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Montevideo

XX SLALM

Model Theory of non-elementary classes: the role of covers

*: state of the art

- Brief overview of Stability Theory for AECs — up to stable / simple. *
  Landmarks of this development, the NIP area.
- Most important examples so far:
  - modules *
  - covers *

① covers
② why AEC?
③ Some of what there is

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① Covers.

MT	Geom.
FO categoricity / stability	Alg. Geom.
?	Analytic (complex) geom.
AEC categ./stab.	?
$L_{\omega_1,\omega}(\cdot)$ categ.	Analytic theory

A kind of description (Zilber, S. Cou. GALOIS)

Prototypical example: $\mathbb{C}_{exp} = (\mathbb{C}, +, \cdot, exp)$

This required $L_{\omega_1,\omega}(Q)$, involved transc. number theory (eg period conjecture)

check

More relevant / more basic: COVER structures.

$$\begin{tikzcd} \mathbb{U} \arrow[d, "f"] \\ \mathbb{X} \end{tikzcd}$$

(for complex algebraic geometry, Cover structures)

+ alg. subvar. of $\mathbb{X}^n$ ($\forall n$), definable.

The cover structure: $\boxed{CS}$

$$\begin{tikzcd}[column sep=small] & \Gamma \curvearrowright \mathbb{U} \arrow[dl, "f_m"'] \arrow[dr, "f_n"] \arrow[dd, "f"] & \\ \Gamma/\Gamma(m) \curvearrowright \mathbb{X}_m \arrow[rr, "pr_{m,n}"] \arrow[dr, "pr_{m,1}"'] & & \mathbb{X}_n \curvearrowleft \Gamma/\Gamma(n) \arrow[dl, "pr_{n,1}"] \\ & \mathbb{X} & \end{tikzcd}$$

For example,
$\mathbb{U} = \mathbb{C}$
$f_n(z) = \exp(z/n)$
$\mathbb{X}_n = \mathbb{C}^*$ $\quad \Gamma/\Gamma(n) \simeq \mathbb{Z}_n$

Point: the CS holds info on the metric topology of $X(\mathbb{C})$
↘ stable [alg. geom.]

W/o the cover, only have the étale topology on $\mathbb{X}$ — X up to abstract autom. of $\mathbb{C}$.

Rmk: Existence of non-trivial covers is a strong requirement on $\mathbb{X}$. Other?

hyper Theorem (Zilber, Kirby, Gavrilovich, Bays, Harris, Daw, Hart, Haykazian, Hyttinen, Eterović ...)

The natural theory (in $L_{\omega_1,\omega}$) of covers

for $\left\{ \begin{array}{c} \exp \\ j \\ \text{modular/Shimura curves} \\ \text{- - - - - var.} \\ \text{Abelian varieties} \\ \text{fmr groups} \\ \text{smooth varieties} \end{array} \right\}$ $Th_{\omega_1,\omega}(j)$ is categorical in unctble cardinals.

Proofs (2005, 06, 11, 17, 20, 22)

require model th. of AECs
plus strong arithmetic theorems
e.g. Mumford-Tate Conj. (proved versions)
Galois / Kummer theory (extended)

[the theorem is being extended — see our monograph with Baldwin — by Harris (M.), Cruz, V., Baldwin, Zilber mirror maps]

Moreover, if $Th^*_{\omega_1,\omega}(p)$ (natural) is unctbly categorical, the arithmetic hard facts follow!

Cat. of $Th_{\omega,\omega}(j)$... $\longleftrightarrow$ M.Th. GAGAGA Arithm. Facts (Mumford-Tate...)

M. Th. A. G

Complete Topological invariants

$$\Sigma(\mathbb{X}_a)$$

$\Sigma(\mathbb{X}_a) = \Sigma(\mathbb{X}_a)^\sigma$
$\Downarrow$
$\mathbb{X}_a^\sigma \approx \mathbb{X}_a$ or
$\mathbb{X}_a^\sigma \approx \mathbb{X}_a^{cc}$

for complex (Shimura, Abelian) varieties $\mathbb{X}_a$

Already: for all smooth complex proj. curves $\mathbb{X}$ of genus ≥ 1, there is an $L_{\omega_1\omega}(Q)$-ax.

$$\Sigma(\mathbb{X})$$

categorical in all unctble cardinals.

Ingredients: o-minimal structure on the cover + Bays-Hart-Hyttinen-Kesälä-Kirby

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2. AEC's

- Early days: "alg.-minded model th." (Shelah 1974)
- Analy. Classes (Fraïssé, Jónsson)

<u>Baldwin, to Libgober:</u> for what's worth, AEC are a style of model theory that approaches mathematics in a more familiar fashion. Instead of syntax & semantics, one investigates class of structures that satisfy certain properties (like closure under limits) — indeed there is a pure category theory def(n).

<u>AEC</u>: smoothly forward closed

M_i $\leq_K$-chain $\Rightarrow$ $\bigcup M_i \in K$

$M_i \leq \bigcup M_i$

$\bigcup M_i = \sup M_i$

Löwenheim–Skolem–Tarski # exists

generative

coherent (e.c.)

$$\begin{array}{ccc} & \tilde{N} & \\ M \nearrow^{K} & & \nwarrow^{K} \\ M & \subseteq & N \end{array} \Rightarrow M \leq_K N$$

model classes

K, $\leq_K$

K: τ-structures

$\leq_K$: order of K

$M' \cong M \in K \Rightarrow M' \in K$

$$\begin{array}{ccc} N' & \cong & N \\ \cup\!\mathrm{I} & & \geq_K \\ M' & \cong & M \end{array} \Rightarrow \begin{array}{c} N' \\ \geq_K \\ M' \end{array}$$

<u>ex</u>:

- $(\mathrm{Mod}(T), \preceq)$ — FO, compl.
- $(\mathrm{Mod}(\varphi), \subseteq)$
- $L_{\kappa\omega}$ $(\mathrm{Mod}(\varphi), \preceq_F)$
- (loc. fin. groups, $\leq_F$)
- (Noetherian rings, $\leq_F$)
- (Part I examples)
- (R-Mod, $\leq_p$)
- ⋮

6]

Although no (directly) def'ble sets,

strong grip on typespaces $gS(M)$.

[Galois types]

In some AECs (AP, NMM, JEP)

$$\mathrm{gtp}(a/_{M},N) = \mathrm{gtp}(b/_{M},N)$$

$$\longleftrightarrow \exists f \in \mathrm{Aut}(\mathfrak{M}) \quad \begin{matrix} f(a)=b \\ f\restriction M = \mathrm{id}_M \end{matrix}$$

* Superstability now pinned down.
(tame) (2017)

(uniqueness of limit models) V.
" VD
stability in large spectra Grossb.
" ...

* stable forking $M_a \underset{K}{\perp} N$

has a Kim-Pillay "charact.

(Grossberg...) 2016

* Simple AECs started

* NIP AECs started [u def.]

Mention Generic Pairs!