Grothendieck and model theory: five characters in search of a theme

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Contents

Prelude: Grothendieck on mysteries and Galois

Grothendieck's voice, speaking on mysteries, on Galois theory, on the vast program in front of his eyes late in his life, seems to fit the five themes of this paper. His fascination for the mysteries of motives, of geometric descriptions of the absolute Galois group of the rationals, is described very accurately in this passage. The connections to model theory seem to be contained implicitly in his words.

> *Ce qui m'intéresse dans ce passé, ce n'est nullement ce que j'y ai fait (la fortune qui est ou sera la sienne), mais bien plutôt ce qui n'a pas été fait, dans le vaste programme que j'avais alors devant les yeux, et dont une toute petite partie seulement s'est trouvée réalisée par mes efforts et ceux des amis et élèves qui parfois ont bien voulu se joindre à moi. Sans l'avoir prévu ni cherché, ce programme lui-même s'est renouvelé, en même temps que ma vision et mon approche des choses mathématiques. (…) les* ***mystères*** *qui m'ont le plus fasciné, tel celui des « motifs », ou celui de la description « géométrique » du groupe de Galois de* $\overline{\mathbb{Q}}$ *sur* $\mathbb{Q}$…
>
> **A. Grothendieck** / L'aventure solitaire[13, p. 363]

Introduction

The work of Alexander Grothendieck is not frequently associated with model theory or mathematical logic. His enormous influx in geometry (of various sorts: arithmetic, algebraic, non-commutative, symplectic, and many links between them) and in functional analysis in his earlier work, the amount of seminal ideas and constructions and methods he developed usually seem to bear on parts of mathematics remote from logic. In this note, we offer a different perspective. Not only did Grothendieck's vast output had connections with parts of mathematical logic (mostly with model theory) but those connections were of a deep kind; perhaps not immediately apparent but central.

We will briefly describe five *Grothendieckian connections* in Model Theory. The first Grothendieckian connection we describe is at the genesis of the main dividing line (stability). There are many characterizations of stable theories, and the main theory was developed initially by Shelah in the 1970s. However, for the specific case of Banach spaces, Grothendieck has an early version of the same dividing line that would prove fruitful and influential in the general case two decades later (this was clarified by Ben Yaacov[2]). The connections between dividing lines or properties from functional analysis and their more general model-theoretic versions is further explored by Khanaki and Pillay[19] and other works of Khanaki.

The second theme is less direct, but potentially deeper: we will describe a line originally opened by Poizat in his article *Une théorie de Galois imaginaire*[29] from 1983. We will land in a part of the "stability hierarchy" (inaugurated implicitly by Morley in the early 1960s, but really established by Shelah in his monumental *Classification Theory*[33] starting around 1970) where the view of model theory as a very extended version of Galois theory works well. This zone of the map of all first order theories (called "stable theories") gave the name *stability theory* to a whole area of classification theory, and for decades was the main dividing line in that theory. The second Grothendieckian connection is then the presence of Galois theory, of very extreme extensions of Galois theory, in both a good part of Grothendieck's work and in model theory.

The third of these connections is truly a later variant of the presence of Galois theory in model theory, in the work of Kamensky[15, 16]: recasting model theoretic versions of Galois theory in the Grothendieck functorial variant of Galois theory (expressed in terms of internal covers following SGA1[8]). This line makes explicit some properties of the Galois correspondence.

The fourth connection is of a different kind: the development of model theoretic ideas for accessible categories. Here, the story is more convoluted. Saharon Shelah opened the line toward the model theory of *abstract elementary classes* starting in the 1980s; this led to enormous developments initially led by the categoricity conjecture and to a whole development of stability theory for non-elementary classes. Later on, this was connected to *accessible categories* initially by Lieberman[23] and then also by Rosický and others[21]. The connection here is perhaps surprising: accessible categories were originally introduced by Grothendieck in SGA4[1] and later linked to categorical logic by Makkai and Paré[27].

Finally, the fifth theme we signal marks a return to the Galois theory connection, linked to some later spin-off of Grothendieck's work. The fifth connection is looser than the previous four and really marks what I will call the Galois/Grothendieck/Shelah ascent of model theory, the development of the right notion of Galois theory for arbitrary theories in recent work by Hrushovski[14] and the beginning of the development of "higher dimensional versions" of model theory, somewhat paralleling the development of "higher category theory" in recent years, a different kind of spin-off of Grothendieck's work in the work of Chernikov, Hempel, Palacín and Takeuchi[5, 4].

I wish to thank Marco Panza for many interesting philosophical interactions, and for having invited me to speak on these topics at Chapman University in California in 2022. This was the first trigger for this article. My thanks also go to my colleague and friend Fernando Zalamea. Without the many observations, perspectives, seminars and conversations we have had, my own awareness of the connections I signal in this article would be much dimmer. The philosophical bearing of these model theoretic themes (that I discuss in the end of this paper) is also linked to the study of *Formalism Freeness* situations initiated by Juliette Kennedy[17]. Her notion is extremely fruitful for the philosophical study of the connections mentioned here. Frequent mathematical discussions with John Baldwin have also been essential to part of this project. I thank him for providing so much back and forth in our joint projects, and for bringing up the fourth connection in this paper (to accessible categories). For other conversations bearing on these topics, I also thank Artem Chernikov, Zoran Škoda, Misha Gavrilovich, Alexander Cruz and Boris Zilber. My awareness of many of these connections was also triggered during very long walks around Te Anau (South Island, New Zealand) with Alex Usvyatsov and Sharon Hollander in 2014. The blend of lakes, Milford Sound/Piopiotahi, woods was the perfect setup to start in my mind to try to understand this blend of model theory, Galois theory and geometry.

1 Stability: an early Grothendieckian theme?

In 1952, studying reflexivity and compactness of Banach spaces, Grothendieck[12] provided a criterion for those two properties. Much later, this criterion was seen to be an early version of what would become the most important dividing line in model theoretic classification theory: stability. Ben Yaacov[2] explains this connection. Further connections between the *stability hierarchy* and properties coming from functional analysis are explored by Khanaki[18] and Pillay[19]. We first describe a bit the role of stability and dividing lines, before addressing the connection with Grothendieck.

1.1 Classification theory

Consider the "formless magma" of all *possible* mathematical structures. Is this pure random noise? Our mathematical knowledge accumulated over millennia tells us quite the opposite! Groups, fields, algebraic varieties, Sobolev spaces, inner models of set theory, function spaces, are certainly not "random" structures, and were arrived at in natural ways. Some contemporary mathematicians, notably Boris Zilber, have engaged in explanations of *mathematical a priori* situations for these structures[7] and narrowed in the centrality of the notion of **categoricity.**

The world of (first order) theories is certainly not a homogeneous world. To the question *What kind of classification is there?*, model theory has provided a strong response in the form of a **classification** of all first order structures via the appearance of invariants and dividing lines. The most important of all these dividing lines is stability/order: every first order theory is either *stable* or has a definable *infinite linear order*.

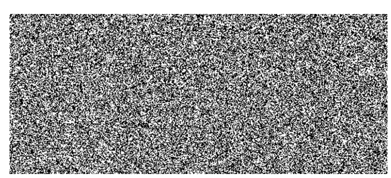


Figure 1: Random noise? No!

Shelah's edifice classifies *all* first order theories by providing «dividing lines» (properties that split the class of all theories by properties that are significant on both sides of the line: an emblematic example being the line stability/order: every theory T is either stable (controlled number of types in some cardinality) or has a definable infinite linear order).

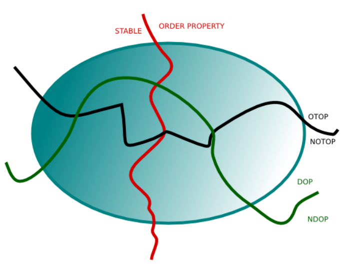
Structures such as $\langle \mathbb{N}, +, <, \cdot, 0, 1\rangle$ for arithmetic, $\langle \mathbb{C}, +, \cdot, 0, 1\rangle$ for algebraic geometry, $\langle \mathbb{R}, +, <, \cdot, 0, 1\rangle$ for real algebraic geometry, vector spaces (modules, etc.), elliptic curves, some combinatorial graphs, Hilbert spaces, ℓ_2, etc. are…

Words like "dimension", "rank", "degree", "density character" - seem to appear attached to those structures, and control them and allow us to capture them. These in turn are instances of more general (model theoretic) notions of independence; these are one of the guiding principles behind the generalization brought by model theory.

1.2 Model Theory: perspective and fine-grain

Model theory has therefore provided a global perspective and fine grain to the question of classification of all possible mathematical theories (and their model classes). Model theory deals with arbitrary **structures**. It provides a whole hierarchy of types of structures (or their theories): Stability Theory. In the "tamest" part of the hierarchy: generalized Zariski topology - *Zariski Geometries* due to Hrushovski and Zilber: algebraic varieties - "arbitrary" structures whose place in the hierarchy ends up automatically giving them strong similarity to elliptic curves. And beyond direct control by a logic: the hierarchy extends rather well (abstract elementary classes).

1.2.1 Taxonomies



Here are some of the main *dividing lines;* this has been described by Shelah as a kind of "taxonomy" of the universe of first order theories; in a recent interview, Shelah has discussed the role of *categoricity* and dividing lines[36]; here, we are mainly concerned with stability as a dividing line. Other dividing lines are summarized in the following table. Notice that each property corresponds to some combinatorial feature (order property, tree properties, etc.). These combinatorial properties turn out to have an effect first on the mere possibility of having a well-behaved notion of *abstract independence* for the theory in question; what really has made model theoretic methods so powerful (more recently, starting around 1990, after two decades of deep work in the pure aspects of stability theory) in other parts of mathematics is the deep connection between *structural aspects* of the theories and their counterparts coming from the part of mathematics in which the theory is originally coined.

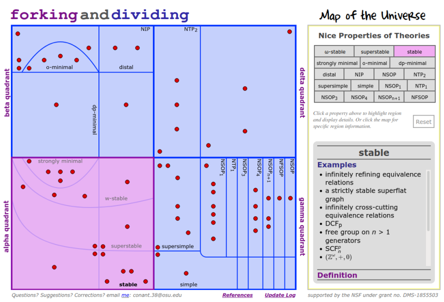
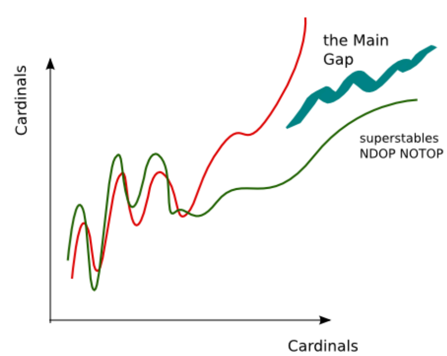


Figure 2: A “high resolution”, interactive, map of first order theories

stable	unstable	order property
NDOP	DOP	dimensional order property
NOTOP	OTOP	(omitting types) order property
superstable	unsuperstable	local control of
depend. (NIP)	IP	codifying $a \in b \subset \omega$
etc. (NTP_2)	TP_2	tree properties...

Figure 2 shows an interactive “map” of first order theories, a very useful tool where current knowledge of many theories (the “red dots”), dividing lines (pink areas in the map, clickable) and structural properties are linked. The map is due to Conant[6].

A very general question framing stability theory is Shelah’s Main Gap; the assertion that, depending on combinatorial aspects of the class of models of a theory T, its spectrum function $I(T, \kappa)$ counting the number of models of T of cardinality κ, up to isomorphism, really has only two possible asymptotic behaviors.



Given a countable theory T, the spectrum function $I(T, \cdot)$ either always achieves the maximum values, else it has a “low” bound:

$$I(T, \aleph_\alpha) < \beth_{\omega_1}(|\omega + \alpha|)$$

Here is a list of later results that were developed within the framework originally set up by stability theory. The fact that model theoretic (i.e., logical) methods came to have such an imprint on algebraic geometric issues, or even on the transcendence of solutions of differential equations, is perhaps a surprising connection with areas closer to what is usually acknowledged as Grothendieckian mathematics.

- Hrushovski's proof of the Mordell-Lang Conjecture (ca. 1990)
- Model-Theoretic Analysis of the André-Oort Conjecture
- Proof (Casale-Freitag-Nagloo) of a Conjecture by Painlevé from ca. 1895, using the model theory of Differentiably Closed Fields
- Model Theoretic Analysis of analytic functions and Grothendieck's Standard Conjectures (Zilber, since ca. 2000) - this part not only in First Order Model Theory
- Other topics developing now.

1.3 The Main Dividing Line: Stability

The original dividing line, from whose name the whole subject inherited its name, is the notion of stability.

1.3.1 Grothendieck, 1952: early version of stability

Itaï Ben Yaacov has explained[2] how the Fundamental Theorem of Stability (the equivalence between *not having an order* and *definability of types*) follows from this theorem from 1952 by Grothendieck in the context of his early study of Banach spaces:

Theorem 1 (Grothendieck, 1952) *Given a topological space X, $X_0 \subseteq X$ a dense subset, then the following are equivalent (for $A \subseteq C_b(X)$, the Banach space of bounded, complex-valued functions on X, equipped with the supremum norm):*

- *A is relatively weakly compact in $C_b(X)$,*
- *A is bounded, and for all sequences (f_n), $f_n \in A$ and (x_n), $x_n \in X_0$,*

$$\text{CRUCIAL POINT} \longrightarrow \quad \lim_n \lim_m f_n(x_m) = \lim_m \lim_n f_n(x_m),$$

when both limits exist.

The statement is interesting for us in many ways. The criterion is given in terms of *exchangeability* of order in the limit computation: $\lim_n \lim_m f_n(x_m) = \lim_m \lim_n f_n(x_m)$ is a specific form of a *non-definability* of order. In Shelah's terminology, this is the negation of the *order property*; Shelah proved that this precisely characterizes stable theories.

2 Galois Theory of Model Theory

The second connection is ample: various aspects of model theory articule extensions of Galois theory to various areas much more general than its original incarnation in fields. As Grothendieck proposed in his *Séminaire de Géométrie Algébrique 1* (SGA1), Exposé I[8], there is a far-reaching view of Galois theory in categorical terms (essentially, shifting from the automorphism group of a field to the automorphism group of the *fibre functor* that assigns to a cover its fibre over a fixed point[1]).

2.1 Model Theory as a natural Galois-theoretic framework

In contrast with the previous developments, model theory, especially after Shelah's development of stability theory, became a natural ground for generalizations of Galois theory of a different kind. The centrality in model theory of understanding interactions between definable sets enabled a different kind of generalization, not relying on Artinian developments of Galois theory, nor (initially) on the Grothendieck kind of generalization[2].

2.1.1 Poizat makes the connection explicit

In his seminal article *Une théorie de Galois imaginaire*[29], Poizat proposed a reading of Galois theory in modern model theoretic language. In his words, *…cette démonstration a le mérite d'être beaucoup plus proche des preuves, ou des indications de preuve, qu'on trouve dans les manuscrits de Galois (…) elle est plus directe que celle qu'on enseigne habituellement dans les cours d'algèbre*[3]. This claim by Poizat takes us up to some point that turns out to be quite far from the kind of generalization proposed after Grothendieck. However, the simplification it provides, initially in terms of exposition of the material, and the *naturality* it signals, would later on be linked more directly to generalizations of Galois theory, in particular to that proposed by Grothendieck (see later section 3).

Figure 3 is a graphical summary of the classical Galois correspondence, with field extensions on one side and their corresponding automorphism groups on the other side. This

[1] The long story following Grothendieck's turn in Galois theory has included generalizations to the so-called *étale* realization, *topological* realization and *de Rham* realization of ever more general objects. We only explore this line here in the third connection (section 3). See Szamuely's book[34] for a very detailed and insightful exposition.

[2] A very good source for the translation initiated by Poizat was given in 2010 by Medvedev and Takloo-Bighash[28]. We refer the reader to their paper for details.

[3] …this proof has the merit of being much closer to the original (indications of) proofs one may find in Galois's manuscripts (…) it is more direct than [the proof] usually taught in algebra courses.

basic, but extremely momentous, correspondance, is played out again quite reasonably not just for fields, but for models of certain first order theories T[4].

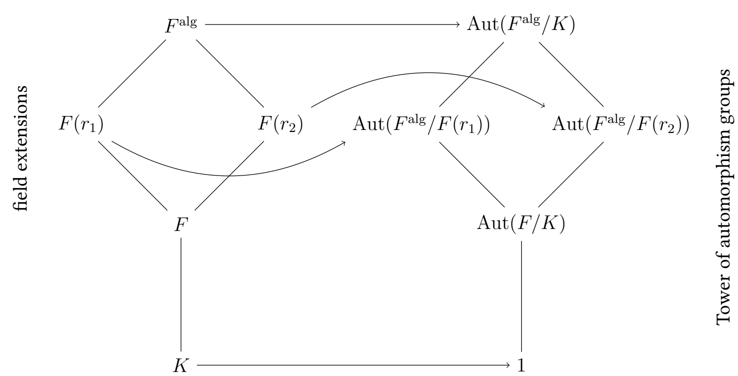


Figure 3: Classical Galois correspondence

2.1.2 In first order, the key role of imaginaries

Around 1980, Shelah introduced the notion of *imaginary* elements of arbitrary first order theories T. These are equivalence classes of definable equivalence relations

$$a/E = \{b \in \mathbb{M} : aE^{\mathbb{M}}b\},$$

where $\varphi(a, b) \iff aEb$ for some formula $\varphi(x, y)$ of the language.

In 1983, Poizat noted that Shelah's imaginaries are *exactly* what is needed for a general Galois theory (for models of a more general T than algebraically closed fields).

Poizat wrote in particular: *the model theoretic proof of the Galois correspondence is much closer to the (indications of a) proof in Galois's manuscripts!*

This is a rather bold claim, in terms of the history of reception of Galois's ideas, and of the different approaches to the proof of his claim. In what sense *closer*? With which obstructions?

We do not develop that point here, but refer on the one hand to Poizat's discussion in the first part of his paper[29], and on the other hand to later explanations. We point out, however, that the use of Shelah's imaginaries (on the one hand) and of the stability of the theory of algebraically closed fields are essential points of Poizat's approach.

The line he opened begs the natural question of how to remove the original limitations for more general first order theories; this line has been taken by Hrushovski and others[14].

[4]The specific theories T for which this Galois correspondence is successful was initially linked to the stability hierarchy. Much more recent work by Hrushovski[14] has provided the right kinds of extensions.

2.2 Some translations (following Medvedev/Takloo-Bighash)

Medvedev and Takloo-Bighash[28] explain some of the translations from specific to more general Galois theory:

Arbitrary first order theory T	ACF (algebraically closed fields)
$\mathbb{M} \models T$ some sufficiently saturated model of T	$(\mathbb{C}, +, \cdot, 0, 1)$
$\mathfrak{A} \prec \mathbb{M}$	$F \leq \mathbb{C}$

So, from the specific case of ACF we switch to the more general case of *arbitrary* first order theories T. The classical role of the complex numbers $\mathbb{C}$ is now played by sufficiently saturated models $\mathbb{M}$ of T, and the role of the subfields of $\mathbb{C}$ is now played by elementary submodels $\mathfrak{A}$ of $\mathbb{M}$.

Many additional details may be found in the Appendix to this article (section A).

2.3 Summary of the first rapprochement: the two sources

Two faraway sources merge in an unlikely way (at first glance): Krasner's version of Galois theory (developed in the 1950's and 60's; Bélair and Poizat have written a description[3]) and Saharon Shelah's *Classification Theory*[33]. These two apparently remote sources are blended in Poizat's treatment and generalization of Galois theory. There is, however, a caveat for the smoothness of the translation: the hypothesis on the stability of the theory.

The key hypothesis to the possibility[5] of defining a good Galois group of a theory was *stability:* roughly, a solid theory of definability of orbits (Galois-types) of the action of the automorphism group of a large structure $M \models T$. We now take a détour.

Stability allows good control of (algebraically closed, in the model theoretic sense) definable sense, in a way that would take us away from our main path in this article.

Together with the elimination of imaginaries mentioned above, the key to the Galois correspondence was, for a long time, stability of the theory T: this allowed good control of algebraic closures of sets, and the Galois correspondence between normal subgroups of the automorphism group of a monster model $\mathbb{M}$ of T and algebraically closed subsets of $\mathbb{M}$ worked (and had as a special case classical Galois theory, for T the theory ACF_0 of algebraically closed fields of characteristic zero).

Many more details of this development are given in the survey by Medvedev and Takloo-Bighash[28]; in Appendix A, we provide some of the essential steps.

[5] At least, in earlier versions of this model theoretic Galois theory.

3 Galois à la Grothendieck, in model theory

Grothendieck recast Galois theory in a category theoretical frame. He replaced the quest for automorphism groups of a field or space by the study of automorphism groups of *fiber functors* (functors assigning to a cover its fiber over the base point). As Szamuely says[34], *"this point of view permits a great clarification of earlier concepts on the one hand, and the most general definition of the fundamental group in algebraic geometry on the other. One should by no means regard it as mere abstraction: without working in the general setting many important theorems about curves could not have been obtained."*

I will not further address here the consequences of this momentous generalization done by Grothendieck[1]; other articles in this volume fulfill this analysis. I will rather signal the way this started to influence model theory after Poizat's early connection of Galois theory with parts of stability theory, in the work of Kamensky and indirectly in the work of Hrushovski as well.

3.1 Interpretations and Stability

One of the most general comparisons between structures (or their theories) is through the notion of *interpretation.* In some senses, this is the most basic and more general notion in model theory; it is really used implicitly in many works. Most theorems require the stricter notions of elementary equivalence or isomorphism between models. For our purposes, it is worth taking a look at the notion of *stable interpretation.*

Following the work of Makkai and Reyes[26] (ultimately derived from that of Grothendieck and Lawvere[20]), we first explain briefly the very basic approach to *models as functors* and then zoom in to the notion of interpretation. We also refer the reader to García-Vargas[10] for a detailed treatment.

Let T be a first order theory in a vocabulary L, and call $\mathcal{T}$ the category of T-**definable** sets and functions. Formally, the objects of $\mathcal{T}$ are equivalence classes between formulas (in the theory T) and morphisms are given by (equivalence classes) of definable functions. See Appendix B.

The crucial point of their approach is considering the **solution set** of each formula, in each L-structure $\mathfrak{M}$,

$$\varphi(\mathfrak{M}) = \{a \in M : \mathfrak{M} \models \varphi(a)\}.$$

This yields a functorial correspondence, for each $\mathfrak{M}$. Thus, each model $\mathfrak{M}$ may be regarded as a functor from $\mathcal{T}$ to the category **Set** of all sets and functions. Natural transformations between these model-functors are elementary maps $\equiv$.

Categories of the form $\mathcal{T}$ have good closure properties (they are Boolean; see Appendix B).

The crucial transfer from models of a theory T_0 to models of T is given by morphisms ι between the two related categories,

$$\iota : \mathcal{T}_0 \to \mathcal{T},$$

preserving their Boolean structure. Note that the vocabularies L_0 of T_0 and L of T *need not be the same;* moreover, many interesting cases arise precisely in the case of *different* vocabularies.

3.2 Interpretation functor between classes of models

Now, something more interesting happens: when we have an interpretation $\iota : \mathcal{T}_0 \to \mathcal{T}$ between theories, we may lift it to classes of models, by *dualizing* the functor:

Given $\iota : \mathcal{T}_0 \to \mathcal{T}$, we get

$$\iota^* : Mod(T) \to Mod(T_0)$$
$$\mathfrak{M} \models T \mapsto \iota^*(\mathfrak{M}) = \mathfrak{M}_0$$

where

$$\mathfrak{M}_0 = \mathfrak{M} \circ \iota : \mathcal{T} \to \mathbf{Set}.$$

Notice that ι^* takes the opposite direction: it transforms model-functors of T into model-functors of T_0. And it does by composing a model of T with the interpretation ι: to evaluate "from the perspective of" T_0, we first interpret via ι and then evaluate from the resulting model of T.

Of course, we also need to lift the interpretation to elementary maps between models of T. This is done in the natural way:

if $\sigma : \mathfrak{N} \to \mathfrak{M}$ is an elementary embedding ($\sigma = (\sigma_Y)_{Y \in \mathcal{T}}$) then

$$\iota^*(\sigma) : \mathfrak{N}_0 \to \mathfrak{M}_0 : \iota^* \sigma_X = \sigma_{\iota X}$$

for each $X \in \mathcal{T}_0$.

The very well-known example of interpretation (slowly uncovered by Wessel, Argand, Gauß around the turn of the 19th Century) of the complex numbers in the real numbers may be described as follows (ACF denotes the first order theory of *algebraically closed fields,* the complex numbers $\mathbb{C}$ are a model of ACF; RCF denotes the theory of *real closed fields,* of which the real numbers $\mathbb{R}$ are a model - the categories of definables are thus denoted $\mathcal{ACF}$ and $\mathcal{RCF}$)[6].

Now, if $R \models RCF$, we compose the model-functor R with ι to obtain $R \circ \iota \models ACF$. This corresponds to the more familiar notation

$$\iota^*(R) = R[\sqrt{-1}].$$

There are many other natural examples of interpretations (including retracts, boolean algebras in boolean rings, etc.).

Stability, that theme we saw in our first section connected to early Grothendieck, is reflected in a natural way in interpretations:

An interpretation $\iota : \mathcal{T}_0 \to \mathcal{T}$ is **stable** if for each model $\mathfrak{M}$ of T, the "expanded interpretation" $\iota^{\mathfrak{M}} : \mathcal{T}_0^{\mathfrak{M}_0} \to \mathcal{T}^{\mathfrak{M}}$ is an immersion. This means each definable in ιX ($X \in \mathcal{T}_0$) using parameters from is the image of a definable set in X using parameters from $\mathfrak{M}_0$. Moreover, if T is a stable theory and $\iota : \mathcal{T}_0 \to \mathcal{T}$ is an interpretation, then ι is a stable interpretation and T_0 is a stable theory.

In[15], Kamensky goes as far as reframing a full-framed Galois theory of model theory for internal covers - Galois theory à la Grothendieck (SGA 1), thus closing the ring opened originally by Poizat (using Shelah's theory), and ideas connected with Grothendieck's approach to model theory.

This is a beautiful third Grothendieckian theme in model theory, really expanding the first one.

There are even notions of the "Galois group" of a first order theory; in connection with the fifth theme, it is worth mentioning that for this notion to support a robust Galois correspondence, the theory must have elimination of imaginaries and be stable.

[6]The interpretation is given by $\iota : \mathcal{ACF} \to \mathcal{RCF}$, where $\iota(z = z) := (x = x \wedge y = y)$, $\iota(z = 0) := (x = 0 \wedge y = 0)$, $\iota(z = 1) := (x = 1 \wedge y = 0)$, $\iota(z \cdot z = 1) := (x = 0 \wedge y = 1)$, $\iota(z_1 + z_2 = z_3) := (\iota_1(z_1) + \iota_1(z_2) = \iota_1(z_3) \wedge \iota_2(z_1) + \iota_2(z_2) = \iota_2(z_3))$, $\iota(z_1 \cdot z_2 = z_3) :=$
$([\iota_1(z_1) \cdot \iota_1(z_2) - \iota_2(z_1) \cdot \iota_2(z_2) = \iota_1(z_3)][\iota_1(z_1) \cdot \iota_2(z_2) + \iota_2(z_1) \cdot \iota_1(z_2) = \iota_2(z_3)])$,
(here, ι_1 and ι_2 are the x-component and the y-component of the map ι).

The previous express the idea that we map "complex" into their two "real components", that addition is defined componentwise, and multiplication follows the rule $(a, b)(\iota\cdot)(c, d) = (ac - bd, bc + ad)$.

4 Categories and Abstract Elementary Classes: the great reversal

4.1 Abstract Elementary Classes: model theory's semantic essence

An unexpected reversal started during the 1980s in model theory. A few years after the first edition of *Classification Theory*[33] in 1980, Shelah started work in the classification theory of *abstract elementary classes.* These constitute a semantic setting for model theory, not immediately depending on any specific logic, but rather building on the abstract properties of a "generalized elementary embedding" notion $\prec$. In their independence from any logical context given a priori, they generalize at the same time many properties of first order logic, of many infinitary logics, and of many natural constructions normally done in algebraic contexts. The reader may check Grossberg's survey[11] for an overview of the details.

The development of model theory in abstract elementary classes had a slow beginning: during the first two decades it remained a rather marginal subarea of model theory. However, around the turn of the century, an explosion of work occurred, and many earlier developments of model theory in first order logic either were generalized, or gave rise to several refinements of the original theory.

4.2 Accessible Categories and Grothendieck

Another surprising connection with Grothendieck lay in store for this part of model theory. Around 2010, Lieberman[24] started a line connecting the model theory of abstract elementary classes to *accessible categories.* An initially surprising connection, its naturality became evident after the work of several members of the Czech school of category theory in combination with experts in abstract elementary classes.

Initially defined by Grothendieck, accessible categories were studied extensively by Makkai and Paré in their *Accessible categories: The foundations of categorical model theory*[27]. Accessible categories are endowed with an internal notion of "smallness" that enables them to capture notions of *generalized cardinality* robust enough to express in a natural way parts of the axioms of abstract elementary classes dealing with bounding cardinalities (technically, the "Löwenheim-Skolem axioms"). Strictly speaking, accessible categories are slightly more general than abstract elementary classes, and may lose some of their strong model-theoretic properties when taken at their fullest; however, the richness of interactions and possibilities opens a fantastic bridge of connections between a very Shelahian world and a very Grothendieckian one.

4.3 Opening toward the future

Recently, Espíndola[9] has started a new path toward infinitary categorical logic, that has a serious potential of results in pure model theoretical problems (approaches to Shelah's Categoricity Conjectures, etc.). Beyond the details of what has been so far achieved, the most interesting aspect of this approach is (in my view) the connection between generalized Grothendieck topologies, abstract elementary classes, infinitary logics and accessible categories.

Other works that partake of this *blend* between very Grothendieckian ideas and problems and constructs coming from model theory is the use of *quantales* in very recent work of Lieberman, Rosický and Zambrano[22]. They generalize *metric* abstract elementary classes in combination with categories stemming from Grothendieck's work on accessible categories.

5 Two Ascents: Hrushovski's Core, Higher Stability

This last section, closing the map of our Grothendieckian themes, is perhaps more tentative than the previous four. It consists of two brief ascents, two descriptions of themes that have started to emerge rather recently in model theory, with overtones connecting them to some of Grothendieck's intuitions.

5.1 Ascent 1: Definability patterns (some features)

In 2019, Hrushovski circulated his paper *Definability patterns and their symmetries*[14]. The paper blends many different lines (structural Ramsey theory, computing better bounds for a problem in certain groups, etc.) around a central issue: what is the *mathematical object* of which an automorphism group would correspond to the Galois group of an arbitrary first order theory T? In this sense, this central problem addresses issues left open by our first Grothendieckian theme, the development of Galois theory of model theory by Poizat and Lascar: in the absence of stability, the earlier theory fails, and the correct notion of Galois group of a theory was still left open.

Hrushovski introduces a new object: the *core* of a theory T, an object that expands models of the theory with a language now aware of model-theoretic *types* (roughly: orbits under the action of automorphism groups of large models), and blending the well-known topology of these types with so-called *definability patterns.* These are predicates measuring the degree of *failure of definability* of types. A technical issue is worth mentioning: one of the landmark equivalent formulations of stability (a theme that has recurred in these notes) is precisely that all types over rich enough (algebraically closed in the model theoretic sense) sets are definable (i.e., a formula $d_p\varphi$ of the theory may be used to detect, for each formula φ, which parameters enter a given type p). Therefore, in the absence of stability, there are non-definable types, types p for which the operator d_p is not well-defined. The addition of new predicates (for each type and finite tuple of formulas) gives a large expansion of the original language, blending the vocabulary of the original theory T with *topological* information (the space of types). The resulting *core* of T, $\mathcal{G}_T$ turns out to have enough automorphisms to support a Galois correspondence that was missing in the original case.

Hrushovski's core of a theory has already been expanded by Segel[30], and is amenable to situations where enrichments of the language (e.g., Galois Morleyizations for abstract elementary classes in the work of Vasey[35]). All of these enhancements of Galois theory have direct and less direct links to the work of Grothendieck and seem to be opening a wealth of new connections, possibly linked to versions of Grothendieck's late conjectures.

Appendix C contains a more detailed development of some of these ideas.

5.2 Ascent 2: Higher Stability?

A recent line of work involves another theme related to some of Grothendieck's ideas, a theme that we could call **higher** stability for lack of a better name.

Stability may be recast as a question on *recovering information* about binary relations $R(x,y)$ given with some obstruction, from unary relations $U_1(x)$ and $U_2(y)$. Several classical dividing lines have the format *"if $R(x,y)$ satisfies obstruction $(*)$ then, it may be 'approximated' by unary relations $U_1(x)$ and $U_2(y)$."*

In[4, 5], Chernikov, Hempel, Palacín and Takeuchi explore higher-dimensional versions of this phenomenon. Their work reveals interesting parallels with also recent work on higher categories; the subject was developed primarily by Lurie in his book *Higher Topos Theory*[25].

This is deeply related to much earlier work originally due to Shelah: excellent classes, from the early 1980s[31, 32]. Excellent classes had a major impact on the development of both the deepest theorems in Classification Theory (the so-called Main Gap Theorem) and the understanding of many phenomena in the infinitary logic $\mathbb{L}_{\omega_1,\omega}$ and various other non-elementary contexts (AECs).

Conclusive remarks

The latest "second ascent" to higher stability closes an interesting circle of themes linking Grothendieck and model theory. I left out many other possible connections (perhaps the most salient of all the model theoretic treatment of *motivic integration*) and focused on five themes closer to my heart. At the center of them all, the notion of stability seems to emerge, in strange ways that link in a deep and mysterious way the work of the two central mathematicians of the past decades.

A Model Theoretic Galois Theory (Appendix)

A.1 Normal and Splitting Extensions

The translation may be extended to normal and splitting extensions. For this, model theoretic versions of the degree of an extension, of normal and splitting extensions need to be defined.

Arbitrary first order theory T	ACF (algebraically closed fields)
$\mathbb{M} \models T$ some sufficiently saturated model of T	$(\mathbb{C}, +, \cdot, 0, 1)$
$\mathfrak{A} \prec \mathbb{M}$	$F \leq \mathbb{C}$
$\mathfrak{A} \prec \mathfrak{B}$ **degree** of the extension	$F \leq F_1$ $deg\,(F_1/F)$
$\mathfrak{B}$ a **normal** extension of $\mathfrak{A}$ $\mathfrak{B}$ a **splitting** extension of $\mathfrak{A}$	F_1 a **normal** extension of F F_1 a **splitting** extension of F

But, under the constrain $\mathcal{B} \succ \mathcal{A}$ a normal extension, corresponding the normal extension $L \geq F$, we may extend the Galois correspondence.

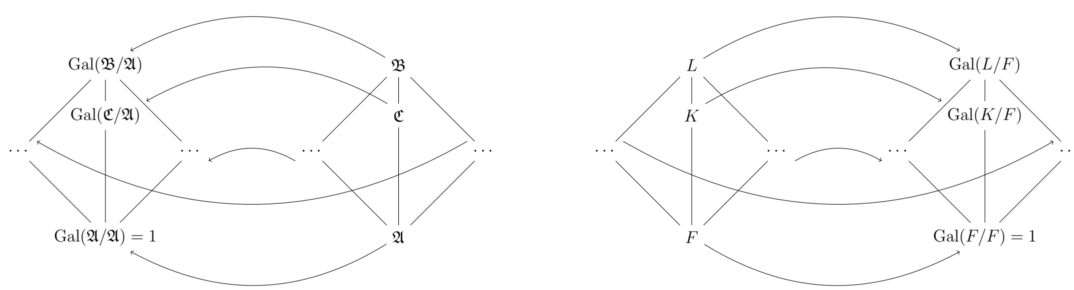


Figure 4: Galois correspondence (left: model theory, right: classical)

A.2 Some differences (lost in translation)

What is lost in this translation? Without entering the details, we mention the following:

- The language $\mathcal{L}$ might not have function symbols (so, the notion of "generated" subfield has to be adapted).
- The role of polynomials in classical Galois theory is now played by formulas. But then the question of quantifier elimination becomes central.
- The notion $\mathfrak{B} \succ \mathfrak{A}$ may be very different from being a linear (or algebraic) extension of $\mathfrak{A}$.
- The *degree* of an extension might not be given by linear dimension.
- There is no substitute for *norms, traces, determinants* (from the usual Artinian presentation of Galois theory).

A.3 A couple of notions for the translation

Here are some notions crucial to the translation:

- $\mathbb{M} \models T$ is a sufficiently *saturated* model. In a saturated model $\mathbb{M}$, all types over subsets of parameters smaller than $\mathbb{M}$ are realized.
- If $\varphi(x, y)$ is an $\mathcal{L}$-formula, $b \in \mathbb{M}$ is a **solution** of $\varphi(a, y)$ if $\mathbb{M} \models \varphi(a, b)$.
- We have notions of algebraic closure acl and definable closure dcl:

  $$b \in \mathrm{acl}(A) \iff \mathrm{orb}\,(b/A) \text{ is finite},$$

  where $\mathrm{orb}(b/A)$ is the orbit of b under the action of the automorphism group of $\mathbb{M}$ fixing A pointwise, $Aut(\mathbb{M}/A)$.

A.3.1 Normal extensions

Let $\mathfrak{A} \prec \mathfrak{B} \prec \mathbb{M}$. Then $\mathfrak{B}$ is a *finite extension* of $\mathfrak{A}$ if for some tuple $\vec{b}$ from $\mathfrak{B} \cap \mathrm{acl}(\mathfrak{A})$, $\mathfrak{B} \subseteq \mathrm{dcl}\,(\mathfrak{A} \cup b)$.

$$\mathfrak{B} \text{ is } \textit{normal over } \mathfrak{A} \iff \forall c \in \mathfrak{B}\,(\mathrm{orb}(c/\mathfrak{A}) \subseteq \mathfrak{B})\,.$$

A.3.2 Splitting extensions

In the same context as before, $\mathfrak{B}$ is a *splitting extension* of $\mathfrak{A}$, over $\mathrm{irr}(\vec{b}/A)$, if $\mathrm{orb}(\vec{b}/A) \subseteq \mathfrak{B}$ and $\mathfrak{B} \subseteq \mathrm{dcl}\,(\mathfrak{A} \cup \mathrm{orb}(b/A))$. Here, $\mathrm{irr}(\vec{b}/A)$ is a formula that witnesses that $\vec{b}$ is algebraic over $\mathfrak{A}$, with minimal number of witnesses (solutions).

A few facts:

- $\mathfrak{B}$ definably closed and splitting over $\mathfrak{A}$ implies that $\mathfrak{B}$ is normal over $\mathfrak{A}$.
- $\mathfrak{A} \prec \mathfrak{B} \prec \mathfrak{C}$ all these extensions of finite degree implies that

$$\deg(\mathfrak{C}/\mathfrak{A}) = \deg(\mathfrak{C}/\mathfrak{B}) \cdot \deg(\mathfrak{B}/\mathfrak{A}).$$

A.4 A key step: coding finite sets

A crucial step in all the previous translation consisted in *coding finite sets.* $\mathfrak{C} = \mathrm{dcl}(\mathfrak{C})$ is a normal extension of $\mathfrak{A} = \mathrm{dcl}(\mathfrak{A})$. If we let G be the group $Aut(\mathfrak{C}/\mathfrak{A})$, **subgroups** $H \leq G$ correspond to **definably closed** intermediate extensions $\mathfrak{A} \leq \mathrm{Fix}(H) = \{c \in \mathfrak{C} : \forall h \in H\,[h(c) = c]\}$. See Figure 5.

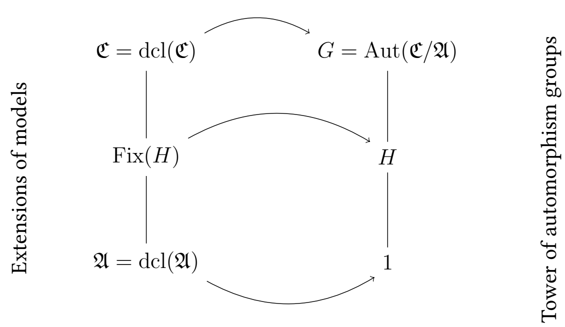


Figure 5: Galois correspondence for models of T

A.4.1 The crucial notion

The theory T **codes finite sets of tuples** if for every natural number n, for every finite $F \subseteq M^n$, there exists some b such that for every $\sigma \in \mathrm{Aut}(\mathbb{M})$,

$$\sigma(b) = b \iff \sigma(F) = F.$$

B Makkai-Reyes, stable interpretations

B.1 The Makkai-Reyes approach: models as functors

(Makkai-Reyes)

- Let us fix a first order theory T in a vocabulary L, and let us consider the category $\mathcal{T}$ of **the definables** of T.
- Objects are equivalence classes between L-formulas mod T. $A :: \varphi(x)$, etc.
- Morphisms correspond to definable functions: if $A :: \phi(x)$ and $B :: \psi(y)$, a definable morphism $f : A \to B$ is a definable $f :: \chi(x, y)$ such that $T \models \forall x \forall y(\chi(x, y) \to \varphi(x) \wedge \psi(y))$ and $T \models \forall x(\varphi(x) \to \exists y \chi(x, y))$.
- Given any L-structure $\mathfrak{M}$ and a formula $\varphi(x)$, the **solution set** is

$$\varphi(\mathfrak{M}) = \{a \in M_x \mid \mathfrak{M} \models \varphi(x)\} .$$

- With this, **we regard models of** T **as functors** from $\mathcal{T}$ to **Set**: $\mathfrak{M}(A) = \varphi(\mathfrak{M})$. Natural transformations $\equiv$ elementary maps.

The category $\mathcal{T} = Def(T)$ is Boolean (regular, and with Boolean algebras of subobjects) and extensive (co-products exist and they form an equivalence between the categories $Sub(X) \times Sub(Y)$ and $Sub(X \sqcup Y)$).

$$\text{Boolean categories} \qquad \longleftrightarrow \qquad \text{First Order}$$

An **interpretation** between T_0 and T is a Boolean and extensive morphism

$$\iota : \mathcal{T}_0 \to \mathcal{T}$$

between the categories $\mathcal{T}_0$ and $\mathcal{T}$ (in the vocabularies L_0 and L).

(ι preserves finite limits, induces homomorphisms of Boolean algebras in subobjects and respects images - and respects co-products)

B.2 Interpretation functor between classes of models

Given $\iota : \mathcal{T}_0 \to \mathcal{T}$,

$$\iota^* : Mod(T) \to Mod(T_0)$$
$$\mathfrak{M} \models T \mapsto \iota^*(\mathfrak{M}) = \mathfrak{M}_0$$

where

$$\mathfrak{M}_0 = \mathfrak{M} \circ \iota : \mathcal{T} \to \mathbf{Set}$$

and if $\sigma : \mathfrak{N} \to \mathfrak{M}$ is an elementary embedding ($\sigma = (\sigma_Y)_{Y \in \mathcal{T}}$) then

$$\iota^*(\sigma) : \mathfrak{N}_0 \to \mathfrak{M}_0 : \iota^* \sigma_X = \sigma_{\iota X}$$

for each $X \in \mathcal{T}_0$.

B.3 Examples - ACF, RCF

An interpretation we have known for some 200 years is the following:

$$\iota : Def(ACF) \to Def(RCF)$$

$$\iota(K) = R^2 \text{ , componentwise sum}$$
$$\text{multiplication } (a, b)(\iota\cdot)(c, d) = (ac - bd, bc + ad)$$

if $R \models RCF$

$$\iota^*(R) = R[\sqrt{-1}].$$

Many other natural examples: retracts, boolean algebras in boolean rings, etc.

B.4 Stable Interpretations - a bit on Galois theory

Stability is reflected in a natural way in interpretations:

An interpretation $\iota : \mathcal{T}_0 \to \mathcal{T}$ is **stable** if for each model $\mathfrak{M}$ of T, the "expanded interpretation" $\iota^{\mathfrak{M}} : \mathcal{T}_0^{\mathfrak{M}_0} \to \mathcal{T}^{\mathfrak{M}}$ is an immersion. This means each definable in ιX ($X \in \mathcal{T}_0$) using parameters from is the image of a definable set in X using parameters from $_0$.

If T is a stable theory and $\iota : \mathcal{T}_0 \to \mathcal{T}$ is an interpretation, then ι is a stable interpretation and T_0 is a stable theory.

Kamensky, in his thesis (with Hrushovski) went as far as reframing a "Galois theory" of model theory for internal covers - Galois theory à la Grothendieck (SGA 1).

B.5 The Galois group of a first order theory

(Assuming that T eliminates imaginaries), A definably closed,

$$Gal(T/A) := Aut(M)/Autf(M)$$

where M is a saturated model of T and

$$Autf(M) = \langle \bigcup_{A \subset N \prec M} Aut_N(M) \rangle$$

This is an invariant of the theory, allowing a Galois connection between definably closed submodels of M and closed subgroups of the Galois group.

C On Hrushovski's definability patterns

C.1 The pattern language: first obstruction

Given $M \models T$, $\mathcal{L}$ consists of predicates Def_t, $t = (\varphi_1, \dots, \varphi_n; \alpha)$, interpreted in $S = S(M)$ as

$$\mathrm{Def}_t^S = \{(p_1, \dots p_n) : \forall a \in \alpha(M) \bigvee_{1 \le i \le n} (\varphi_i(x, a) \in p_i)\}.$$

For $n = 1$, the predicate $\mathrm{Def}_{\varphi;\alpha}$ captures those 1-types of T for which α acts as a (partial) definition scheme for φ.

First obstruction beyond First Order: Which formulas to use for definitions???

C.2 Possible workarounds

The pattern theory $\mathcal{T}$ of T is the set of all (local) primitive universal $\mathcal{L}$-sentences true in $S(M)$ for some $M \models T$.

Galois-types have very good behaviour in AECs... However, the collection of all Galois-types is not necessarily well-equipped with a "standard" topology!

Definability (of types) has been treated (Shelah, Grossberg, Vasey, VanDieren, Boney, V.) in a weak, abstract way in AECs through non-splitting. Shelah even calls non-splitting extensions in the NIP theories context *weakly definable* types.

We may use sentences of new logics $\mathbb{L}^{1,\mathrm{aec}}$, to test syntactic definability patterns to build $\mathcal{L}$. Work in progress with my students in Bogotá and with Shelah.

C.3 Galois Morleyizations

Vasey in 2016 introduced "Galois Morleyizations" for AECs. Essentially, expanding L by adding predicates for all Galois types (orbits). He proved under "tameness" assumptions that part of the content of an AEC $\mathcal{K}$ may be read *functorially* from a SYNTACTIC counterpart of the AEC $\mathcal{K}$. In particular, *stable* AECs have canonical forking relations defined both semantically and syntactically. So far, there is (as far as I have seen) no study of definability of types in that context. But that should enter the picture...

C.4 Abstract cores (Hrushovski)

A *core* for T is an $\mathcal{L}$-structure $\mathcal{J}$ such that

- For any (orbit-bounded) $M \models T$, there is an $\mathcal{L}$-embedding
$$j : \mathcal{J} \to S(M).$$
- For any j as above, there is a retraction $r : S(M) \to \mathcal{J}$ such that $r \circ j = Id_{\mathcal{J}}$.

Cores exist, are unique up to isomorphism. $Aut(\mathcal{J})$ has a natural locally compact topology (basic closed sets of the form $W(R : a, b) = \{g : \mathrm{Def}_t(ga_1, \dots, ga_n, b_1, \dots, b_m)\}$

Calibrating the *existence* of such cores for additional contexts is doable: choice of logic or plain selection of predicates behaving as if coming from a concrete definability pattern.

C.5 The core of j?

Example: the core of j (the j-mapping), as axiomatized by Boris Zilber and Adam Harris in $\mathbb{L}_{\omega_1,\omega}$
$$((\mathbb{H}, \sigma)_{\sigma\in\Gamma}, j, (\mathbb{C}, +, \cdot, 0, 1))$$
is an interesting case for study (here, the quasiminimality of the structure, plus the axiomatization in $\mathbb{L}_{\omega_1,\omega}$ are key).

References

[1] Michael Artin, Alexander Grothendieck, J. L. Verdier, N. Bourbaki, P. Deligne, and Bernard Saint-Donat, editors. *Séminaire de géométrie algébrique du Bois-Marie 1963–1964. Théorie des topos et cohomologie étale des schémas. (SGA 4). Un séminaire dirigé par M. Artin, A. Grothendieck, J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne, B. Saint-Donat. Tome 1: Théorie des topos. Exposés I à IV. 2e éd.*, volume 269 of *Lect. Notes Math.* Springer, Cham, 1972.

[2] Itaï Ben Yaacov. Model theoretic stability and definability of types, after A. Grothendieck. *The Bulletin of Symbolic Logic*, 20(4):491–496, 2014.

[3] Luc Bélair and Bruno Poizat. *Œuvres choisies de Marc Krasner - un mathématicien hors du commun.* L'Harmattan, 2019.

[4] Artem Chernikov and Nadja Hempel. On n-dependent groups and fields. II. *Forum of Mathematics, Sigma*, 9:51, 2021. Id/No e38.

[5] Artem Chernikov, Daniel Palacin, and Kota Takeuchi. On n-dependence. *Notre Dame Journal of Formal Logic*, 60(2):195–214, 2019.

[6] Gabriel Conant. Interactive webpage (https://www.forkinganddividing.com/), 2023.

[7] John A. Cruz-Morales, Andrés Villaveces, and Boris Zilber. Around logical perfection. *Theoria*, 87:971–985, 2021.

[8] Michel Demazure and Alexander Grothendieck. Schémas en groupes. Fasc. I. Exposés I à IV. Séminaire de Géométrie Algébrique de l'Institut des Hautes études Scientifiques, 1963, dirigé par Michel Demazure et Alexander Grothendieck. 2ème éd. Bures-Sur-Yvette (Essonne): Institut des Hautes Études Scientifiques. v, 252 p. (not consecutively paged) (1963)., 1963.

[9] Christian Espíndola. Infinitary first-order categorical logic. *Annals of Pure and Applied Logic*, 170(2):137–162, 2019.

[10] Johan Felipe García-Vargas. Internalidad: Un camino de la Teoría de Modelos hacia la Teoría de Galois. Master's thesis, Universidad Nacional de Colombia, 2017.

[11] Rami Grossberg. Classification theory for abstract elementary classes. In *Logic and algebra*, pages 165–204. Providence, RI: American Mathematical Society (AMS), 2002.

[12] A. Grothendieck. Critères de compacité dans les espaces fonctionnels généraux. *American Journal of Mathematics*, 74:168–186, 1952.

[13] Alexander Grothendieck. *Récoltes et semailles.* TEL. Gallimard, 2021.

[14] Ehud Hrushovski. Definability patterns and their symmetries, November 2019. ArXiv: 1911.01129.

[15] Moshe Kamensky. A categorical approach to internality. In *Models, logics, and higher-dimensional categories: A tribute to the work of Mihály Makkai. Proceedings of a conference, CRM, Montréal, Canada, June 18–20, 2009*, pages 139–156. Providence, RI: American Mathematical Society (AMS), 2011.

[16] Moshe Kamensky. Model theory and the Tannakian formalism. *Transactions of the American Mathematical Society*, 367(2):1095–1120, 2015.

[17] Juliette Kennedy. *Gödel, Tarski and the lure of natural language. Logical entanglement, formalism freeness.* Cambridge: Cambridge University Press, 2021.

[18] Karim Khanaki. Stability, the NIP, and the NSOP: model theoretic properties of formulas via topological properties of function spaces. *Mathematical Logic Quarterly (MLQ)*, 66(2):136–149, 2020.

[19] Karim Khanaki and Anand Pillay. Remarks on the NIP in a model. *Mathematical Logic Quarterly (MLQ)*, 64(6):429–434, 2018.

[20] F. W. Lawvere. The category of categories as a foundation for mathematics. Proc. Conf. Categor. Algebra, La Jolla 1965, 1-20 (1966)., 1966.

[21] Michael Lieberman, Jiří Rosický, and Sebastien Vasey. Forking independence from the categorical point of view. *Advances in Mathematics*, 346:719–772, 2019.

[22] Michael Lieberman, Jiří Rosický, and Pedro Zambrano. Tameness in generalized metric structures. *Archive for Mathematical Logic*, 62(3-4):531–558, 2023.

[23] Michael J. Lieberman. Category-theoretic aspects of abstract elementary classes. *Annals of Pure and Applied Logic*, 162(11):903–915, 2011.

[24] Michael J. Lieberman. Category-theoretic aspects of abstract elementary classes. *Annals of Pure and Applied Logic*, 162(11):903–915, 2011.

[25] Jacob Lurie. *Higher topos theory*, volume 170 of *Ann. Math. Stud.* Princeton, NJ: Princeton University Press, 2009.

[26] M. Makkai and G. E. Reyes. Model-theoretical methods in the theory of topoi and related categories. I. *Bulletin de l'Académie Polonaise des Sciences, Série des Sciences Mathématiques, Astronomiques et Physiques*, 24:379–384, 1976.

[27] Michael Makkai and Robert Paré. *Accessible categories: The foundations of categorical model theory*, volume 104 of *Contemp. Math.* Providence, RI: American Mathematical Society, 1989.

[28] Alice Medvedev and Ramin Takloo-Bighash. An invitation to model-theoretic Galois theory. *The Bulletin of Symbolic Logic*, 16(2):261–269, 2010.

[29] Bruno Poizat. Une théorie de Galois imaginaire. *The Journal of Symbolic Logic*, 48:1151–1170, 1983.

[30] Ori Segel. Positive definability patterns. July 2022. ArXiv 2207.12449.

[31] Saharon Shelah. Classification theory for nonelementary classes, I. The number of uncountable models of $\psi \in L_{\omega_1,\omega}$. Part A. *Israel Journal of Mathematics*, 46:212–240, 1983.

[32] Saharon Shelah. Classification theory for nonelementary classes, I. The number of uncountable models of $\psi \in L_{\omega_1,\omega}$. Part B. *Israel Journal of Mathematics*, 46:241–273, 1983.

[33] Saharon Shelah. *Classification Theory and the Number of Non-Isomorphic Models.*, volume 92 of *Stud. Logic Found. Math.* Amsterdam etc.: North-Holland, 2nd rev. ed. edition, 1990.

[34] Tamás Szamuely. *Galois groups and fundamental groups*, volume 117 of *Camb. Stud. Adv. Math.* Cambridge: Cambridge University Press, 2009.

[35] Sebastien Vasey. The categoricity spectrum of large abstract elementary classes. *Selecta Mathematica. New Series*, 25(5):51, 2019. Id/No 65.

[36] Andrés Villaveces. Categoricity, External and Internal: An Excerpt from a Conversation with Saharon Shelah. *Theoria*, 87:1001–1012, 2021.