7. Are there transitive models V and W such that $V \subset W$, $\wp(\omega) \cap V = \wp(\omega) \cap W$, but CH holds in one of the models and not in the other? (Analyze the two possibilities.)
8. (Kunen) Let κ be strongly inaccessible and α be the minimum ordinal such that $V_\alpha \equiv V_\kappa$. Show that V_α is not an elementary submodel of V_κ. *Hint: Notice that $Th(V_\kappa) \in V_{\omega+1}$.*
9. A subset C of κ is called **club** if C is **cl**osed and **u**n**b**ounded in κ. ["C is closed" means that every supremum of a bounded subset of C belongs to C]. Show that if κ is regular and uncountable, then there exists a club $C \subset \kappa$ such that $L_\alpha \prec L_\kappa$ for all $\alpha \in C$.
10. κ is **strongly Mahlo** iff κ is strongly inaccessible and every club $C \subset \kappa$ contains some regular cardinal. Show that if κ is strongly Mahlo, there exists some **regular** $\lambda < \kappa$ such that $V_\lambda \prec V_\kappa$.
11. κ is **weakly compact** iff given $A \subset V_\kappa$, there exists $N \supsetneq V_\kappa$ and $B \subset N$ such that $(V_\kappa, A, \in) \prec (N, B, \in)$. Prove that if κ is weakly compact, then κ is strongly hyperhyperhyperhyperhyperhyperhyperhyperhyperhyperhyperhyperhyperhyperMahlo[2].
12. Assume that $S \subset H_\kappa$, with $\kappa > \omega$, κ regular and $|S| \leq 2^{\aleph_0}$. Prove that there exists A such that $S \subset A \prec H_\kappa$, $|A| = 2^{\aleph_0}$, y $A^\omega \subset A$.
13. (R. Pol) Prove the following theorem due to Arkhangel'skiĭ, using elementary submodels.

    If X is a Lindelöf 1-countable Hausdorff space, then $|X| \leq 2^{\aleph_0}$.

    Let τ be the family of open sets of X. Let $S = \{X, \tau\}$, and let κ be large enough so that $S \in H_\kappa$. Let A be as in the previous problem. Show that $X \subseteq A$. You may start by checking that $X \cap A$ is closed (since in our situation, closures are determined by ω-sequences); then, if $p \in X \setminus A$, a countable cover of $X \cap A$ by sets in $\tau \restriction A$ would contradict that A is an elementary submodel of H_κ.

[2] κ is **strongly Mahlo** iff κ is strongly inaccessible and every club $C \subseteq \kappa$ has some regular cardinal. κ is **weakly Mahlo** iff κ is regular, uncountable, and every club $C \subseteq \kappa$ has some regular cardinal. κ is **strongly hyperMahlo** iff κ is strongly inaccessible and every club $C \subseteq \kappa$ has some Mahlo cardinal.

14. $\mathbb{P}$ has **precalibre** ω_1 iff given $\{p_\alpha \in \mathbb{P} \mid p \in \omega_1\}$, there exists $X \in [\omega_1]^{\omega_1}$ such that $\{p_\alpha \mid \alpha \in X\}$ has the f.i.p. (i.e. given $s \in [X]^{<\omega}$, the p_α with $\alpha \in s$ have a common extension)[3]. Prove that under MA_{ω_1} every *ccc* partial order has precalibre ω_1.

15. Let K_α for $\alpha < \omega_1$ be closed subsets of $[0,1]$ such that $\mu(K_\alpha) > 0.22$ (μ is the ordinary Lebesgue measure).

    (a) Assume MA_{ω_1}. Prove that there exists $X \in [\omega_1]^{\omega_1}$ such that $\mu(\bigcap_{\alpha \in X} K_\alpha) \geq 0.22$. *Hint:* Let $\mathbb{P} = \{p \in [\omega_1]^{<\omega} | \mu(\bigcap_{\alpha \in p}) > 0.22\}$. Show that $\mathbb{P}$ is *ccc* and apply the previous problem with $p_\alpha = \{\alpha\}$.

    (b) Show that the previous fails under CH. *Hint:* start by enumerating $[0,1]$ as an ω_1-sequence. Choose the sets K_α by making use of the enumeration.

16. Prove that the three following theories are equiconsistent:

    (a) $ZFC+$ there exists a strongly Mahlo cardinal.

    (b) $ZFC+$ there exists a weakly Mahlo cardinal $\kappa < \mathfrak{c}$.

    (c) $ZFC + \mathfrak{c}$ is weakly Mahlo.

    Hint: notice that every weakly Mahlo cardinal is strongly Mahlo in L. Then, show that weakly Mahlo cardinals are still weakly Mahlo in *ccc* extensions. Show first that if $C \subseteq \kappa$ is a club in $M[G]$, then there exists $C' \subseteq \kappa$ a club in M such that $C' \subseteq C$.

17. Let M be a ctm of ZFC. Build a sequence $\langle M_n | n < \omega \rangle$ such that

    (a) $M_0 = M$.

    (b) $M_{n+1} = M_n[G_n]$ for some $\mathbb{P}_n \in M_n$ and some G_n that is $\mathbb{P}_n$-generic over M_n.

    (c) There is no transitive model N such that $o(N) = o(M)$ and $\bigcup_n M_n \subseteq N$.

    Hint: do collapse extensions.

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[3]For example, when we proved that $Fn(I,2)$ is *ccc*, we really proved it has **precalibre** ω_1.

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A ZFC vs HoTT: a crisis in foundations?

A.1 New foundations?

An apparent crisis... again?

At the beginning of the 20th century, the crisis in foundations of mathematics opened intense avenues of discussion inside the mathematical community. Russell's paradox generated serious interest among mathematicians and philosophers.

A century later: mathematical logic is a full-fledged branch of mathematics, extremely sophisticated (four main subareas, problems in interaction with noncommutative geometry, mathematical physics, etc.).

To most logicians, foundations have been relegated to an absolute background.

In 2006 a new crisis is looming, after Voevodsky's work.

Voevodsky - A story of complicated back-and-forths

Voevodsky (1966-2017) in his short life received many accolades: Fields Medal in 2002 (age 36), for his new cohomology theories for algebraic varieties. He proved conjectures by Milnor and Bloch-Kato, connecting K-theoretic groups of fields, and Galois cohomology.

From the IAS (Princeton) obituary: "He had a deep understanding of classical homotopy theory, where the objects considered are flexible, meaning **continuous deformations are neglected**, and was able to transpose..."

Voevodsky - A story of complicated back-and-forths

- 1990: Voevodsky & Kapranov: "∞-Groupoids as a Model for a Homotopy Theory" - they applied his ideas to motivic cohomology
- 1992: "Cohomological Theory of Presheaves with Transfers" - the base of later work with Suslin, Friedlander, etc.
- ...
- 1999-2000: lecture series at the IAS: he identified a mistake in a key lemma from his 1992 paper - more problematic situations
- ...
- 2009: univalent models ... computational verification

Two different kinds of pathologies: in Objects... The previous crisis, a century ago, was mainly a crisis on the (ontological) status of mathematical **objects:**

- What are real numbers? What's their cardinality, really?
- What are sets, really? - Russell's Paradox
- How do we "anchor" the reals, the continuum, on the natural numbers?
- What's the nature of Cantor's infinities?
- How do we base the rest of mathematics on set theory?

Voevodsky unveiled a crisis of mathematical **proofs,** rather than objects!
Two different kinds of pathologies: in Proofs...
Voevodsky unveiled a crisis of mathematical **proofs,** rather than objects!

- How do we weed out a wrong lemma?
- How to reduce our creation to something "programmable" - so as to avoid mistakes?
- How to ground mathematics (at least, the part close to Voevodsky's work) in a different way, close to the two previous questions?

A possible reaction

ZFC $\rightarrow$ Univalent Foundations (UF)

Set Theory, as seen from outside In 2016, during an event in Bielefeld where ZFC was being compared with UF. Set Theorist Mirna Džamonja summarized the view from outside of set theory:

- ZFC axioms plus possible the existence of large cardinals
- Important for foundations of mathematics, since many classical notions may be axiomatized in set theory and can be represented in von Neumann's cumulative hierarchy.
- Hilbert thought that all mathematics could be formulated in basic set theory...

In a volume from 2015 celebrating the work of the logician Jouko Väänänen, his colleague Roman Kossak summarized the absolute success of ZFC: **a measure of the success of these foundations is that mathematicians do not care about these matters anymore**. (I would add mathematical logicians to Kossak's claim!)

Set Theory, seen from Within

- By Gödel's Incompleteness Theorem, it is better to concentrate on what *can be done* in ZFC and study what cannot, through independence proofs (Shelah, *The Future of Set Theory*)
- Continue the search for "natural" axioms in addition to ZFC, with the hope of proving questions such as CH, etc. (Gödel's program, the California Set Theory school, Woodin, etc.)
- Use axioms such as $V = L$, etc. (too restrictive)
- Forcing Axioms (MA, MM, PFA, etc.) - Magidor, Viale, etc.
- Look for structural features of the continuum through detailed combinatorial analysis, or pcf theory, or careful use of ultraproducts, etc.

How crises unfold, and whence they come

We may (as mathematicians!) look at the kind of questions that gave rise to the crisis of a century ago and what kind of responses appeared (and keep appearing):

- In Set Theory, the role of the Continuum Problem and some of its answers.
- In HoTT+UF, the role of Grothendieck's question on deformations, deformations of deformations, etc.

A.2 The first crisis: Why ZFC? What does ZFC achieve?

A.2.1 The continuum problem - properties of the reals

Cantor - Hilbert - Gödel

Cantor 1878: Does there exist an infinite, uncountable $A \subset \mathbb{R}$, not in bijection with $\mathbb{R}$?

$$2^{\aleph_0} \stackrel{?}{=} \aleph_1$$

Hilbert 1900: First problem: prove (or refute) the Continuum Hypothesis.

Gödel 1940: The Continuum Hypothesis cannot be refuted: in L, a "model of ZFC", the Continuum Hypothesis holds. Key point: the *rigidity* of L (*Condensation*).

Really capturing the reals
Only restriction *in ZFC* on the size of the continuum:

Kőnig $\mathrm{cf}\, 2^{\aleph_0} \neq \omega$.

And further structural questions:
Suslin: How can we characterize the structure $(\mathbb{R}, <)$?
More precisely: Which properties of an abstract ordering $(X, <)$ imply that $(X, <) \approx (\mathbb{R}, <)$?

$$\text{total dense order without endpoints} + \text{ Dedekind-complete?} \tag{1}$$

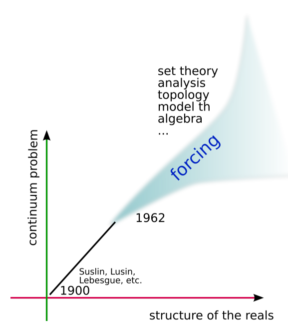
this is not enough! the "long line" (ω_1-many copies of the interval $]0, 1]$, with the usual order) also satisfies (1).

Theorem A.1. **(Cantor, 189x)** If $(X, <) \models (1)$ and there exists a countable dense $D \subset X$ such that $(D, <) \approx (\mathbb{Q}, <)$ then $(X, <) \approx (\mathbb{R}, <)$.

The two axes: cardinal vs structure

Suslin (1920): Is (1) + ccc sufficient?

Suslin's Hypothesis (SH): yes. But... SH is independent!

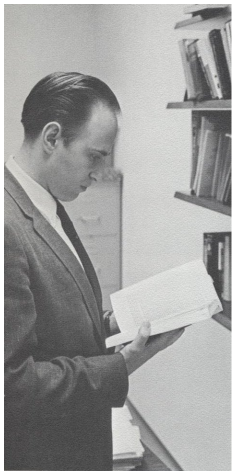


A.2.2 Cohen's times

Set Theory Zeitgeist c. 1960 / the wild sixties

- Dana Scott: No measurable cardinals in L.
- Azriel Lévy: $L(\mathbb{R})$ - relative constructibility.
- Alfred Tarski: model theory with a very strong structural emphasis in Berkeley. The "West Coast" style.
- Michael Morley: categoricity transfer - combinatorial set theory used in model theory (thesis supervisor: S. Mac Lane).

- Jerry Keisler: initial work in models of set theory and invariants of large cardinals.



THE INDEPENDENCE OF THE CONTINUUM HYPOTHESIS

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Communicated by Kurt Gödel, September 30, 1963

This is the first of two notes in which we outline a proof of the fact that the Continuum Hypothesis cannot be derived from the other axioms of set theory, including the Axiom of Choice. Since Gödel[3] has shown that the Continuum Hypothesis is consistent with these axioms, the independence of the hypothesis is thus established. We shall work with the usual axioms for Zermelo-Fraenkel set theory,[2] and by Z-F we shall denote these axioms without the Axiom of Choice, (but with the Axiom of Regularity). By a model for Z-F we shall always mean a collection of actual sets with the usual ϵ-relation satisfying Z-F. We use the standard definitions[3] for the set of integers ω, ordinal, and cardinal numbers.

THEOREM 1. *There are models for Z-F in which the following occur:*

(1) *There is a set a, $a \subseteq \omega$ such that a is not constructible in the sense of reference 3, yet the Axiom of Choice and the Generalized Continuum Hypothesis both hold.*

(2) *The continuum (i.e., $\mathcal{P}(\omega)$ where $\mathcal{P}$ means power set) has no well-ordering.*

(3) *The Axiom of Choice holds, but $\aleph_1 \neq 2^{\aleph_0}$.*

(4) *The Axiom of Choice for countable pairs of elements in $\mathcal{P}(\mathcal{P}(\omega))$ fails.*

Only part 3 will be discussed in this paper. In parts 1 and 3 the universe is well-ordered by a single definable relation. Note that 4 implies that there is no simple

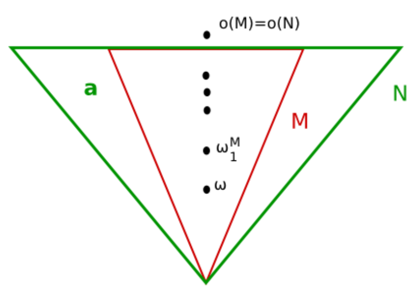
"There were no techniques for constructing models of ZFC beyond L and levels of the von Neumann hierarchy!" **Paul Cohen (1934-2007)**

Here is a description on Cohen's achievement: *"(Cohen) had done work that should long outlast our times. For mathematical logic, and the broader culture that surrounds it, his name belongs with that of Gödel. Nothing more dramatic than their work has happened in the history of the subject."* **Angus MacIntyre**

A.2.3 Cohen's forcing: Expanding the universe?

Adding reals - adjoining roots How to obtain $N \models ZFC + \neg CH$?

Let M be a countable transitive model of ZFC. In principle, we could have $M \models CH$ (as maybe…so far…$ZFC \vdash CH$). ω_1^M, ω_2^M, etc. are countable ordinals in V.



Adding reals or roots
In V:

♣ Fix $\vec{a} = \langle a_\xi | \xi < \omega_2^M \rangle$ a sequence of ω_2^M different reals (or subsets of ω, or elements of ${}^\omega 2$, which are not in M).

♣ Add $\vec{a}$ to M... to get $N = M[\vec{a}]$. Then

$$N \models c \geq \omega_2$$

(because of $\vec{a}$).

Problem with the previous naïve scheme: adding $\vec{a}$ to M and getting a model of ZFC, with the same ordinals and cardinals. Indeed, we need

$$\omega_2^N = \omega_2^M.$$

Try something like $N = M \cup \{\vec{a}\}$?

What does this model?

Not much! But we should certainly add whatever is constructible from $\vec{a}$. For example, we should have that $\{\xi | (a_\xi)^2 > 8\} \in N$.

Adding reals or roots

Allegory:

Pick a field (e.g. $\mathbb{Q}$ or $\mathbb{F}_n$)	Extend it to $\mathbb{Q}[\pi]$ or $\mathbb{Q}[\sqrt{2}]$	We don't have $\mathbb{Q}[\pi] = \mathbb{Q} \cup \{\pi\}$	Close it under...
Start with $M \models ZFC + CH$	Add $\aleph_2$ new reals ($\vec{a}$)	We are *far* from a model of ZFC	What does that mean in our case?

Same ordinals: $M \cap On = N \cap On$.

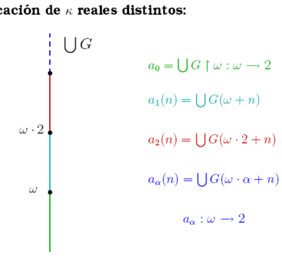
Same cardinals: if $\alpha \in M \cap On$ and (α is a cardinal)^M then (α is a cardinal)^N. In N: $\neg \exists \beta < \alpha (f : \beta \xrightarrow{onto} \alpha)$.

Adding κ-many different reals To get $N \models \neg CH$, we use $\mathbb{P} = Fn(\aleph_2^M, 2)$. So, $N \models 2^{\aleph_0} \geq \aleph_2$.

As ($\mathbb{P}$ is ccc)^M, N and M have the same cardinals.

$\bigcup G$ codifies a κ-sequence of different reals $\vec{a}_G$.

La codificación de κ reales distintos:



Why are all those $\aleph_2$ reals different? By genericity: if $\alpha < \beta < \aleph_2$, the set

$$E_{\alpha\beta} = \{p \in \mathbb{P} | \exists n [p(\omega \cdot \alpha + n) \neq$$

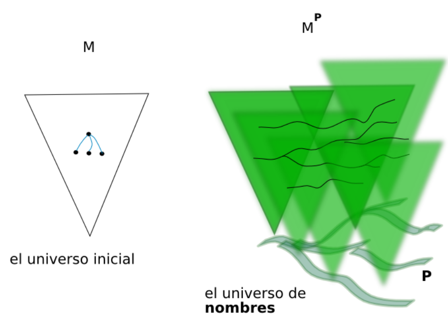
$$p(\omega \cdot \beta + n) \wedge \omega \cdot \alpha + n, \omega \cdot \beta + n \in \text{ dom } (p)]\}$$

is dense.

A.2.4 Names (sections) - control of the generic

The extension must be generic Cohen, on genericity: *thus* **a** *must have certain special properties…Rather than describe* **a** *directly, it is better to examine the various properties of* **a** *and determine which are desirable and which are not. The chief point is that we do not wish* **a** *to contain "special" information about M, which can only be seen from the outside…The* **a** *which we construct will be referred to as a "generic" set relative to M. The idea is that all the properties of* **a** *must be "forced" to hold merely on the basis that* **a** *behaves like a "generic" set in M. This concept of deciding when a statement about* **a** *is "forced" to hold is the key point of the construction.*

The universe of names $M^{\mathbb{P}}$

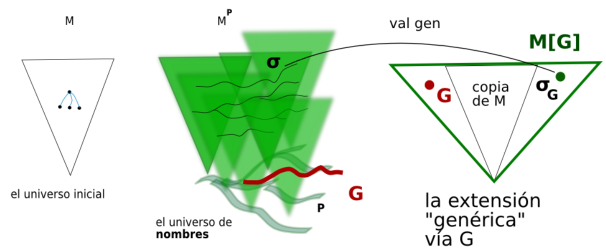


Definition A.2 ($\mathbb{P}$-names). τ is a $\mathbb{P}$-*name* iff τ is a relation such that if $\langle \sigma, q \rangle \in \tau$, σ is a $\mathbb{P}$-name and $q \in \mathbb{P}$. $V^{\mathbb{P}}$ is the class of all $\mathbb{P}$-names.

A.2.5 The Cohen model - control

The generic model $M[G]$ - two steps!

Definition A.3 (The generic extension). Given a filter G, a $\mathbb{P}$-name τ, $\tau_G = val(\tau, G) = \{\sigma_G | \langle \sigma, q \rangle \in \tau \wedge q \in G\}$.
$M[G] = \{\tau_G | \tau \in M, \tau \text{ is a } \mathbb{P}\text{-name}\}$.



Truth and Definibility Lemmas

How does one control what is forced? A larger part of the *technical* construction goes through the tools to control *in* M what is forced to hold in a model $M[G]$. This is not a trivial step: in the interesting cases, $G \notin M$.

The control of "what's forced" is achieved through a "forcing logic" very close to sheaves over partially ordered sets, and through a "Generic Model Theorem": the Definability Lemma roughly says that $\Vdash$ is definable in M.

($p \Vdash \varphi$ as a predicate of two variables can**not** be defined in M, for much the same reason that $M \models \varphi$ is not definable in M [Tarski]).

The Truth Lemma says exactly how truth $\models$ works in the generic extension $M[G]$.

Forcing theorems - iterated forcing and MA During the first years after Cohen's work, Solovay and Silver at Berkeley refined the theory and turned it into a tool of enormous power to build models of many interesting statements.

Additionally, Martin captured many questions of the early 20th century in terms of a unique "forcing axiom" (called Martin's Axiom), with many interesting consequences under the failure of the Continuum Hypothesis.

The idea to prove the Consistency of Martin's Axiom used *Iterated Forcing* (just as forcing but controlling the expansions of the universe after (transfinite) iterations of forcing). This is not trivial: at limit stages many objects may appear that could alter seriously the properties of the universes involved.

Some additional directions

- (Kennedy, Magidor, Väänänen - current work): Logics in between first and second order, with generalized quantifiers, and their connection with inner models of set theory.
- (Woodin, 1999) Ω-logic (this would imply that CH is false)
- (Woodin, 2010) Ultimate-L (if it exists, CH is true)

- (Dzamonja, Väänänen - currently) $\beth_\omega$-compactness of the "chain" logic $L^{ch}_{\kappa,\kappa}$ with κ strong limit singular - this would imply the Singular Cardinal Hypothesis (Shelah) for those cardinals.
- (Shelah, 1995) A theory that allows *in ZFC* to provide robust answer to the Continuum Hypothesis, with a flavor akin to localizations in number theory: **pcf** theory.
- (Väänänen, Villaveces - current work): Logics in between first and second order, associated to singular cardinal and some control of interpolation.

A.3 HoTT + UF (Homotopy Type Theory + Univalent Foundations)

A problem due to Grothendieck Shulman motivated the subject of Homotopy Type Theory via this Grothendieck problem:

> ...the study of n-truncated homotopy types (of semisimplicial sets, or of topological spaces) [should be] essentially equivalent to the study of so-called n-groupoids. . . . This is expected to be achieved by associating to any space (say) X its "fundamental n-groupoid" $\prod_n(X)$. The obvious idea is that 0-objects of $\prod_n(X)$ should be the points of X, 1-objects should be "homotopies" or paths between points, 2-objects should be homotopies between 1-objects, etc.

Grothendieck, 1983

A.3.1 Type Theories

Types vs sets - Extensionality

Recall the Axiom of Extensionality of set theory:

$$\textbf{Ext: } \forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \to x = y)$$

The role of **equality** is in some sense over-defined. Mixed with the Axiom of Choice and the Excluded Middle Principle, working in ZFC, we get the extremely powerful theory we use in a large part of mathematics...

but we (seem to) lose the possibility of thinking equality as equivalence.

We thus *mainly* abandon the idea of extensionality.

Dependent Type Theory (Martin-Löf)

We thus *mainly* abandon the idea of extensionality…

and we replace first order logic by a type theory. The origins date back to Russell, but in the 1960s Per Martin-Löf constructed (for reasons that were back then linked to the study of randomness and probability) his **dependent type theory**. This may also be regarded (de Bruijn) as a computational language:

Basic expressions (contexts) are of the form

term : Type

for example, $x : \mathbb{N}$ o $y : \mathbb{R}$...

Dependent Type Theory (Martin-Löf)

Basic expressions (contexts) are of the form

term : Type

for example, $x : \mathbb{N}$ o $y : \mathbb{R}$...

And from contexts we may pass on to *proofs* or **judgments** according to certain rules:

$$x : \mathbb{N}, y : \mathbb{R} \vdash (5x^2 + 2, x \times y) : \mathbb{N} \times \mathbb{R}$$

Classifying Category

We may integrate all of these items into the (so called) "Classifying" Category (of contexts) **Ctx**. In this category:

- Objects are contexts, and
- Morphisms are **judgments** $\vdash$.

$$\Gamma \vdash a : A$$

This looks like…a *deduction* ???

Dependence

When we write

$$\Gamma \vdash a : A$$

we think of a as a morphism from Γ into A.

And when writing

$$\Gamma \vdash B \quad \textbf{type}$$

we think of B as a family of contexts. In this sense

- Contexts with variables are **types** and
- those with no variables (after substitution by constants) are **morphisms**.

Dependent Type Theory centers on **how** to treat those substitutions.

Deductions - A Proof System There are several other rules and *constructions*:

Type Format	Notation	(special case)
Inhabitant	$a : A$	
Dependent type	$x : A \vdash B(x)$	
Sigma (sum)	$\sum_{(x:A)} B(x)$	$A \times B$
Pi (product)	$\prod_{(x:A)B(x)}$	$A \to B$
Coproduct	$A + B$	
Identity	$Id_A(a,b), a = b$	
Universe	U	
Base	Nat, Bool, 1, 0	

and extensionality axioms on morphismos...

$$\frac{\Gamma \vdash A \quad \mathbf{type}; \Gamma; n : A \vdash B \quad \mathbf{type}}{\Gamma \vdash \prod_{(n:a)} B \vdash \mathbf{type}}$$

Connection with propositional logic and sets

The format numerator (premise) / denominator (conclusion) is very close to formal languages and was implemented for proof assistants. The most famous of them in this context is **Coq** (No excluded middle!)

Level	1	2	3	$\cdots$	n	$\cdots$
Type	V/F	Set	Morph	$\cdots$	n-types	$\cdots$

In this sense type theory could seem to be a generalization of set theory, but this is NOT the case (set theory does not merely consist of its objects but also of axioms, and first order logic - Džamonja).

A.3.2 Univalence - Synthetic and Analytic

Identity - non-identity - ¿two identities? In Martin-Löf's theory the type $Id_A(x,y)$ captures the idea that **the propositions** x and y are equivalent. Therefore we may *prove* their equivalence in the system.

All this opens the way to **two different kinds** of identity:

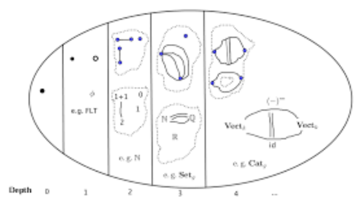
- **Definitional** Identity $A = B$ (types) and $x = y : A$ for objects of a given type, and
- **Propositional** Identity $Id_A(x,y)$.

Clearly, definitional identity implies propositional identity; the converse usually does not hold.

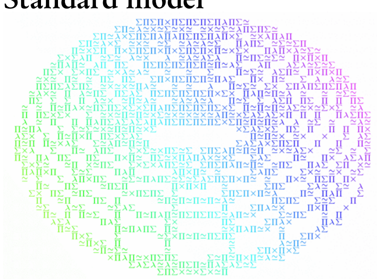
Identity vs deformation

- Martin-Löf: some ways of perceiving the problem of the two identities add further judgments but may have Russell-like paradoxes (Girard)…
- enter Voevodsky with his Univalence Axiom: $Id = Eq$.

Building carefully the Univalence Axiom is quite delicate (a frequent reading of this is **the univalence type is inhabited**).



Standard model



In topology: ∞-groupoids The so-called ∞-groupoids of homotopy theory end up providing the first "standard model" of Hott+UF. Voevodsky basically

- Interprets the category **Ctx** as the homotopic category of (Kan) simplicial complexes,
- Types are then spaces and morphisms,
- But most importantly, propositional equality is now **exactly** *homotopic equivalence*, and the stratified structure is an ∞-groupoid.

Voevodsky's Theorem

Theorem A.4 (Voevodsky - 2012). If we assume the existence of two inaccessible cardinals, then it is consistent that the category $sSet/W$ forms a model of Martin-Löf's type theory together with the Univalence Axiom.

Notes:

- Two inaccessible cardinals is stronger than ZFC but very weak among axiomatics of set theory.
- In the proof he uses the Excluded Middle at the level of Propositions (1) and the Axiom of Choice at the level of Sets (2).

Some more recent results in synthetic homotopy theory The following facts of Algebraic Topology have been proved (and verified) in HoTT:

- $\pi_1(S^1) = \mathbb{Z}$ (Shulman, Licata)
- $\pi_k(S^n) = 0$ if $k < n$ (Brunerie, Licata)
- $\pi_n(S^n) = \mathbb{Z}$ (Brunerie, Licata)
- The exact sequence of a fibration (Voevodsky)
- The van Kampen Theorem (Shulman)
- Covering Spaces Theory (Hou)
- ...

An "internal" critical position: Lurie
Two critiques to the project:

- From outside: almost all theorems are algebraic topology - less so of algebraic geometry, and really very few are of other disciplines.
- More internally: Jacob Lurie in various posts shows scepticism –the sort of scepticism of one of the great specialists of Higher Topos Theory– and asks tough questions to HoTT proponents. For instance, computing the fundamental group of $S^1 \times S^1$ took them years of work.

A.3.3 Conclusions/Beginnings

Pluralist perspective... Džamonja's mathematical pluralist perspective may be summarized as follows:

- Univalent Foundations are really a foundation for the *constructive* part of mathematics –the key point was to note the connection between homotopy theory and type
- The use of proof assistants (Coq, Agda) may formalize an important part of mathematics, and verify proofs.

Pluralist perspective...

- (Voevodsky) HoTT+UF is consistent modulo the consistency of ZFC.

- Set Theory is still the important standard of consistency (Voevodsky, Logic Colloquium 2013).

Summarizing…

- **Set Theory:** foundations for an important part of mathematics in a format consistent with usual practice.
- **Category Theory:** a way of modelling parts of mathematics that depend on proper classes and where universal properties are essential –such as algebraic geometry.
- **Univalent Foundations:** a novel way of *discussing* proofs –obviously a central and very important topic.

Bourbaki - a "historical" vision of the rôle of ZFC
Bourbaki in the first volume of *Théorie des ensembles* says:

> We know that, logically speaking, it is possible to derive all current mathematics from a unique source, set theory. When doing this, we do not pretend to write in stone a law; some day may come when mathematicians will reason in a completely different way that is not formalizable in the language we adopt here, and according to some, recent progress in homology suggests that day is not far along. In that case we will have to enlarge the syntax, even if it is not necessary to completely change the language. The future of mathematics will decide.

Axiom Systems: Constitutions?

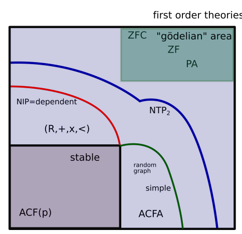
- Rami Grossberg (in private communications) has described ZFC as "a good constitution": it may perhaps last long but it does not have to be eternal.
- In light of the recent proofs and announcements (Mochizuki on the **abc** Conjecture in 2014, controversy with Scholze and Stix), and the lack of *informed consensus* on the matter among the mathematical community at large, Voevodsky's worry is particularly relevant.
- A the time of writing (2021) the mathematical community has no clarity of **where** the formalization of IUTT (Interuniversal Teichmüller Theory) takes place. Scholze and Stix's isolating a key lemma and reducing it to more classical algebraic geometry is a possibility, but the stakes seem open.

B Tra teoria dei modelli e teoria degli insiemi

Teoria dei modelli / teoria degli insiemi: Mura / ponti

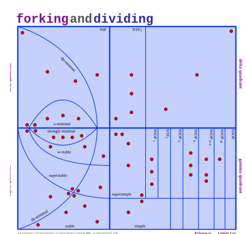
Prima del 1970: La teoria dei modelli stava diventando troppo "insiemistica" secondo qualche specialisti. . . (teoremi dei due cardinali - Morley, Chang, . . .)

Circa 1970: Shelah fa i primi passi verso la sua *teoria della stabilità*



"mappa" dell'universo (modellistico, in Primo Ordine)

Ovvero



(ved. forkinganddividing.com / G. Conant)

B.1 Teoria dei modelli (Categoricità, . . .)

B.1.1 Categoricità - Perché?

Łoś, Morley, Shelah...

All'inizio del secolo scorso, Steinitz dimostrò che «la geometria algebrica è categorica»: più precisamente, egli dimostrò che ogni coppia di corpi algebricamente chiusi con stessa caratteristica e la stessa cardinalità *devono* essere isomorfi.

Negli anni 1920 e 1930, Gödel, Carnap, Skolem, ... hanno studiato i risultati ben noti dell'incompletezza delle strutture «fisse» e la completezza quando si passa alle classi di strutture - e si abbandona l'idea di struttura fissa - la *categoricità* è emersa come una versione di completezza peculiare e molto speciale.

A metà degli anni 1950, basandosi su molte altre osservazioni, Łoś congetturò che ogni teoria *del primo ordine* in un vocabolario numerabile ha soltanto quattro tipi di spettro di categoricità:

$$\emptyset \qquad (\aleph_0) \qquad (> \aleph_0) \qquad (Card_\infty).$$

La congettura di Shelah (versione iniziale)

Un «test problem» centrale per un'ampia parte della teoria dei modelli fin dagli anni Novanta: trovare versioni del teorema de Morley e del teorema di *trasferimento di categoricità* di Shelah, per contesti più ampi: ad esempio, le *classi elementari astratte* (estensioni definite semanticamente della teoria dei modelli di $L_{\lambda^+,\omega}(Q), \ldots$).

Congettura B.1 (Shelah). Per ogni cardinale λ, esiste un μ_λ tale che se ψ è un enunciato della logica $L_{\omega_1,\omega}$ che soddisfa un teorema di «Löwenheim-Skolem» verso il basso fino a λ e se anche è categorica in *qualche* cardinale $\geq \mu_\lambda$, allora è categorica in *tutti* i cardinali oltre μ_λ.

B.1.2 Altre regioni della "mappa" del universo

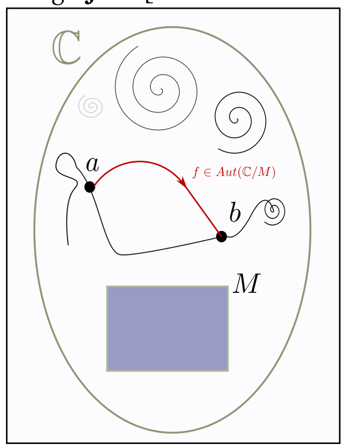
Classi Elementari Astratte

Fissiamo un vocabolario τ.
Sia $\mathcal{K}$ una classe di τ-strutture, $\prec=$ una relazione binaria su $\mathcal{K}$.

Definizione B.2. Diciamo che $(\mathcal{K},)$ è una *classe elementare astratta* (AEC) se

- $\mathcal{K}$, sono **chiuse per isomorfismi**,
- $M, N \in \mathcal{K}, MN \Rightarrow M \subset N$,
- è un ordine parziale,
- (**Tarski-Vaught**) $M \subset N\bar{N}, M\bar{N} \Rightarrow MN$, e…
- (**↘LS**) $\exists\kappa = LS(\mathcal{K}) \geq \aleph_0$ tale che $\forall M \in \mathcal{K}$, $\forall A \subset |M|$, $\exists NM$ con $A \subset |N|$ e $\|N\| \leq |A| + LS(\mathcal{K})$,
- (**Unioni di -catene**) Un'unione di una -catena in $\mathcal{K}$ appartiene a $\mathcal{K}$, è una -estensione di tutti i modelli della catena e risulta essere anche l'estremo superiore della catena.

I tipi di Galois (anche chiamati «tipi orbitali»)

La corretta nozione di *tipo* in una AEC (con le proprietà di amalgamazione **AP** e di «joint embedding» **JEP** [immersioni congiunte], senza modelli massimali (**NMM**)) è: 

1. Per le proprietà AP, JEP e NMM, è possibile la costruzione di un «monster model», un modello mostro (universale, modello-omogeneo) $\mathbb{C}$ nella classe.
2. Poi, definiamo $ga{-}tp(a/M) = ga{-}tp(b/M)$ sse esiste $f \in Aut(\mathbb{C}/M)$ tale che $f(a) = b$.
3. Poi, dichiariamo (anche sotto le ipotesi aggiuntive AP, JEP, NMM) che i **tipi di Galois** su M sono le *orbite* dell'azione del gruppo $Aut_M(\mathbb{C})$ (gli automorfismi del mostro $\mathbb{C}$ che fissano M *puntualmente*).
4. Possiamo dimostrare che questo generalizza la nozione (sintattica) di tipo. Inoltre, notiamo che esiste una nozione di «tipo di Galois» in situazioni molto più generali (senza AP, JEP o NMM); tuttavia, la loro definizione è meno diretta, siccome non necessariamente esiste in quei casi un «modello mostro».

Congettura di Categoricità di Shelah

- Un problema centrale nella teoria dei modelli delle Classi Elementari Astratte (AEC): dimostrare versioni del Teorema di Morley (Congettura di Łoś) per AEC - *Trasferire la Categoricità*.
- "Versioni semantiche" di teoria dei modelli di $L_{\lambda^+,\omega}(Q)$.

Congettura B.3 (La stessa Congettura di Shelah, riformulata intorno al 1980 nel contesto allora nuovo di AEC). Per ogni λ, esiste μ_λ tale che **se** $\mathcal{K}$ è una AEC con $LS(\mathcal{K}) = \lambda$, categorica in *qualche* cardinale $\geq \mu_\lambda$, **allora** $\mathcal{K}$ è categorica in *tutte le* cardinalità oltre μ_λ.

B.1.3 Cronologia della dimostrazione

Cronologia della dimostrazione (ca. 1980 a 2015)

- Il problema è aperto per frammenti numerabili di $L_{\omega_1,\omega}$ (il problema originale, anni 1970). Qui, la
- congettura è specificamente che $\mu_{\aleph_0} = \beth_{\omega_1}$. Shelah, Jarden, Grossberg, Vasey hanno dei risultati parziali.
- Makkai-Shelah (1985): vale la Congettura per classi assiomatizzate in $L_{\kappa,\omega}$ per κ **fortemente compatto**.
- Kolman-Shelah (c. 1990): categoricità «all'ingiù» per classi assiomatizzate in $L_{\kappa,\omega}$ per κ **misurabile**.
- Boney (2013) consistenza dell'intera congettura, sotto ipotesi dell'esistenza di una classe propria di cardinali fortemente compatti. Qualche risultato adizionale di Vasey (più recente - forking per AEC).

B.2 Cardinali fortemente compatti, tameness

B.2.1 Localizzare tipi

Grossberg e VanDieren: la docilità viene isolata

Intorno all'anno 2000 Grossberg e VanDieren hanno dimostrato il seguente

Teorema 1. Sia $\mathcal{K}$ una AEC con AP, JEP e senza modelli massimali (NMM). Allora
se $\mathcal{K}$ è χ-*docile* e λ^+-categorica per qualche $\lambda \geq LS(\mathcal{K})^+ + \chi$, anche $\mathcal{K}$ deve essere μ-categorica per tutti i $\mu \geq \lambda$.

La loro dimostrazione è fondata su una dimostraziona precedente di trasferimento «all'ingiù» di categoricità, da Shelah; G e VD hanno aggiunto un elemento cruciale, isolando la nozione di *docilità*, in inglese *tameness*, «sotterrata» nella dimostrazione di trasferimento «all'ingiù» da Shelah - estrarre la nozione permette a G e VD anche di dimostrare la categoricità «verso l'alto».

B.2.2 Docilità

Docilità: «localizzare la differenza» fra tipi

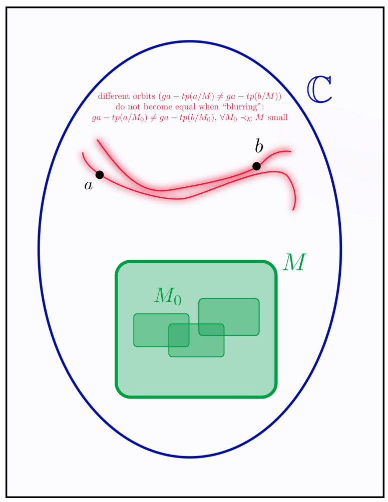
Idea: «localizzare» la condizione di...
estendere una funzione f che fissi un modello M in una AEC $\mathcal{K}$ fino a ottenere una $\mathcal{K}$-immersione:

- se non esiste immersione f che fissa M e invia qualche a sopra b allora abbiamo che

$$\mathrm{gatp}(a/M) \neq \mathrm{gatp}(b/M)$$

- *vogliamo*: localizzare questa richiesta per controllare che esiste un sottomodello $M_0 \preceq_{\mathcal{K}} M$ tale che

$$\mathrm{gatp}(a/M_0) \neq \mathrm{gatp}(b/M_0).$$



Ottenere la docilità da grandi cardinali

Nel 2013, W. Boney ha aperto una strada nuova per capire la congettura: perché non concentrarsi sull'impatto dei grandi cardinali sulla *docilità* o nozioni correlate?

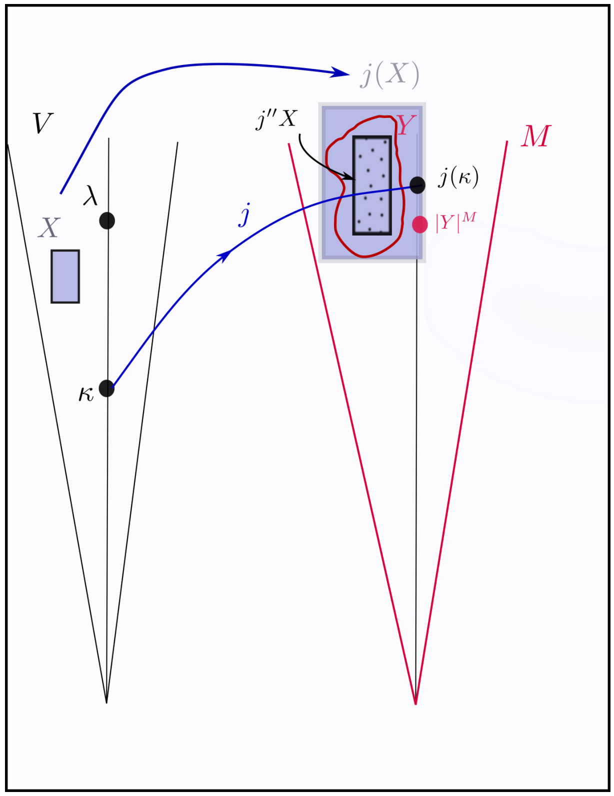
Teorema 2 (Boney). Se κ è fortemente compatto e $\mathcal{K}$ è essenzialmente sotto κ (i.e. $LS(\mathcal{K}) < \kappa$ ovvero $\mathcal{K} = Mod(\psi)$ per qualche $L_{\kappa,\omega}$-enunciato ψ) allora $\mathcal{K}$ è $(<\kappa, \kappa)$-docile.

La dimostrazione è piuttosto diretta, data la forza dell'ipotesi. Boney e Unger hanno anche dimostrato che sotto l'inaccessibilità forte di κ, la $(<\kappa, \kappa)$-docilità di tutte le AEC (quasi) implica la compattezza forte di κ.

B.2.3 La dimostrazione, leggermente riformulata

Riformuliamo la dimostrazione di Boney

Un cardinale κ è *fortemente compatto* sse per ogni $\lambda > \kappa$ esiste un'immersione elementare $j : V \to M$ con punto critico κ, ed esiste un insieme $Y \in M$ tale che $j''\lambda \subseteq Y$ e $|Y|^M < j(\kappa)$.



Definizione B.4. Sia $j : V \to M$ un'immersione elementare. Diciamo che j soddisfa la *proprietà di copertura* (κ, λ) se per ogni X tale che $|X| \leq \lambda$ esiste $Y \in M$ tale che $j''X \subseteq Y \subseteq j(X)$ e $|Y|^M < j(\kappa)$.

κ misurabile	j soddisfa la (κ, κ)-pc
κ λ-fortemente compatto	j soddisfa la (κ, λ)-pc

B.2.4 $j(\mathcal{K})$...

L'«immagine» di una AEC sotto $j : V \to M$

Sia $(\mathcal{K}, \prec_{\mathcal{K}})$ una AEC in τ.

Un teorema famoso di Shelah (Presentation Theorem) ci dà:

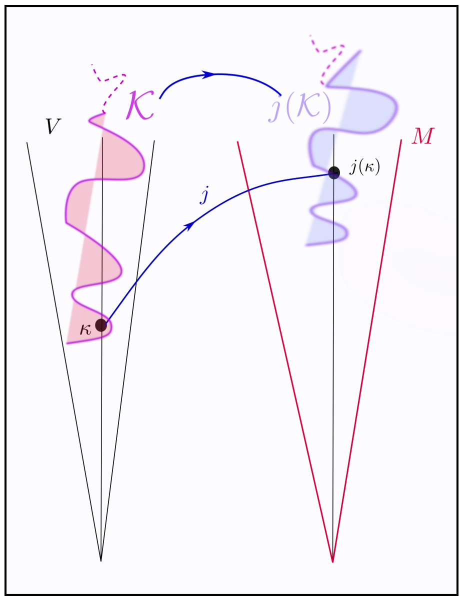
- $\tau' \supset \tau$,
- T' una τ'-teoria, e
- Γ' un insieme di T'-tipi

tali che

$$\mathcal{K} = PC(\tau, T', \Gamma') =$$
$$\{M' \restriction \tau \mid M' \models T' \text{ e } M' \text{ omette } \Gamma'\},$$

Definiamo $j(\mathcal{K})$ la classe $PC^M(j(\tau), j(T'), j(\Gamma'))$.

Per l'elementarità di j, $M \models j(\mathcal{K})$ è una AEC con numero di LS uguale a $j(LS(\mathcal{K}))$.

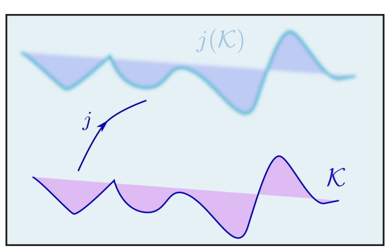


Come paragoniamo $\mathcal{K}$ e la sua «immagine» $j(\mathcal{K})$?

Tentiamo di ottenere $j(\mathcal{K}) \subset \mathcal{K}$ e $\prec_{j(\mathcal{K})} \subset \prec_{\mathcal{K}}$. La definizione importante è «j **rispetta** $\mathcal{K}$»:

Definizione B.5. Sia $\mathcal{M} \in \mathcal{K}$ (una τ-AEC); dunque $j(\mathcal{M})$ è una $j(\tau)$-struttura. Diciamo che j *rispetta* $\mathcal{K}$ se valgono le seguenti condizioni:

- Per ogni $\mathcal{M} \in j(\mathcal{K})$, $\mathcal{M} \restriction \tau \in \mathcal{K}$,
- per ogni $\mathcal{M}, \in j(\mathcal{K})$, $\mathcal{M} \prec_{j(\mathcal{K})}$ implica $\mathcal{M} \restriction \tau \prec_{\mathcal{K}} \restriction \tau$,
- per ogni $\mathcal{M} \in \mathcal{K}$, $j''\mathcal{M} \prec_{\mathcal{K}} j(\mathcal{M}) \restriction \tau$.



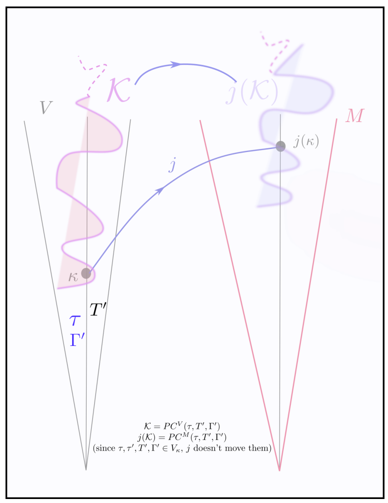
Due situazioni nelle quali «j rispetta $\mathcal{K}$»:

1. ($\mathcal{K}$ data **sotto** κ.) Sia $j : V \to M$ con punto critico κ; $\mathcal{K}$ una AEC con $LS(\mathcal{K}) < \kappa$. **Allora,** $\mathcal{K} = PC(\tau', T', \Gamma')$, con $|\tau'| + |T'| + |\Gamma'| < \kappa$; spdg $\tau', T', \Gamma' \in V_\kappa$; dunque

$$j(\mathcal{K}) = PC^M(\tau, T', \Gamma')$$

$$= (\mathcal{K} \cap M, \prec_{\mathcal{K}} \cap M).$$

2. $\mathcal{K} = Mod(\varphi)$, $\varphi \in L_{\kappa,\omega}$, con $\prec_{\mathcal{K}} = \subset^{TV}_{\mathcal{F}}$, $\mathcal{F}$ frammento di $L_{\kappa,\omega}$.



Ottenere la docilità

Dimostriamo allora che se $\mathcal{K}$ è una AEC con $LS(\mathcal{K}) < \kappa < \lambda$, e $j : V \to M$ ha la proprietà di copertura (κ, λ) e rispetta $\mathcal{K}$ allora $\mathcal{K}$ è $(< \kappa, \lambda)$-docile.

Siano dunque $\mathcal{M} \in \mathcal{K}_\lambda$ e $p_1 = \mathrm{gatp}(\vec{a}/\mathcal{M},_1)$, $p_2 = \mathrm{gatp}(\vec{b}/\mathcal{M},_2)$ due tipi tali che per ogni $\prec_{\mathcal{K}} \mathcal{M}$ di cardinalità $< \kappa$ abbiamo

$$p_1 \restriction = p_2 \restriction .$$

(Qui, $\vec{a} = (a_i)_{i \in I}, \vec{b} = (b_i)_{i \in I}$.)

Sia adesso $Y \in M$ tale che $j''|\mathcal{M}| \subset Y \subset j(|\mathcal{M}|)$ e $|Y|^M < j(\kappa)$.

Bisogna ricordare che in M, $LS(j(\mathcal{K})) = j(LS(\mathcal{K})) < j(\kappa)$, dunque esiste $\mathcal{M}' \in j(\mathcal{K})$ tale che $Y \subset |\mathcal{M}'|$, $\|\mathcal{M}'\| < j(\kappa)$ e $\mathcal{M}' \prec_{j(\mathcal{K})} j(\mathcal{M})$; per la transitività, $\mathcal{M}' \prec_{j(\mathcal{K})} j(_i)$, $i = 1, 2$.

Per l'elementarità, $M \models j(p_1) \restriction \mathcal{M}' = j(p_2) \restriction \mathcal{M}'$ (in $j(\mathcal{K})$)
da cui

$$\begin{aligned} p_1' &= \mathrm{gatp}(j(\vec{a})/\mathcal{M}' \restriction \tau, j(_1) \restriction \tau) \\ &= \mathrm{gatp}(j(\vec{b})/\mathcal{M}' \restriction \tau, j(_2) \restriction \tau) = p_2' \end{aligned}$$

in $\mathcal{K}$ (ancora, per la nostra ipotesi su j).

Siccome $j''\mathcal{M} \prec_{\mathcal{K}} j(\mathcal{M})$ possiamo concludere che $j''\mathcal{M} \prec_{\mathcal{K}} \mathcal{M}' \restriction \tau$ (assioma di coerenza), restringendo allora abbiamo che

$$\mathrm{gatp}(j(\vec{a})/j''\mathcal{M}, j_1'') = \mathrm{gatp}(j(\vec{b})/j''\mathcal{M}, j_2'').$$

Restringendo ancora, otteniamo

$$\mathrm{gatp}(\vec{a}/j''\mathcal{M}, j_1'') = \mathrm{gatp}(\vec{b})/j''\mathcal{M}, j_2''),$$

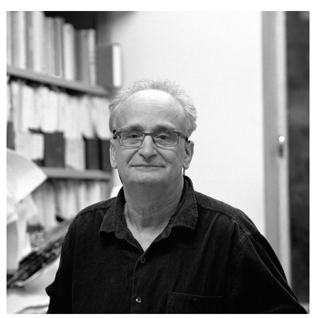
e possiamo concludere che

$$p_1 = p_2. \qquad \square$$

B.3 Altre interazioni

Altre interazioni
Modelli / Insiemi

Un aneddoto di Shelah



> Oh... I had a very strange referee report on the (proper forcing) paper. I think Moschovakis was the editor. So he thought "Saharon is a model theorist" well, he knew me - I was even a year in UCLA before, so he sent it to a model theorist. And the problem was in model theory, [of the form] "the consistency of...", and the referee report said "well, there is very little model theory". . .

Saharon Shelah, in un'intervista (AV), 2017.

B.3.1 Assolutezza?

Un comportamento dicotomico

- Sotto «diamante debole», cioè $2^{\kappa} < 2^{\kappa^+}$ (conseg. di GCH!):

  Teorema 3 (Shelah, circa 1984). (Sotto $2^{\kappa} < 2^{\kappa^+}$, oppure GCH.) Ogni AEC $\mathcal{K}$ con $LS(\mathcal{K}) \leq \kappa$, categorica in κ, che **non** soddisfa AP per modelli di cardinalità κ, ha necessariamente la massima quantità possibile, 2^{κ^+}, di modelli non-isomorfi di cardinalità κ^+.

- Purtroppo, sotto MA_{ω_1} il contrario accade: (MA_{ω_1}) Si può costruire una classe (assiomatizzabile nella logica $L_{\omega_1,\omega}(Q)$) che è $\aleph_0$-categorica, non soddisfa AP in $\aleph_0$ ed *anche è* categorica in $\aleph_1$ (la minima quantità possibile!).

Forzare l'isomorfismo / categoricità

Teorema 4 (Asperó, V.). L'esistenza di una AEC debole, categorica in $\aleph_1$ e in $\aleph_2$, in cui non vale AP in modelli di cardinalità $\aleph_1$, è consistente con $ZFC + CH + 2^{\aleph_1} = 2^{\aleph_2}$.

Per dimostrare questo, facciamo un'iterazione di forcing di lunghezza ω_3 e lavoriamo con «risoluzioni» di modelli.

La situazione ha connessioni con certe possibilità di «fallimenti» del teorema di Morley: situazioni in cui una classe soddisfa categoricità fino ad una certa cardinalità oltre cui il numero di modelli diventa il massimo (Hart-Shelah 1985 per $L_{\omega_1,\omega}$, Shelah-V. 2021 per $L_{(2^\lambda)^{++},\omega}$). Il nostro risultato è (paragonato a risultati positivi di Vasey e Shelah) la situazione più generale possibile di fallimento di Morley!

B.3.2 Proprietà dell'albero / Collasso della docilità

Il collasso e le sue limitazioni

Far collassare grandi cardinali mantenendo *alcune* delle loro proprietà ha una lunga storia di risultati interessanti. Per esempio,

- *Mitchell* ha fatto collassare cardinali debolmente compatti fino a $\aleph_2$ **mantenendo la proprietà dell'albero**. Questo è stato poi generalizzato (facendo collassare molto di più) per ottenere la proprietà dell'albero in tutti gli $\aleph_n$ ($n > 1$) e/o in $\aleph_{\omega+1}$ (Magidor, Cummings, Neeman, Fontanella, etc.)
- Per le proprietà «forti» o «super» dell'albero la forza di consistenza sarebbe prossima a un cardinale fortemente compatto / supercompatto rispettivamente (Weiss, Viale, Fontanella, Magidor).

Immersioni generiche

- Queste sono versioni di proprietà generali di riflessione/compattezza. Anche la docilità è una proprietà generalizzata di compattezza.
- Il collasso diretto di (per esempio) un cardinale fortemente compatto κ (dove già sappiamo che c'è $(<\kappa,\kappa)$-docilità) a $\aleph_2$ non funziona:
- Le classi risultanti $j(\mathcal{K})$ e (quando $\mathcal{K} = PC(L, T', \Gamma')$ le classi $\mathcal{K}^{V[G]} = PC^{V[G]}(L, T', j(\Gamma'))$ presentano una «docilità residua» interessante…
- tuttavia, addattare il collasso di Lévy (iterazione di Easton) o le costruzioni più sofisticate menzionate non può dare la piena docilità; risulta soltanto quella residuale.