On Bill Lawvere's seminal 'The Category of Categories' (1964-1966)

Or Grounding What's To Be Grounded, If. . .

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Homage to Bill Lawvere / Homenaje a William Lawvere

Universidad Nacional de Colombia, May '23

Our path today

The Foundational Aim in CCFM - The Zeitgeist of the mid-1960s

Lawvere in Colombia: a brief interlude

The AfterMath / The Origins

Bill Lawvere, in 1997:

We have had to fight against the myth of the mainstream which says, for example, that there are cycles during which at one time everybody is working on general concepts, and at another time anybody of consequence is doing only particular examples, whereas in fact serious mathematicians have always been doing both.

Toposes of Laws of Motion - F. W. Lawvere - 1997

Plan

The Foundational Aim in CCFM - The Zeitgeist of the mid-1960s

- Lawvere - Grothendieck - Cohen - Mac Lane
- Some excerpts from Lawvere's paper
- Some surprises (?)

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- Later avenues - Grounding in Categorical Logic
- Earlier groundings

The mid 1960s: some names and works (!)

- Lawvere '65: *The Category of Categories as a Foundation for Mathematics*
- Cohen '63: *The Independence of the Continuum Hypothesis*
- Grothendieck '62 to. . . : *Séminaire de Géométrie Algébrique*
- Mac Lane '65: *Categorical algebra*
- Morley '65: *Categoricity in Power*

Lawvere's paper is part of an intriguing Zeitgeist (partially even in a shared graduate student house in Chicago - Morley, Cohen; Morley as a student of Mac Lane's).

Seminal work in *model theory, set theory, algebraic geometry* and. . . **category theory**

Openings

Categorical Foundations

Forcing: geometry of set theory possibilities

Geometry made functorial

Model Theory's dimensionality started

Categorical algebra

Declaration of intent: relevant properties. . .

The Category of Categories as a Foundation for Mathematics*,**

By

F. WILLIAM LAWVERE

In the mathematical development of recent decades one sees clearly the rise of the conviction that the relevant properties of mathematical objects are those which can be stated in terms of their abstract structure rather than in terms of the elements which the objects were thought to be made of. The question thus naturally arises whether one can give a foundation for mathematics which expresses wholeheartedly this conviction concerning what mathematics is about, and in particular in which classes and membership in classes do not play any role. Here by "foundation" we mean a single system of first-order axioms in which all usual mathematical objects can be defined and all their usual properties proved. A foundation of the sort we have in mind would seemingly be much more natural and readily-useable than the classical one when developing such subjects as algebraic topology, functional analysis, model theory of gene-

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Elementary theory of abstract categories

Lawvere posits from the ground his construction:

- Basic formulas: for any letters x, y, u, A, B

  $\Delta_0(x) = A, \quad \Delta_1(x) = B, \quad \Gamma(x, y; u), \quad x = y$ are formulas, and

- if ϕ, ψ are formulas, then so are $\phi \wedge \psi$, $\phi \vee \psi$, $\phi \rightarrow \psi$ and $\neg\phi$; also,
- if ϕ is a formula and x is a letter, then $\forall x \phi, \exists x \phi$ are formulas.

Axioms

in which every occurence of each letter x is within the scope of a quantifier $\forall x$ or $\exists x$.

The theorems of the elementary theory of abstract categories are all those sentences which can be derived by logical inference from the following axioms (it is understood that Δ_0, Δ_1 are unary function symbols)

Four bookkeeping axioms

$$\Delta_i(\Delta_j(x)) = \Delta_j(x), \quad i, j = 0, 1.$$

$$\Gamma(x, y; u) \quad \text{and} \quad \Gamma(x, y; u') \Rightarrow u = u',$$
$$\exists u[\Gamma(x, y; u)] \Longleftrightarrow \Delta_1(x) = \Delta_0(y),$$
$$\Gamma(x, y; u) \Rightarrow \Delta_0(u) = \Delta_0(x) \quad \text{and} \quad \Delta_1(u) = \Delta_1(y).$$

Identity axiom

$$\Gamma(\Delta_0(x), x; x) \quad \text{and} \quad \Gamma(x, \Delta_1(x); x).$$

Associativity axiom

$$\Gamma(x, y; u) \quad \text{and} \quad \Gamma(y, z; w) \quad \text{and}$$
$$\Gamma(x, w; f) \quad \text{and} \quad \Gamma(u, z; g) \Rightarrow f = g.$$

Besides the usual means of abbreviating formulas, the following (as well as others) are special to the elementary theory of abstract categories:

$$A \xrightarrow{f} B \quad \text{means} \quad \Delta_0(f) = A \quad \text{and} \quad \Delta_1(f) = B,$$

Many usual categorical notions. . .

. . . can be expressed as formulas in the elementary theory of abstract categories.

Express abstractly (here) the basics. Later, commutativity, retracts, generators, etc.

G is a generator if $\forall f \forall g [\Delta_0(f) = \Delta_0(g) \wedge \Delta_1(f) = \Delta_1(g) \wedge f \neq g \Rightarrow \exists x [\Delta_0(x) = G \wedge \Delta_1(x) = \Delta_0(f) \wedge xf \neq xg]]$.

Also, finite limits, products, coproducts, terminal objects, coterminal objects, equalizers, coequalizers, pullbacks and pushouts. . . are elementary.

Infinite limits and colimits are not! Being "finitely generated" is not!

FUNctors

The world of functors is also a category...

The theory "splits" in two levels, a **basic** theory and a **stronger** theory!

A technical point: the adjunction of **two constants** ∂_0 and ∂_1.

$1, 2, 3, \ldots$ - Duality now possible

$$\exists 1 \, \forall A \, \exists ! x [A \xrightarrow{} 1].$$

A functor is called *constant* iff it factors through 1. We also find it a great notational convenience to assume the following "partial skeletal axiom":

$$\forall x [A \xrightarrow{x} A \quad \text{and} \quad \mathrm{Iso}(x) \Rightarrow x = A] \quad \text{and} \quad A \cong B \Rightarrow A = B .$$

That is, if the identity is the only endofunctor of A which is an automorphism, then A is the only category in its isomorphism class. For example, 1 is the unique terminal category. We now state axioms characterizing 2:

∂_0 and ∂_1 are constant .
$\Gamma(\partial_i, \partial_j; \partial_j)$, $\quad i, j = 0, 1$.
$\partial_0 \neq \partial_1$, $\quad \partial_i \neq 2$, $\quad i = 0, 1$.
$\forall x [2 \xrightarrow{x} 2 \Rightarrow x = \partial_0 \quad \text{or} \quad x = \partial_1 \quad \text{or} \quad x = 2]$.
2 is a generator.
If C is any generator, then 2 is a retract of C.

The intuitive validity of the last statement is easily seen with the help of the category E to be defined presently.

Proposition. If C is any generator with exactly three endofunctors, two of which are constant, and which is a retract of any other generator, then $C = 2$.

We remark that a simpler set of properties hoped by Freyd to characterize 2 [*Abelian Categories*, Harper and Row 1964] fails to do so since the following category also has exactly two objects and three endofunctors:

Objects 0, 1, 2, 3, etc., yield notions of functoriality and adjoints.

Discrete cores, skeletons

$$A \cong A \times 1 \xrightarrow{A \times \bar{\partial}_i} A \times 2 \xrightarrow{\bar{\varphi}} B \qquad i = 0,1$$

where $\bar{\varphi} = (A \times \varphi)\, e$, the e being of course the evaluation functor. If $A \xrightarrow{f} B$, we denote by 1_f the object of B^A corresponding to it. In particular 1_A is a distinguished object $2 \to A^A$ in A^A, and $1_{fg} = 1_f \circ 1_g$.

Definition. The category A is said to be *discrete* (or to be a *set*) iff A^τ is an isomorphism $A^2 \cong A^1$. That is, every morphism in a set is an object.

Axiom. *For any category A there is a discrete category A_c with a functor $A \to A_c$ such that for any functor $A \to B$ from A to a discrete category there is exactly one functor making this diagram commute*

$$\begin{array}{ccc} A & \to & A_c \\ & \searrow & \downarrow \\ & & B \end{array} \quad .$$

This A_c is called the *set of components* of A.

Axiom. Dualize the proceeding axiom. Thus every category A has a maximal discrete subcategory $|A|$, called for obvious reasons the *set of objects* of A. The "absolute value" notation for the set of objects will be used consistently. By the *set of morphisms* of A we understand the discrete category $|A^2|$, since A^2 is of course a category whose objects correspond to morphisms in A. In particular

$$|N^2|$$

is called the *set of nonnegative integers*, where N is the monoid of non-

Discrete categories, forgetful functors, etc.

. . . and a kind of completeness theorem for his system

The Category of Categories as a Foundation

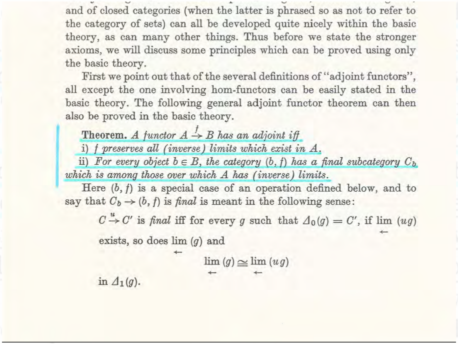
limits holds equally well for metacategories and metafunctors (i.e. "subcategories" of the universe defined by formulas but which, like the full metacategory of all sets, cannot necessarily be represented by an actual category in the universe,) it follows that sets are closed under the formation of finite roots. In fact,

Metatheorem. *Define a relative interpretation of the elementary theory of abstract categories into the basic theory by relativizing all quantifiers to functors between discrete categories. Then in the induced theory, all theorems of the elementary theory of the category of sets* [Lawvere, Proc. Nat. Ac. Sc. USA Dec. 1964] *are provable.*

Thus one could, by referring only to discrete categories, develop on the basis of the axioms we have so far assumed such subjects as number theory, calculus, linear operators in Hilbert space, etc. (such a program, of course, would not make the most efficient use of the functorial method.) In such a development, as well as in our work here, it is convenient to use

Lawvere proves that a functor $f : A \to B$ has an adjoint iff

- f preserves all (inverse) limits which exist in A,
- for every object $b \in B$, the category (b, f) has a final subcategory C_b, which is among those over which A has (inverse) limits.



and of closed categories (when the latter is phrased so as not to refer to the category of sets) can all be developed quite nicely within the basic theory, as can many other things. Thus before we state the stronger axioms, we will discuss some principles which can be proved using only the basic theory.

First we point out that of the several definitions of "adjoint functors", all except the one involving hom-functors can be easily stated in the basic theory. The following general adjoint functor theorem can then also be proved in the basic theory.

Theorem. *A functor $A \xrightarrow{f} B$ has an adjoint iff*

i) f preserves all (inverse) limits which exist in A.

ii) For every object $b \in B$, the category (b, f) has a final subcategory C_b which is among those over which A has (inverse) limits.

Here (b, f) is a special case of an operation defined below, and to say that $C_b \to (b, f)$ is *final* is meant in the following sense:

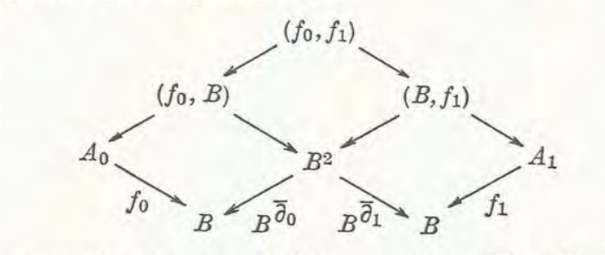
$C \xrightarrow{u} C'$ is *final* iff for every g such that $\Delta_0(g) = C'$, if $\lim\limits_{\leftarrow} (ug)$ exists, so does $\lim\limits_{\leftarrow} (g)$ and

$$\lim_{\leftarrow} (g) \cong \lim_{\leftarrow} (ug)$$

in $\Delta_1(g)$.

The core: the (f_0, f_1) operation

For two functors $f_\ell : A_\ell \to B \qquad \ell = 0, 1$, the category (f_0, f_1) is such that the three squares below are pullbacks:



This gives in specific cases <u>forgetful</u> functors, hom-sets, the well-ordered natural numbers $\omega = (0, N)$, N the additive monoid of non-negative integers (or the coequalizer of $\overline{\partial}_0, \overline{\partial}_1 : 1 \to 2$). . .

Lawvere develops power categories, a robust theory of natural transformations, all in terms of his basic language and ∂_0, ∂_1 (and duality).

He also provides enough tools for Yoneda Lemmas in his context.

And a version of Lévy's and Bernays' Reflection Principle: if $\phi(v_0, \ldots, v_{n-1})$ is any formula, then $\phi(v_0, \ldots, v_{n-1}) \Rightarrow \exists C[C$ is C-complete and full in the universe and $v_0, \ldots, v_{n-1} \in C \wedge C \models \phi(\overline{v}_0, \ldots, \overline{v}_{n-1})]$.

(The $\overline{v}_i$ here denote the morphisms in C corresponding the functors v_i.)

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Lawvere in Colombia - 1983

In 1983, Jesús Hernando Pérez organized a Categorical Logic Meeting in Bogotá with Bill Lawvere and other category theorists. . .

Ieke Moerdijk, Anders Kock, Bill Lawvere, Luis Español, others. . .

More. . . (with many thanks to Clara Helena Sánchez!)

Bill Lawvere, Bogotá

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Xavier Caicedo, Luis Español, Anders Kock, Ellie Anne Duque, Ieke Moerdijk, Bill Lawvere - Villeta, Colombia

More. . . (with many thanks to Clara Helena Sánchez!)

Bill Lawvere - Ieke Moerdijk (in Villeta, Colombia)

Plan

What came after CCFM? And what was its wider context?

I close this short presentation based on Bill Lawvere's CCFM article by looking (not overly systematically) at what came afterwards and before, at the AfterMath and the (historical) Context of his work.

What came after CCFM? And what was its wider context?

I close this short presentation based on Bill Lawvere's CCFM article by looking (not overly systematically) at what came afterwards and before, at the AfterMath and the (historical) Context of his work.

Disclaimer: this is all very tentative, and people in this audience surely may have a lot of precisions to add to my remarks!

The AfterMath...

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- The more recent (2019) work of Lurie in Ultracategories [an attempt to obtain a Completeness Theorem for $(\infty, 1)$-categories]...

For sure, there is much more, but. . .

What emerges from such a list bears witness to at least three widely different directions of research at the crossroads of

logic **computer science** **physics** **philosophy**

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In addition to all the developments mentioned, I would like to mention some much more general human quests that are, in some way, being addressed by Lawvere in his work, in this paper.

Interrelatedness (as described by Florensky in 1921)

. . . it has been established that there is no spirit that is incapable of acting on matter—we know an active, creative, organizing spirit. There is no self-sufficient matter an no separate, purified spirit.

The theory of internal secretion: every tissue, every cell, etc. acts on all others, exhibits internal activity; and when blood flows into it, it processes the latter and then exudes it with a particular imprint to other parts of the body. Thus, all is connected with all, and not just with the most proximate parts of the body: every particle of the body acts upon all the others; change in one place is reflected everywhere. The body is highly interconnected . . .

Pavel Florensky - *At the Watersheds of Thought: The Elements of a Concrete Metaphysics - Moscow, 1921.*

. . . or Leibniz three centuries ago!

. . . It is the confusion of the ideal with the actual which has muddled everything and caused the labyrinth of the composition of the continuum. Those who make up a line from points have looked for the first elements in ideal things or relations, something completely contrary to what they should have done. . .

however, number and line are not chimerical things. . .

From the fact that a mathematical body cannot be resolved into first constituents we can, at any rate, infer that it isn't real, but something mental, indicating only a possibility of parts, not anything actual. Indeed, [in] a mathematical line [. . .] the parts are only possible and completely indefinite.

Leibniz: letter to de Volder (1704) + Note on Foucher's Objection (1695)

We could go on and on, all the way to Plato's cave (and the shadows, and our reconstruction of the world from them
(As S. Weil says: Forme unique d'un objet dont on voit passer plusieurs fois des ombres différentes. Les captifs n'ont même pas idée du rapport objet-ombre.))

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But time to come back to our 21st Century!

¡Gracias! Thanks!!!