Randomizing Infinitary Logic

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Are there other logics at a similar point of equilibrium?

Are there strict strengthenings of $\mathcal{L}_{\omega\omega}$ with a Lindström-type characterization?

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$$\boxed{\mathcal{L}_{\kappa\omega} < \mathcal{L}^1_\kappa < \mathcal{L}_{\kappa\kappa}.}$$

Some **good** properties of $\mathcal{L}^1_\kappa$

- The logic $\mathcal{L}^1_\kappa$ satisfies Interpolation.

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- **First Characterization Theorem for $\mathcal{L}^1_\kappa$, Shelah [2012].** For an uncountable cardinal κ such that $\kappa = \beth_\kappa$, the logic $\mathcal{L}^1_\kappa$ has strong well ordering number κ, Löwenheim-Skolem number[1] κ and is a maximal such logic above $\mathcal{L}_{\kappa\omega}$.

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(As expected, the proofs of these two facts are intertwined. Strong well ordering number κ may be regarded as a weak form of Compactness, combined with a weak form of Löwenheim-Skolem.)

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Interpolation

- Craig($\mathcal{L}_{\kappa^+\omega}, \mathcal{L}_{(2^\kappa)^+\kappa^+}$) (Malitz 1971):
  If $\varphi \vdash \psi$, where φ is a τ_1-sentence and ψ is a τ_2-sentence and both are in $\mathcal{L}_{\kappa^+\omega}$ then there exists $\chi \in \mathcal{L}_{(2^\kappa)^+\kappa^+}(\tau_1 \cap \tau_2)$ such that

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  $$\mathcal{L}_{\kappa^+\omega} \leq \mathcal{L}^* \leq \mathcal{L}_{(2^\kappa)^+\kappa^+}$$
  and Craig($\mathcal{L}^*$).
- Shelah's $\mathcal{L}^1_\kappa$ balances this: $\mathcal{L}_{\kappa\omega} < \mathcal{L}^1_\kappa < \mathcal{L}_{\kappa\kappa}$ and Craig($\mathcal{L}^1_\kappa$).

Why is Interpolation an important property?

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- In the absence (or weakness) of Compactness, variants of Interpolation and its friends (plus Uniform Reducibility of Pairs) has deep connections with the (im-)possibility of defining classes with non-rigid models (Caicedo 2004).
- We also saw an interesting use of Interpolation earlier here in the work of Džamonja and Parente (inconsistency flows and preservation of formulas by sublogics), generalizing Van Benthem's result using Chu spaces.

Further model theory of $\mathcal{L}^1_\kappa$

In more recent work (2021), Shelah has proved that when κ is a strongly compact cardinal, elementary equivalence $\equiv_{\mathcal{L}^1_\kappa}$ has an algebraic characterization (à la Keisler-Shelah for FO logic):

$$\mathsf{M} \equiv_{\mathcal{L}^1_\kappa} \mathsf{N} \ \ \underline{\text{iff}} \ \ \exists\, (\mathcal{U}_i)_{i<\omega} \left(\lim_i \mathsf{M}^\omega/\mathcal{U}_i \approx \lim_i \mathsf{N}^\omega/\mathcal{U}_i \right),$$

where the $\mathcal{U}_i$ are κ-complete ultrafilters.

Where $\mathcal{L}^1_\kappa$ is rather weak

The logic $\mathcal{L}^1_\kappa$ is derived from a game, in the sense that a sentence is, by definition, a class of structures, closed under a certain Ehrenfeucht-Fraïssé type of game.

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The logic $\mathcal{L}^1_\kappa$ is derived from a game, in the sense that a sentence is, by definition, a class of structures, closed under a certain Ehrenfeucht-Fraïssé type of game.
This results in the absence of a *generative syntax*, i.e. a syntax defined in such a way that the set of all formulas can be obtained by closing the set of atomic formulas under negation, conjunction, quantifiers, and possibly other logical operations!

Why would we want a generative syntax?

The lack of generative syntax complicates, on one hand, applying the great model-theoretic properties of $\mathcal{L}^1_\kappa$ in real life situations, and on the other hand, further study of it and logics in its neighbourhood.

- The method of Skolem functions.
- or simply induction on the complexity of formula …

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Why would we want a generative syntax?

The lack of generative syntax complicates, on one hand, applying the great model-theoretic properties of $\mathcal{L}_{\kappa}^{1}$ in real life situations, and on the other hand, further study of it and logics in its neighbourhood.

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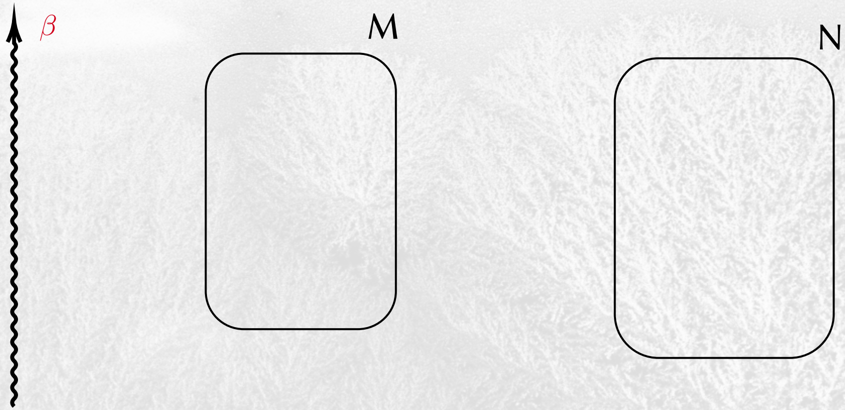
highly require generative syntax.
We believe that much more can be said about $\mathcal{L}_{\kappa}^{1}$ and even about more general questions regarding logics derived from games, by providing a generative syntax for $\mathcal{L}_{\kappa}^{1}$.

Shelah's game $\Game^{\beta}_{\theta}(M, N)$.



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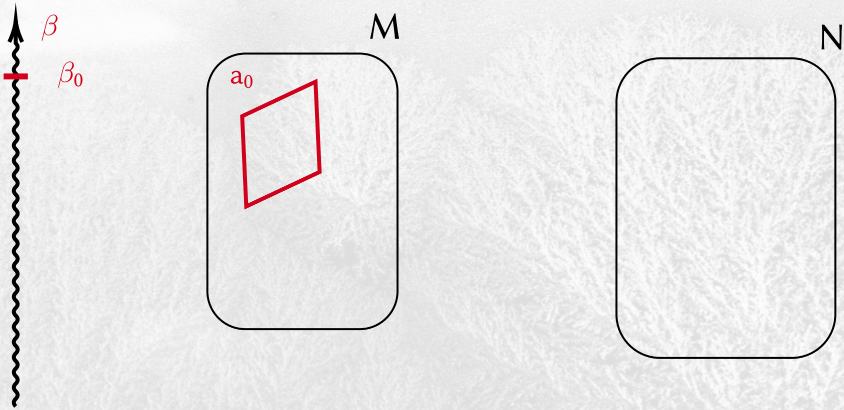

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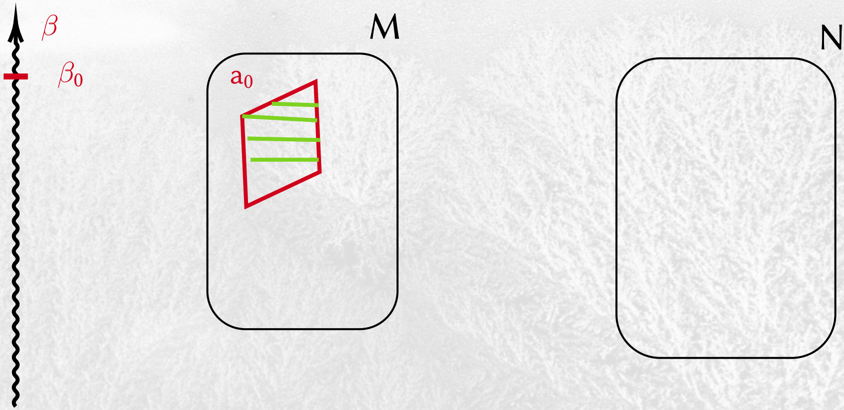

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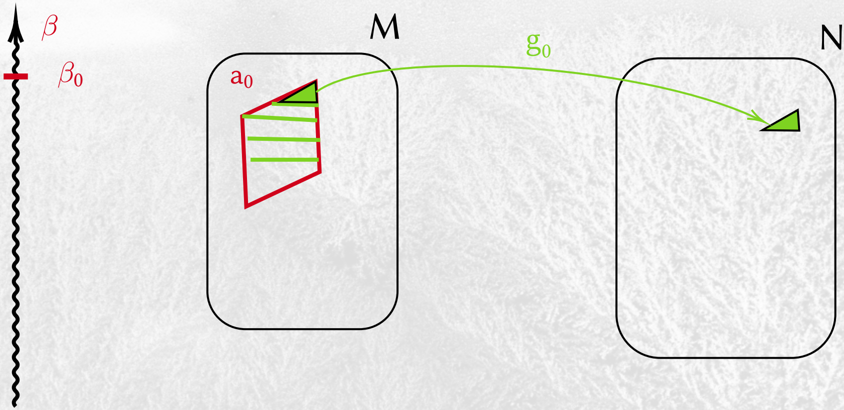

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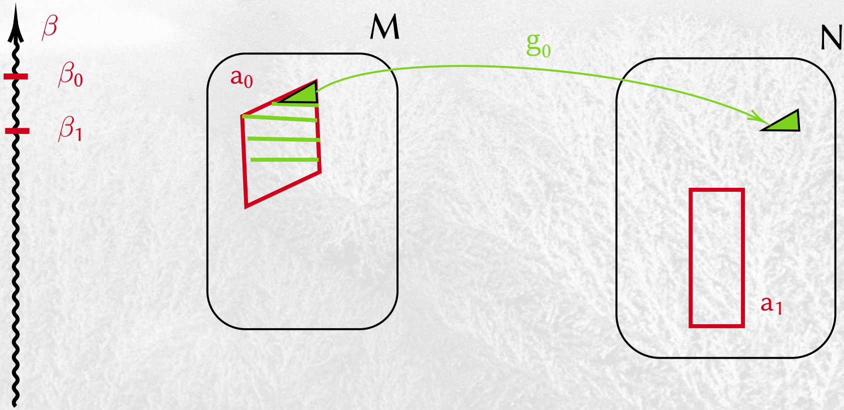

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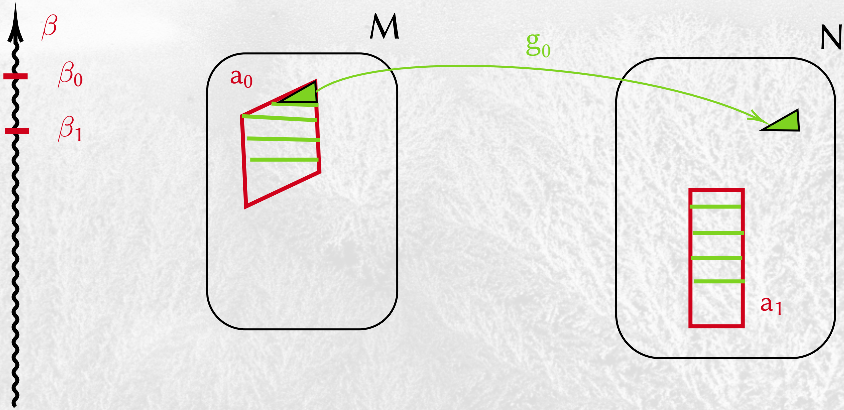

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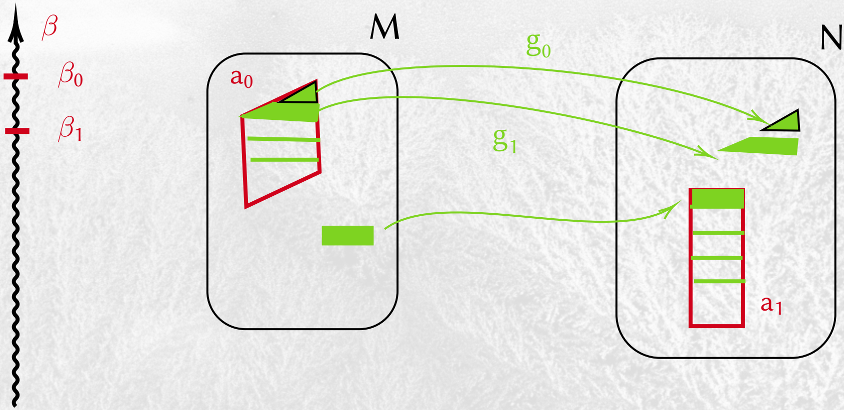

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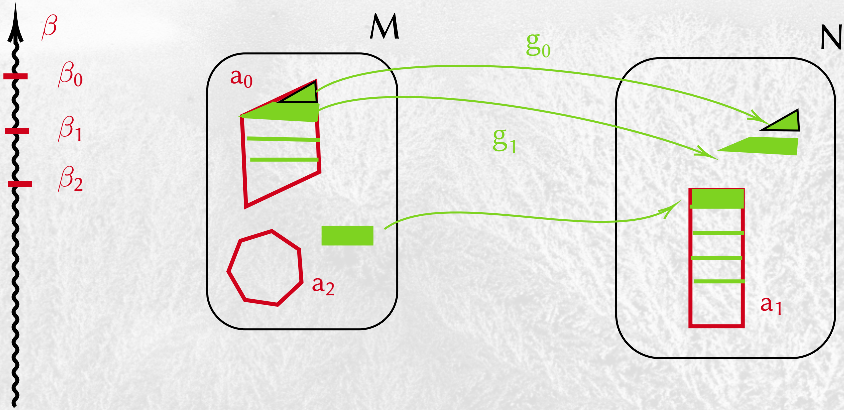

Shelah's game $\eth^\beta_\theta(M, N)$ of ordinal clock β.

$\forall$	$\exists$
$\beta_0 < \beta, \vec{a^0}$	
	$f_0 : \vec{a^0} \to \omega$, $g_0 : M \to N$ a p.i.
$\beta_1 < \beta_0, \vec{b^1}$	$f_1 : \vec{a^1} \to \omega$, $g_1 : M \to N$ a p.i., $g_1 \supseteq g_0$
$\vdots$	$\vdots$

Constraints:

- $\text{len}(\vec{a^n}) \leq \theta$
- $f^{-1}_{2n}(m) \subseteq \text{dom}(g_{2n})$ for $m \leq n$.
- $f^{-1}_{2n+1}(m) \subseteq \text{ran}(g_{2n})$ for $m \leq n$.

$\exists$ wins if she can play all her moves, otherwise $\forall$ wins.

Shelah's game equivalence (not [nec.] transitive!)

- $\mathsf{M} \sim^\beta_\theta \mathsf{N}$ iff $\exists$ has a winning strategy in the game.
- $\mathsf{M} \equiv^\beta_\theta \mathsf{N}$ is defined as the transitive closure of $\mathsf{M} \sim^\beta_\theta \mathsf{N}$.
- A union of $\leq \beth_{\beta+1}(\theta)$ equivalence classes of $\equiv^\beta_\theta$ for some $\theta < \kappa$ and $\beta < \theta^+$ is called a sentence of L^1_κ.

Shelah's game equivalence (not [nec.] transitive!)

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Notice the weirdness!

Plan

Shelah's Logic $\mathcal{L}^1_\kappa$: the good and the missing
- $\mathcal{L}^1_\kappa$ solves old questions
- A weak point of $\mathcal{L}^1_\kappa$: lack of generative syntax
- A quick description of the Shelah game

Cartagena Logic: a solution
- A description of the Cartagena game
- Boolean variables: avoiding tricky subformulas
- A collapsing map from $\mathcal{L}^{\mathsf{Bool}}_{\kappa,\kappa}$ to $\mathcal{L}^{\mathsf{c}}_\kappa$
- Cartagena recovered

A solution: Cartagena Logic (Kivimäki, Väänänen, V.)

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- We first define an expansion $\mathcal{L}^{\mathsf{Bool}}_{\kappa\kappa}$ of $\mathcal{L}_{\kappa\kappa}$: *Boolean expansion* of $\mathcal{L}_{\kappa\kappa}$: exactly the same sentences as $\mathcal{L}_{\kappa\kappa}$, but a new kind of variable allowed in formulas. These new variables allow, in a sense, **quantifying over moves** in a game.

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- We then define the **Cartagena game**, a simplified variant of the game of $\mathcal{L}^1_\kappa$. With the help of $\mathcal{L}^{\mathsf{Bool}}_{\kappa\kappa}$, we are able to build a syntax that completely corresponds to Cartagena game.

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Two logics, one outside $\mathcal{L}_{\kappa\kappa}$ (with a new kind of variables), and its projection inside $\mathcal{L}_{\kappa\kappa}$: $\mathcal{L}^{\mathrm{c}}_\kappa$.

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Moreover:

- We study model theory of Cartagena logic. Some of the good properties of $\mathcal{L}^1_\kappa$ can be proved in a stronger form than in $\mathcal{L}^1_\kappa$. This, as expected, results in a slightly weaker expressive power!

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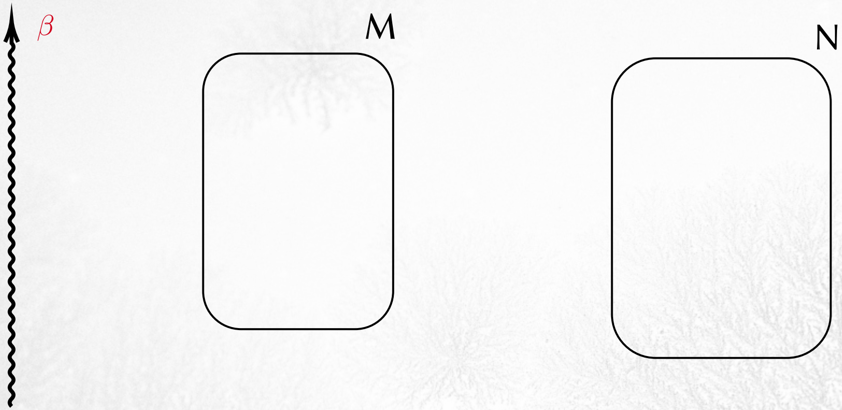
Our main result, thus, is the existence of a good approximation to logic $\mathcal{L}^1_\kappa$, with a simple generative syntax and rich model theory.

The Cartagena game $\eth^{\beta,c}_{\theta}(M, N)$.



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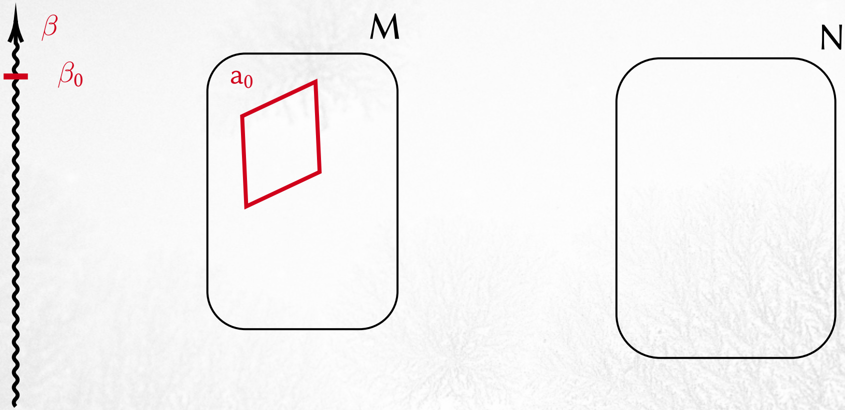

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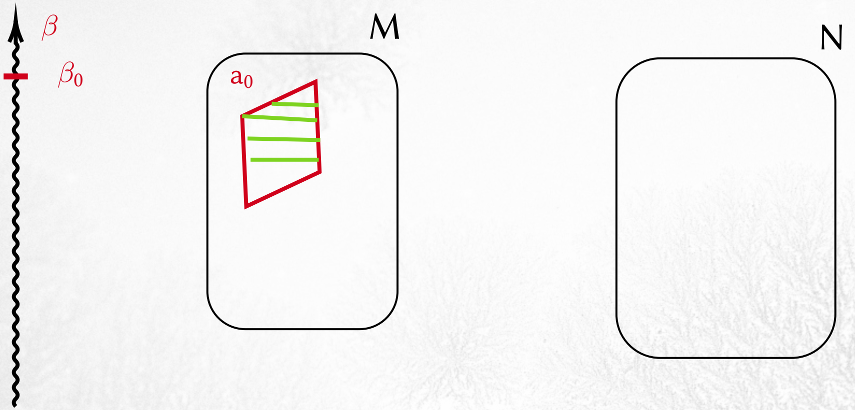

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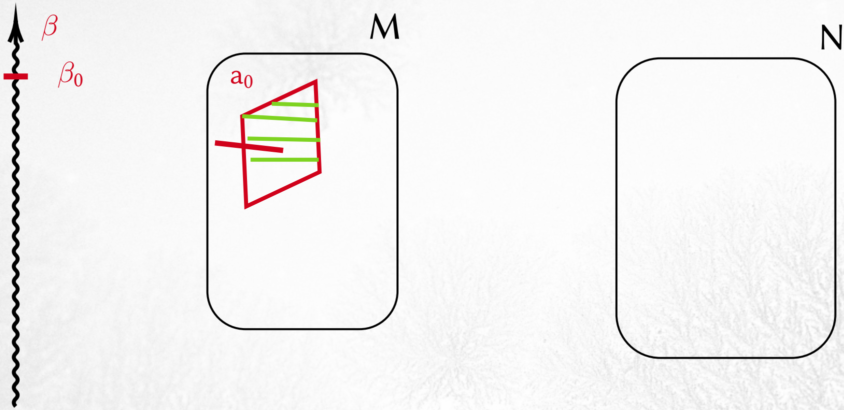

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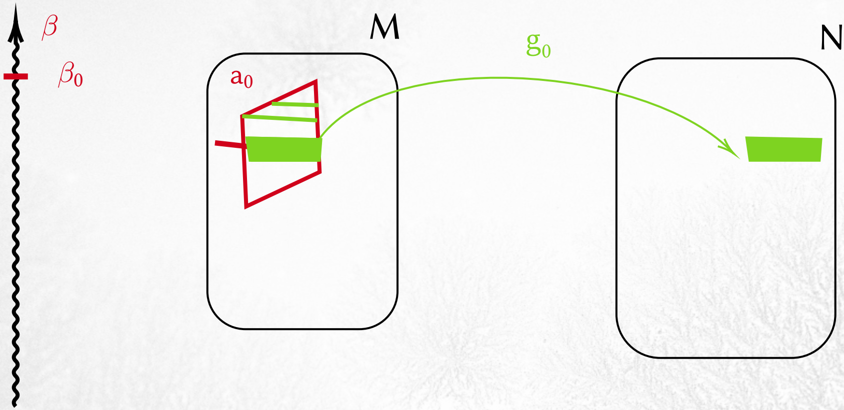

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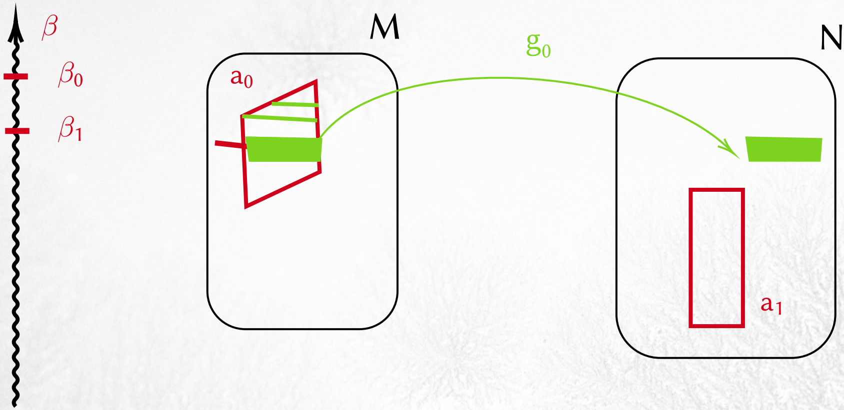

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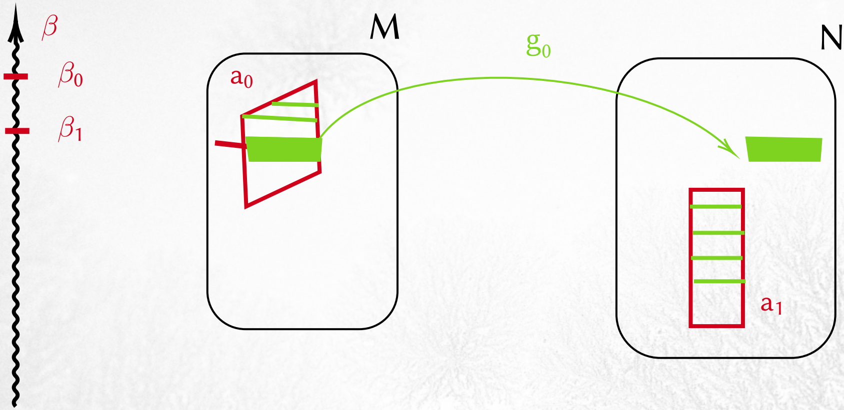

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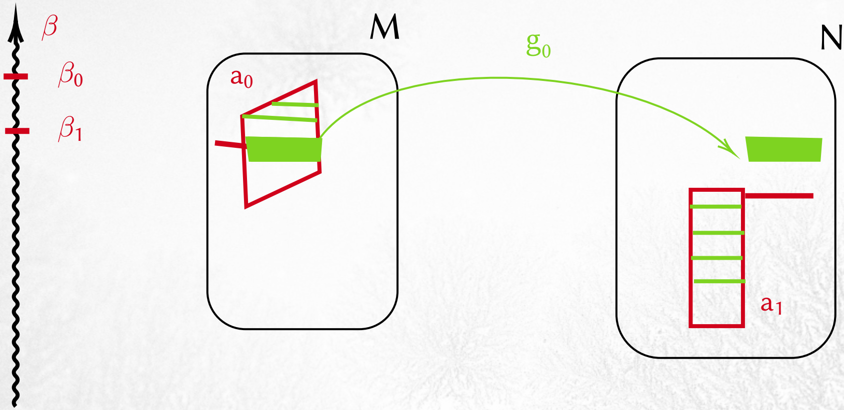

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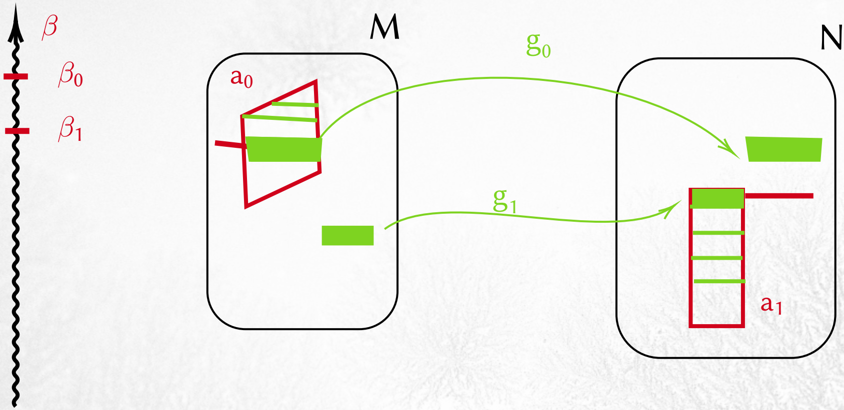

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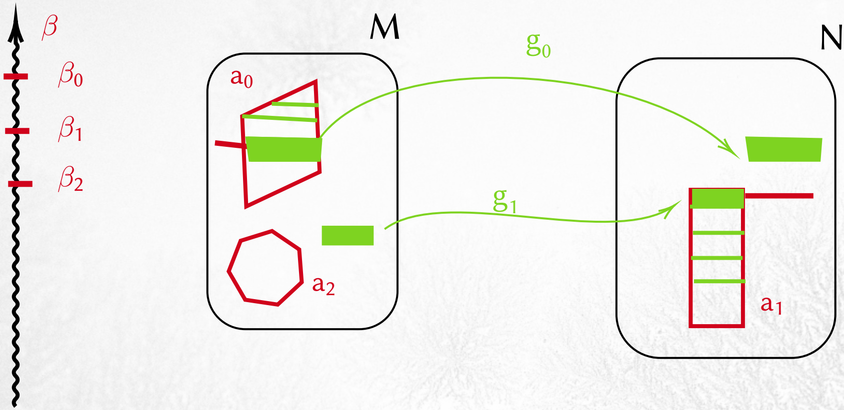

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- Between $\mathcal{L}_{\kappa\omega}$ and $\mathcal{L}_{\kappa\kappa}$, closer to $\mathcal{L}_{\kappa\kappa}$. However, **it does not define well ordering:** the $\mathcal{L}_{\kappa\kappa}$ sentence

$$\neg\exists\overline{x}_\omega \bigwedge_{n\in\omega} x_{n+1} < x_n$$

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- But this sentence is actually in $\mathcal{L}_{\omega_1\omega_1}$, therefore difficult to naively avoid.
- The crucial step to capture with the syntax of Cartagena Logic needs to incorporate the act of partitioning a set into countably many pieces (by player $\exists$).

Boolean variables: randomizing selections

More considerations:

- We need to deal with **partitions** $\mathcal{W}_{\mathsf{f}} = \{\mathsf{f}^{-1}\{\mathsf{n}\} : \mathsf{n} \in \omega\}$,

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- "Delayed conjunctions:"

$$\exists \overline{\mathsf{x}}_\theta \bigvee_{\mathsf{f}:\theta\to\omega} \bigwedge\!\!\!\!\bigwedge_{\mathsf{X}\in\mathcal{W}_\mathsf{f}},$$

one of the two dual forms of **Cartagena quantifiers**

More on Boolean variables

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- Boolean variables X range over subsets of some θ: over subtuples of some long tuple α_θ played by player $\forall$.
- Then, when a Boolean variable $\mathsf{X} \in \mathsf{BVar}^\theta_\kappa$ takes a value $\mathsf{A} \subseteq \theta$, this, in some sense, corresponds to fixing the subtuple

$$\alpha_{\mathsf{A}} \subseteq \alpha_\theta.$$

Main point of Boolean variables

By integrating Boolean variables into the syntax, we allow formulas to be "undecided" about player $\forall$'s choice concerning which piece of partition should the game be continued with.

The Kabuki definition of Boolean extension $\mathcal{L}^{\mathsf{Bool}}_{\kappa\kappa}$

Definition 1.3 (Boolean extension $\mathcal{L}^{\mathsf{Bool}}_{\kappa\kappa}$)**.** Let λ be an infinite regular cardinal. The $\mathcal{L}^{\mathsf{Bool}}_{\lambda\lambda}$-*formulas* are defined as follows.

(1) Every atomic formula with variables in Var_λ is an $\mathcal{L}^{\mathsf{Bool}}_{\lambda\lambda}$-formula.
(2) If φ is an $\mathcal{L}^{\mathsf{Bool}}_{\lambda\lambda}$-formula then so is

$$\neg\varphi.$$

(3) If Φ is a set of $\mathcal{L}^{\mathsf{Bool}}_{\lambda\lambda}$-formulas of size $< \lambda$, then

$$\bigvee \Phi,$$
$$\bigwedge \Phi,$$

are $\mathcal{L}^{\mathsf{Bool}}_{\lambda\lambda}$-formulas.
(4) If φ is an $\mathcal{L}^{\mathsf{Bool}}_{\lambda\lambda}$-formula and $\bar{x} \subseteq \mathsf{Var}_\lambda$ has length $< \lambda$, then

$$\exists \bar{x}\varphi,$$
$$\forall \bar{x}\varphi,$$

are $\mathcal{L}^{\mathsf{Bool}}_{\lambda\lambda}$-formulas.
(5) If $\{\varphi_u : u \subseteq \theta\}$ are $\mathcal{L}^{\mathsf{Bool}}_{\lambda\lambda}$-formulas, then so are the following:

$$\bigvee\!\!\!\bigvee_{u \in p(X)} \varphi_u,$$
$$\bigwedge\!\!\!\bigwedge_{u \in p(X)} \varphi_u,$$

provided that $\theta < \lambda$, $X \in \mathsf{BVar}^\theta_\lambda$ and $p : \mathscr{P}(\theta) \to \mathscr{P}(\mathscr{P}(\theta))$ is a function.
(6) If φ is an $\mathcal{L}^{\mathsf{Bool}}_{\lambda\lambda}$-formula, then so are the following:

$$\bigvee\!\!\!\bigvee_{X \in \mathcal{W}} \varphi,$$
$$\bigwedge\!\!\!\bigwedge_{X \in \mathcal{W}} \varphi,$$

provided that $\theta < \lambda$ and $\mathcal{W} \subseteq \mathscr{P}(\theta)$.

CARTAGENA LOGIC 7

(7) If φ is an $\mathcal{L}^{\mathsf{Bool}}_{\lambda\lambda}$-formula, $\bar{X} = (X_i)_i \subseteq \mathsf{BVar}_\lambda$ is a tuple of Boolean variables of length $< \lambda$ with $X_i \in \mathsf{BVar}^{\theta_i}_\lambda$ and $\mathcal{W}_i \subseteq \mathscr{P}(\theta_i)$, then

$$\bigvee\!\!\!\bigvee_{\bar{X} \in \prod_i \mathcal{W}_i} \varphi,$$
$$\bigwedge\!\!\!\bigwedge_{\bar{X} \in \prod_i \mathcal{W}_i} \varphi$$

are $\mathcal{L}^{\mathsf{Bool}}_{\lambda\lambda}$-formulas.

For a singular cardinal κ, we let

$$\mathcal{L}^{\mathsf{Bool}}_{\kappa\kappa} := \bigcup_{\lambda<\kappa} \mathcal{L}^{\mathsf{Bool}}_{\lambda^+\lambda^+}.$$

A Boolean variable can occur only as an (uninterpreted) index set of a conjunction or a disjunction. The operation

$$\bigvee\!\!\!\bigvee_{u \in p(X)}$$

from clause 5 is a logical operation that introduces X as a variable in the formula. It is not a "real" disjunction. Later, we will see how to substitute a set A for X, resulting in a "real" disjunction

$$\bigvee_{u \in p(A)} .$$

The definition for substitution is given below, Definition 1.11. Similarly for the conjunction $\bigwedge\!\!\!\bigwedge_{u \in p(X)}$.

Example 1.4. For example, the formula

$$\bigvee\!\!\!\bigvee P(x_i^X)$$

The new stuff

If $\{\phi_\mathsf{u} : \mathsf{u} \subseteq \theta\}$ are $\mathcal{L}^{\mathsf{Bool}}_{\lambda\lambda}$-formulas, then so are all the following:

$$\bigvee\!\!\!\bigvee_{\mathsf{u}\in \mathsf{p}(\mathsf{X})} \phi_\mathsf{u}, \qquad \bigwedge\!\!\!\bigwedge_{\mathsf{u}\in \mathsf{p}(\mathsf{X})} \phi_\mathsf{u},$$

where $\theta < \lambda$, $\mathsf{X} \in \mathsf{BVar}^\theta_\lambda$ and $\mathsf{p} : \mathscr{P}(\theta) \to \mathscr{P}(\mathscr{P}(\theta))$ is a function,

$$\bigvee\!\!\!\bigvee_{\mathsf{X}\in \mathcal{W}} \phi, \qquad \bigwedge\!\!\!\bigwedge_{\mathsf{X}\in \mathcal{W}} \phi,$$

where $\theta < \lambda$ and $\subseteq \mathscr{P}(\theta)$, if $\overline{\mathsf{X}} = (\mathsf{X}_\mathsf{i})_\mathsf{i} \subseteq \mathsf{BVar}_\lambda$ is a tuple of Boolean variables of length $< \lambda$ with $\mathsf{X}_\mathsf{i} \in \mathsf{BVar}^{\theta_\mathsf{i}}_\lambda$ and $\mathcal{W}_\mathsf{i} \subseteq \mathscr{P}(\theta_\mathsf{i})$, then

$$\bigvee\!\!\!\bigvee_{\overline{\mathsf{X}}\in \prod_\mathsf{i} \mathcal{W}_\mathsf{i}} \phi, \qquad \bigwedge\!\!\!\bigwedge_{\overline{\mathsf{X}}\in \prod_\mathsf{i} \mathcal{W}_\mathsf{i}} \phi.$$

Main points of $\mathcal{L}^{\mathsf{Bool}}_{\kappa\kappa}$:

Formula constructions such as $\bigvee\!\!\!\bigvee_{\mathsf{X}\in\mathcal{W}} \phi$, $\bigwedge\!\!\!\bigwedge_{\mathsf{X}\in\mathcal{W}} \phi$

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Formula constructions such as $\bigvee\!\!\!\bigvee_{\mathsf{X}\in\mathcal{W}}\phi$, $\bigwedge\!\!\!\bigwedge_{\mathsf{X}\in\mathcal{W}}\phi$

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- the constructs $\bigvee\!\!\!\bigvee_{\mathsf{u}\in\mathsf{p}(\mathsf{X})}$, $\bigwedge\!\!\!\bigwedge_{\mathsf{X}\in\mathcal{W}}$ are not "real" disjunctions or conjunctions, they are logical operations that introduce X as a variable in the formula
- an involved definition of **substitution** maps these new Boolean variable formulas into "usual" formulas of $\mathcal{L}_{\kappa\kappa}$

The collapse to Cartagena Logic

A **global valuation** is a tuple $\overline{\mathsf{A}} = (\mathsf{A}_\mathsf{X})_{\mathsf{X}\in\mathsf{BVar}_\kappa}$ such that $\mathsf{A}_\mathsf{X} \subseteq \theta$ whenever $\mathsf{X} \in \mathsf{BVar}^\theta_\kappa$.
Every global valuation $\overline{\mathsf{A}}$ uniquely determines a mapping

$$\mathcal{L}^{\mathsf{Bool}}_{\kappa\kappa}\text{-flas} \xrightarrow{\pi_{\overline{\mathsf{A}}}} \mathcal{L}_{\kappa\kappa}\text{-flas}$$

via the substitution

$$\phi(\overline{\mathsf{x}}, \overline{\mathsf{X}}) \longmapsto \phi(\overline{\mathsf{x}}, \overline{\mathsf{A}}).$$

Semantics of Cartagena Logic

With valuations we get semantics:
Let $\phi(\bar{x}, \bar{X})$ be an $\mathcal{L}^{\mathsf{Bool}}_{\kappa\kappa}$-formula. For a structure $\mathcal{M}$, a tuple $\bar{a} \in \mathcal{M}^{\bar{x}}$ and a valuation $\bar{A}$ of $\bar{X}$, we denote

$$\mathcal{M} \models \phi(\bar{a}, \bar{A})$$

if $\mathcal{M} \models \phi(\bar{x}, \bar{A})[\bar{a}/\bar{x}]$, where $\phi(\bar{x}, \bar{A})$ is understood as an $\mathcal{L}_{\kappa\kappa}$-formula in the natural way, and $[\bar{a}/\bar{x}]$ denotes the ordinary substitution of $\bar{a}$ for $\bar{x}$, defined for the logic $\mathcal{L}_{\kappa\kappa}$.

The syntax works

Theorem (Kivimäki, Väänänen, V.)

The syntax thus defined captures the Cartagena game:

$\mathsf{M} \equiv_{\mathcal{L}^c_\kappa} \mathsf{N}$ <u>iff</u> $\exists$ *has a winning strategy for the Cartagena game of* M, N.

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The syntax thus defined captures the Cartagena game:

$$\mathsf{M} \equiv_{\mathcal{L}^c_\kappa} \mathsf{N} \qquad \underline{\text{iff}} \qquad \exists \textit{ has a winning strategy for the Cartagena game of } \mathsf{M}, \mathsf{N}.$$

The proof requires dealing with partitions and their refinements, and uses heavily the notion of formulas being "downwards/upwards correct" - the restriction in the definition of the induction.

Cartagena Syntax: the collapse of Boolean logic

With the previous, the syntax of Cartagena Logic is the image of the <u>collapse</u> logical operator, inside $\mathcal{L}_{\kappa\kappa}$.
Before stating the model-theoretic properties of Cartagena Logic, let us ponder a moment on this fact:

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- The collapse maps $\pi_{\overline{A}}$ suggest interesting "fibers" or maybe even continuity - in the case of first order logic, this is very close to Keisler's randomized models.
- The method of "randomizing" (in first order, continuous first order) has proved extremely fruitful. This is the first situation where (to our knowledge) this happens in infinitary logic!

Model Theory

Shelah's $\mathcal{L}^1_\kappa$ has interpolation, Lindström, strong well-ordering number κ (all of these for uncountable $\kappa = \beth_\kappa$). Moreover, if κ is strongly compact, $\mathcal{L}^1_\kappa$ has a weak version of the "Keisler-Shelah" phenomenon.

Our Cartagena Logic shares the strong well-ordering number κ. It does not satisfy interpolation nor Lindström, but has better behaviour with respect to unions of elementary chains and potentially better control of definability (at least, explicit!).

How explicit?

The following classes of structures are definable in the Cartagena logic $\mathcal{L}^c_\kappa$, for each cardinal $\theta < \kappa$:

1. Models of cardinality θ, as well as models with a predicate or a definable subset of size θ.
2. Graphs with a clique of size θ.
3. Graphs of size θ that admit an ω-coloring.
4. For each cardinal $\theta < \kappa$ of uncountable cofinality: θ-Aronszajn trees.
5. Partially ordered models with an uncountable descending chain.

Explicit examples (two of the previous):

- **Graphs with an ω-coloring** The class of graphs of size θ, colorable by ω colors:

$$\forall_\theta \bigvee_{f:\theta\to\omega} \bigwedge\!\!\!\!\bigwedge_{X\in_f} \left(\bigwedge\!\!\!\!\bigwedge_{\{i,j\}\in[X]^2} \neg E(x_i, x_j) \right)$$

plus graph axioms and the sentence that defines models of size θ.

- **Long descending chains:** Let $\theta := \omega_1$. The sentence

$$\exists_\theta \bigwedge_{f:\theta\to\omega} \bigvee\!\!\!\!\bigvee_{X\in_f} \bigvee\!\!\!\!\bigvee_{u\in[X]^\theta} \bigwedge_{\substack{\{i,j\}\in[u]^2,\\ i<j}} x_j < x_i$$

defines the class of models with an uncountable descending chain.

The distance between Cartagena Logic and $\mathcal{L}^1_\kappa$

For comparison:

- (Väänänen, Veličković): $(\mathbb{R}, <) \not\equiv_{\mathcal{L}^1_\kappa} (\mathbb{R} \setminus \{0\}, <)$,
- (We prove:) $(\mathbb{R}, <) \equiv_{\mathcal{L}^c_\kappa} (\mathbb{R} \setminus \{0\}, <)$.

So, $\mathcal{L}^c_\kappa < \mathcal{L}^1_\kappa$.

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So, $\mathcal{L}^c_\kappa < \mathcal{L}^1_\kappa$.
But we also prove that $\Delta(\mathcal{L}^c_\kappa) = \mathcal{L}^1_\kappa$. In fact, a second Lindström-type theorem (Shelah, 2012) shows minimality of $\mathcal{L}^1_\kappa$ as the logic above $\mathcal{L}_{\kappa\omega}$ which is Δ-closed and in which the class of models with a technical property (shared by the two logics) called ω-covering up to θ is definable, for each $\theta < \kappa$.

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The ω-covering property is heavily used by Shelah in the analysis of $\mathcal{L}^1_\kappa$ and we also prove it for Cartagena logic (and use it).

Open questions

- Computation of **types** in our syntax (and types in Boolean expansion) - for the first order case, this is essentially Keisler (and Fajardo's analysis of typespaces at the start of his extension of Keisler's logic),
- Develop more **stability theory**
- Connections with **AEC**s seem obvious, but limited
- Use this syntax to create **topological analysis** of the logic(s)
- Explore automorphisms (Caicedo 2004: **Int+URP** implies that every definable class must have a non-trivial automorphism, so well-ordering in particular cannot be definable).
- **Hanf numbers** - with Aguilera, Kivimäki and Väänänen, now in progress!

Kiitos paljon, rakkaat joukkoteorian ystävät!