Grandes cardinales virtuales y genéricos (y algo de teoría de modelos)

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Seminario Teoría de Conjuntos - Univ. Andes / Abril de 2025

Universidad Nacional de Colombia / Bogotá

Contents

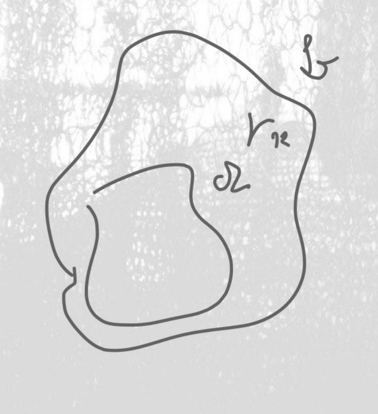
A story of compactness (and what remains in its absence)

Today, two topics:

- The quest for the natural logic for Abstract Elementary Classes (AECs): removing compactness, while keeping a very weak remnant! (The apparent paradox of a rich model theory with “very” pale compactness)
- Another logic similar to First Order, but much stronger: $\mathbb{L}^1_\kappa$. And some connections to strong and weak compactness properties…

AECs: why so much stability theory?

And our first question was the title of this slide!



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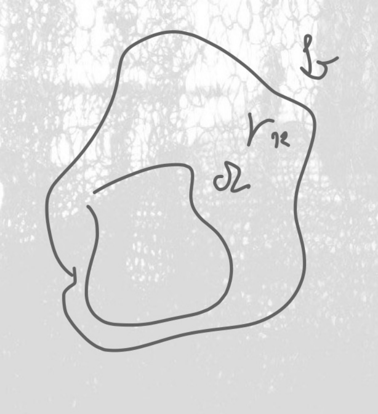


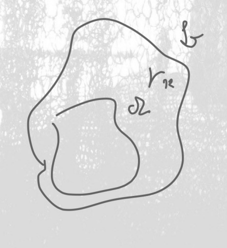
In AECs, we replace from the start the usual extreme emphasis on φ, T, compactness by more **semantical** notions:
$\prec_{\mathcal{K}}$, f a morphism,
f $\in$ Aut($\mathbb{C}$), etc.

φ

T

$T_0 \subseteq^{\mathrm{fin}} T$

$\vdots$



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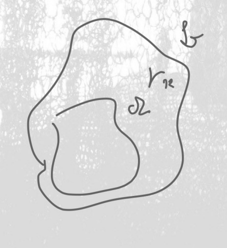
emphasis shift
towards 1980



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: Instead of extracting $\prec$, f, etc. from T, φ, we turn $\prec$, f a strong embedding into the primitive notions!

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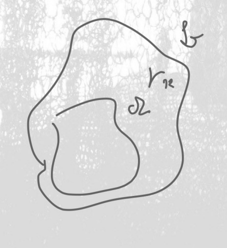

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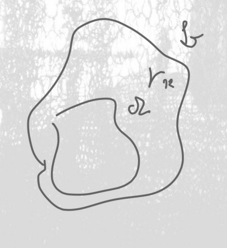

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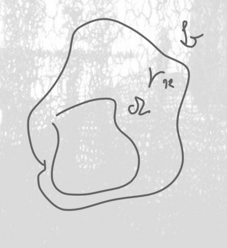

$\mathfrak{A} \prec_{\mathcal{K}} \mathfrak{B}$

"perfect" extension, algebraically closed, etc.

AEC - the axioms, briefly

Fix $\mathcal{K}$ be a class of τ-structures, $\prec_{\mathcal{K}}$ a binary relation on $\mathcal{K}$.

Definition

$(\mathcal{K}, \prec_{\mathcal{K}})$ is an abstract elementary class iff

- $\mathcal{K}$, $\prec_{\mathcal{K}}$ are **closed under isomorphism**,
- $M, N \in \mathcal{K}$, $M \prec_{\mathcal{K}} N \Rightarrow M \subset N$,
- $\prec_{\mathcal{K}}$ is a partial order,
- (**TV**) $M \subset N \prec_{\mathcal{K}} \bar{N}, M \prec_{\mathcal{K}} \bar{N} \Rightarrow M \prec_{\mathcal{K}} N$,
- ($\searrow$**LS**) There is some $\kappa = LS(\mathcal{K}) \geq \aleph_0$ such that for every $M \in \mathcal{K}$, for every $A \subset |M|$, there is $N \prec_{\mathcal{K}} M$ with $A \subset |N|$ and $\|N\| \leq |A| + LS(\mathcal{K})$,
- (**Unions of $\prec_{\mathcal{K}}$-chains**) A union of an arbitrary $\prec_{\mathcal{K}}$-chain in $\mathcal{K}$ belongs to $\mathcal{K}$, is a $\prec_{\mathcal{K}}$-extension of all models in the chain and is the sup of the chain.

AECs, as described by a model theorist to a geometer

> ``Anatoly: For what's worth, AECs are a style of model theory that approaches mathematics in a more familiar fashion. Instead of syntax and semantic, one investigates class of structures that satisfy certain properties (like closure under limits) - indeed there is a purely category theory definition.''

John Baldwin, in an email to Anatoly Libgober (2023)

Abstract Elementary Classes are smoothly forward closed, generative and cumulative/coherent model classes

And really, a lot of examples (and model theory)

Many natural constructions in Mathematics are examples of AECs (or metric AECs)

1. Complete first order theories
2. Excellent, quasiminimal classes
3. Various classes axiomatizable in $L_{\omega_1,\omega}$ or $L_{\kappa\omega}$
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8. AECs of C^*-algebras (Argoty, Berenstein, V.)
9. Zilber analytic classes (pseudoexponentiation, j-map, Shimura)
10. Classes of Valued Fields...

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So, the question of finding **right notions of definability** responsible for all these inner workings is important. . .

Axiomatizing an AEC: attempts (old and new)

- Shelah, V. 2020,
- Leung 2021.

Fix $(\mathcal{K}, \prec_{\mathcal{K}})$ an AEC with $\mathrm{LS}(\mathcal{K}) = \kappa$. We also assume wlog that all models in $\mathcal{K}$ are of cardinality $\geq \kappa$.

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- **Shelah-Vasey:** If $\mathsf{LS}(\mathcal{K}) = \aleph_0$, $\mathcal{K}$ is $\aleph_0$-stable and has the $\aleph_0$-AP, and $\mathsf{I}(\aleph_0, \mathcal{K}) \leq \aleph_0$ <u>then</u> $\mathcal{K}$ is $\mathsf{PC}_{\aleph_0}$.

More

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2020

Shelah-V.

$\mathcal{K} = \mathrm{Mod}\,(\psi_{\mathcal{K}})$

$\psi_{\mathcal{K}} \in \mathbb{L}_{\beth_2(\kappa)^{+3},\kappa^+}$

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(The $\omega \cdot \omega$ refers to quantification of an EF game of length $\omega \cdot \omega$)

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In 2022, a better bound: in $\mathbb{L}_{(2^\kappa)^+,\kappa^+}$

a better bound, with the high price of using $\forall x_0 \exists y_0 \ldots \forall x_i \exists x_i \ldots, \ (i < \omega \cdot \omega)$

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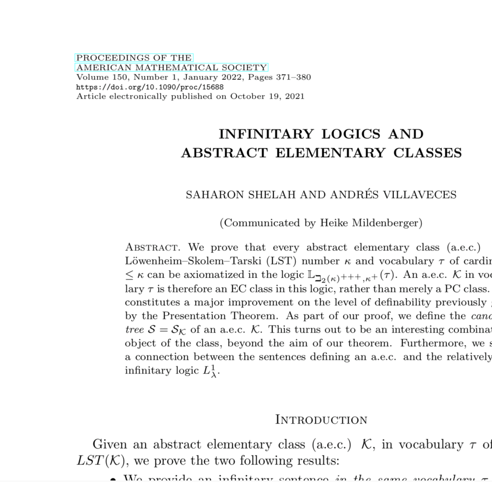
PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 150, Number 1, January 2022, Pages 371–380
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Article electronically published on October 19, 2021

INFINITARY LOGICS AND ABSTRACT ELEMENTARY CLASSES

SAHARON SHELAH AND ANDRÉS VILLAVECES

(Communicated by Heike Mildenberger)

ABSTRACT. We prove that every abstract elementary class (a.e.c.) Löwenheim–Skolem–Tarski (LST) number κ and vocabulary τ of cardin $\leq \kappa$ can be axiomatized in the logic $\mathbb{L}_{\beth_2(\kappa)^{+++},\kappa^+}(\tau)$. An a.e.c. $\mathcal{K}$ in voc lary τ is therefore an EC class in this logic, rather than merely a PC class. constitutes a major improvement on the level of definability previously by the Presentation Theorem. As part of our proof, we define the *cano tree* $\mathcal{S} = \mathcal{S}_{\mathcal{K}}$ of an a.e.c. $\mathcal{K}$. This turns out to be an interesting combina object of the class, beyond the aim of our theorem. Furthermore, we s a connection between the sentences defining an a.e.c. and the relatively infinitary logic L^1_λ.

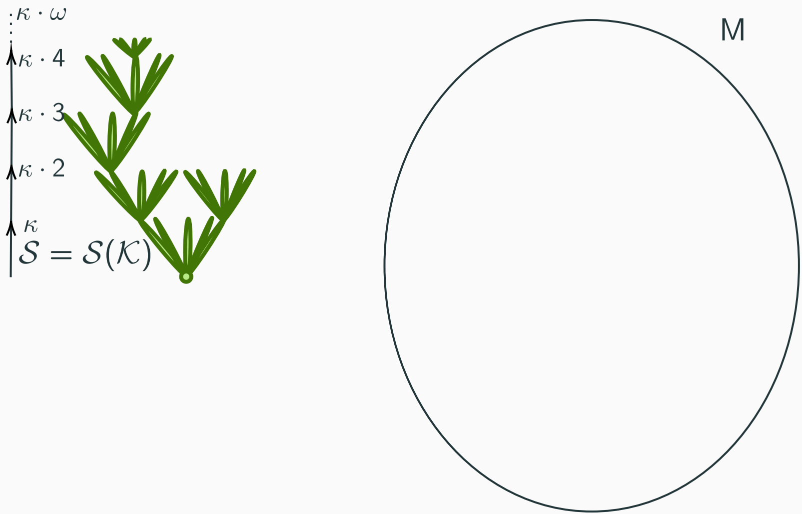
INTRODUCTION

Given an abstract elementary class (a.e.c.) $\mathcal{K}$, in vocabulary τ of $LST(\mathcal{K})$, we prove the two following results:

a better logic,

In 2022, a better bound: in $\mathbb{L}_{(2^\kappa)^+,\kappa^+}$

Testing an arbitrary L-structure M against $\mathcal{S}_{\mathcal{K}}$



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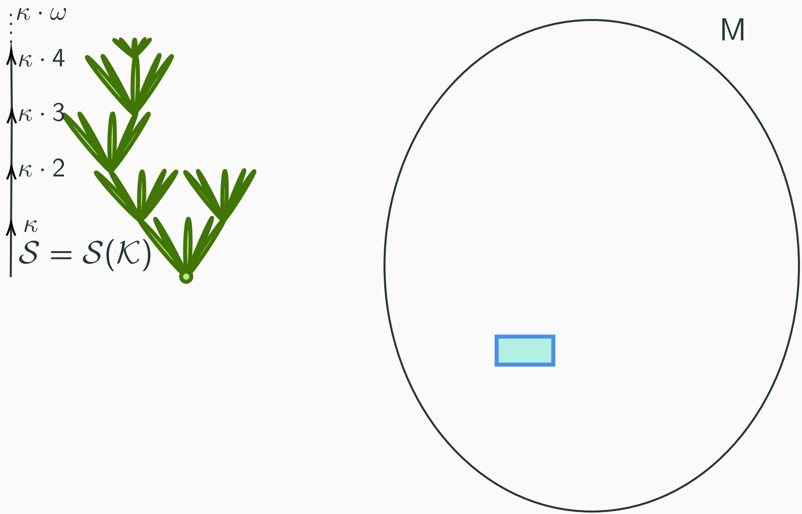

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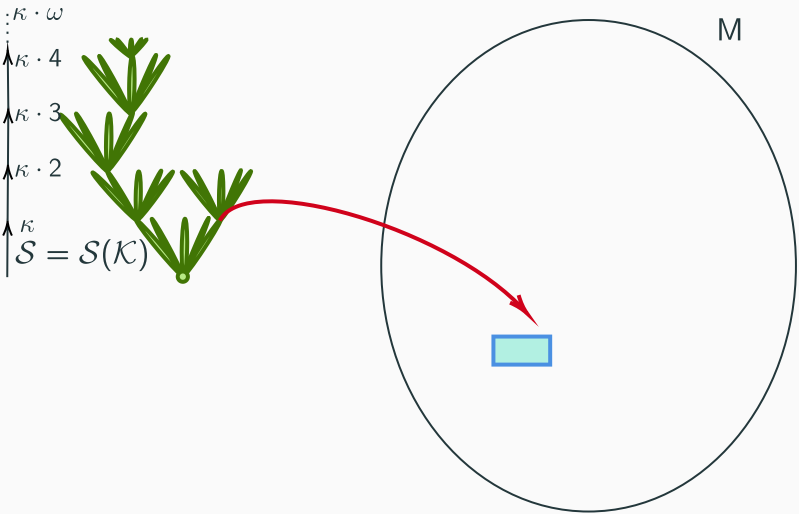

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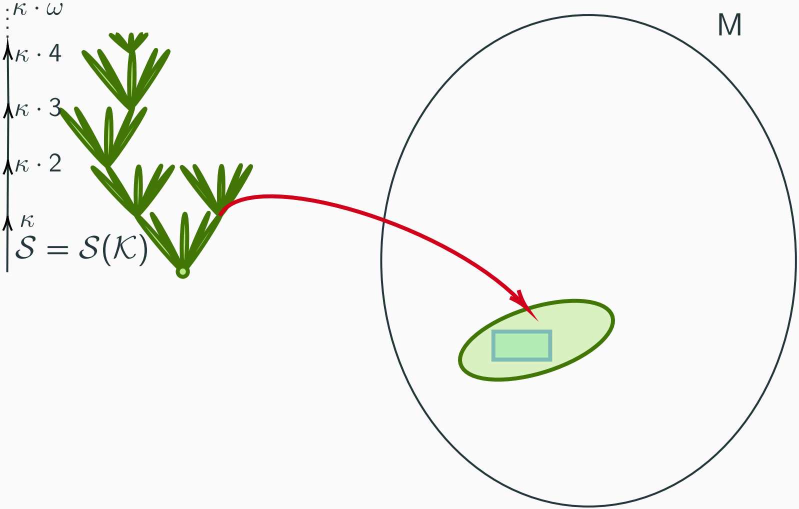

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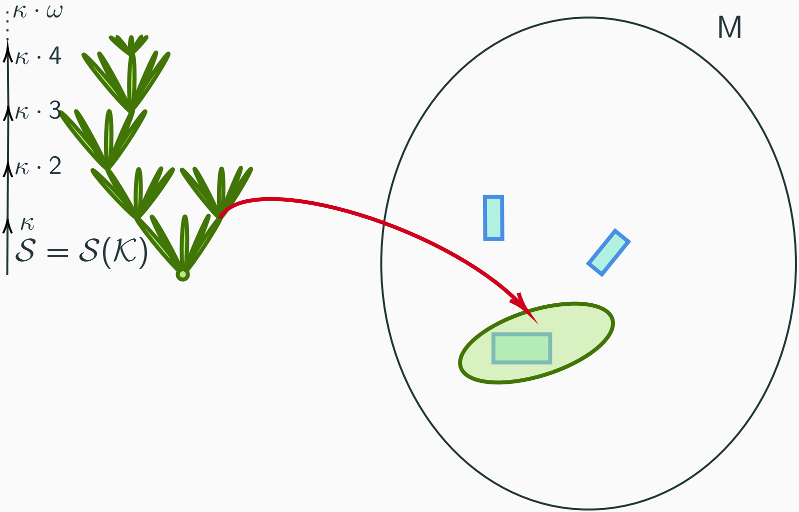

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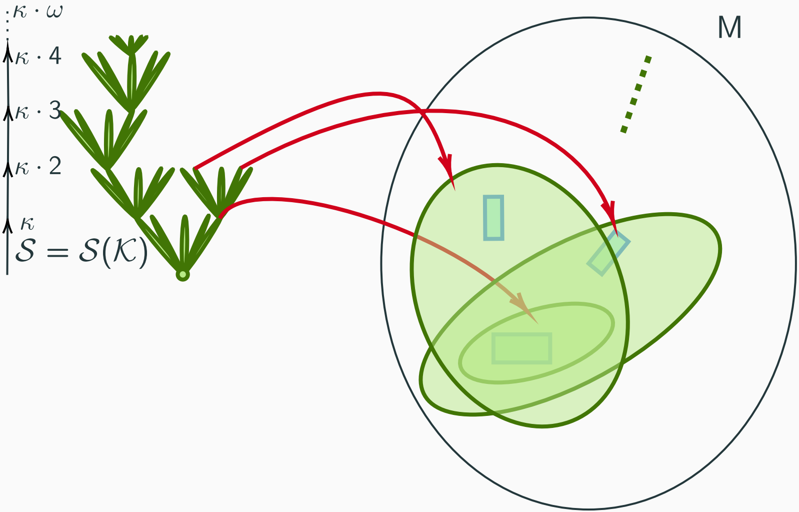

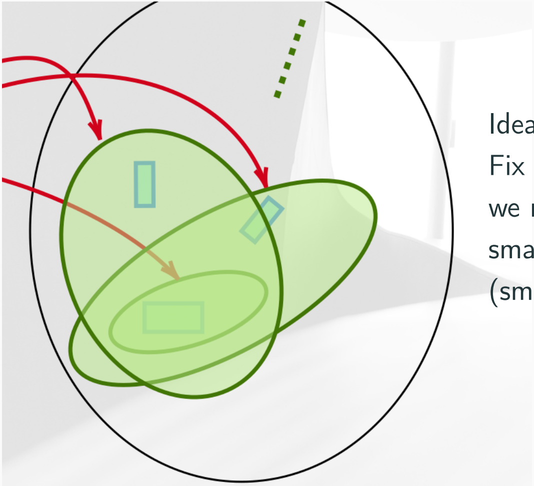

Idea of our axiomatization:

Fix an L-structure M. How can we realize M as a direct limit of small models $N \in \mathcal{K}$?

(small = size $\kappa = LS(\mathcal{K})$)

Realizing an arbitrary model as a limit

$$M = \varinjlim\{N \subseteq M | N \in \mathcal{K}\} \qquad ???$$

(Of course, we need a lot of constraints!)

Towards this goal

We use the canonical tree of $\mathcal{K}$: models of size $\kappa = \mathsf{LS}(\mathcal{K})$, with universes

$$\kappa, \kappa + \kappa, \kappa + \kappa + \kappa, \ldots$$

and a whole "system of $\prec_{\mathcal{K}}$-elementary embeddings" between those models:

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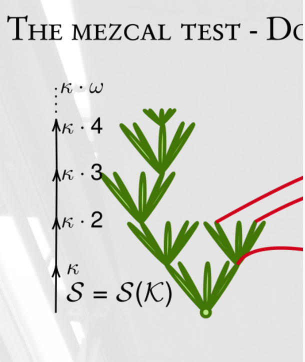
$\mathcal{S}_{\mathcal{K}}$: the canonical tree of $\mathcal{K}$.

In $\mathcal{S}_{\mathcal{K}}$, $N_1 \triangleleft N_2$ iff $N_1 \prec_{\mathcal{K}} N_2$.



We now use syntax to...

... to "test" the model M **- the test membership in** $\mathcal{K}$

M **must "pass"** $\beth_2(\kappa)^{++} + 2$ **tests**

(a newer proof reduces this number to $(2^\kappa)^+$**)**

$I_2(\kappa)^{++}+2$ (2020) / $\alpha < (2^{\kappa})^+$ (2021)

sentences, "approximating" $\mathcal{K}$:

$\varphi_{0,0} = \top$

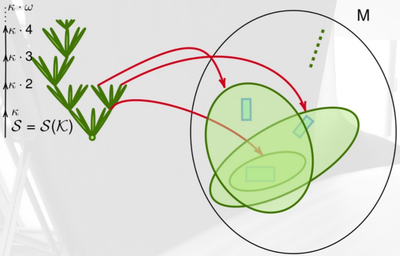
$\varphi_{1,0}$ iterate the "test" against the tree S_α

$\vdots$

$\varphi_{\alpha,0}$

$\vdots$

The mezcal test - Does $M \in \mathcal{K}$?



Formulas $\varphi_{M,\gamma,n}(\bar{x}_n)$

For M in the canonical tree $\mathcal{S}$ at level n, a formula with $\kappa \cdot n$ free variables, defined by induction on γ.

- $\gamma = 0$: $\varphi_{0,0} = \top$ ("truth"). If $n > 0$,

$$\varphi_{M,0,n} := \bigwedge \mathrm{Diag}^n_\kappa(M),$$

the atomic diagram of M in $\kappa \cdot n$ variables.

- γ limit: Then

$$\varphi_{M,\gamma,n}(\bar{x}_n) := \bigwedge_{\beta<\gamma} \varphi_{M,\beta,n}(\bar{x}_n).$$

- $\gamma = \beta + 1$: Then $\varphi_{M,\gamma,n}(\bar{x}_n)$ is the $L_{\lambda^+,\kappa^+}(\tau)$ formula

$$\forall \bar{z}_{[\kappa]} \bigvee_{\substack{N \succ_{\mathcal{K}} M \\ N \in \mathcal{S}_{n+1}}} \exists \bar{x}_{=n} \left[\varphi_{N,\beta,n+1}(\bar{x}_{n+1}) \wedge \bigwedge_{\alpha < \alpha_n[N]} \bigvee_{\delta \in S[N]} z_\alpha = x_\delta \right]$$

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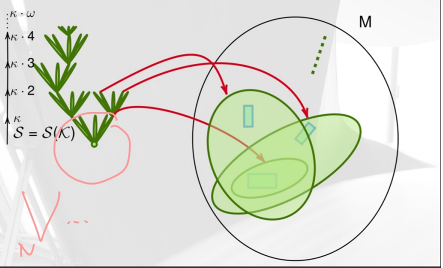

$$\varphi_{1,0} : \forall z \bigvee_{N \in \mathcal{S}_1} \exists \bar{x}_1 \left[\varphi_{N,0,1}(\bar{x}_1) \wedge \bigwedge_{\alpha<\kappa} \bigvee_{\delta<\kappa} z_\alpha = x_\delta \right]$$

size κ

a copy of N

the copy covers z

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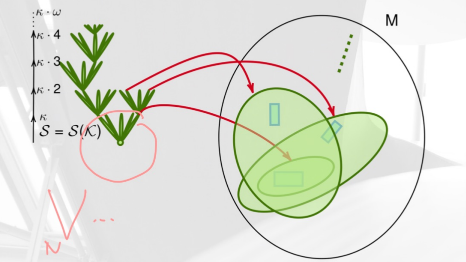

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size κ — "copy of N" — covers z

$M \models \varphi_{1,0}$ if it may "be covered" by levels $N \in \mathcal{K}$, of size κ"

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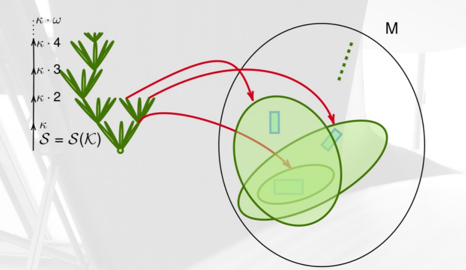
size κ — version of N — covers z

“$\mathcal{S}_1$ covers M”

The mezcal test - Does $M \in \mathcal{K}$?

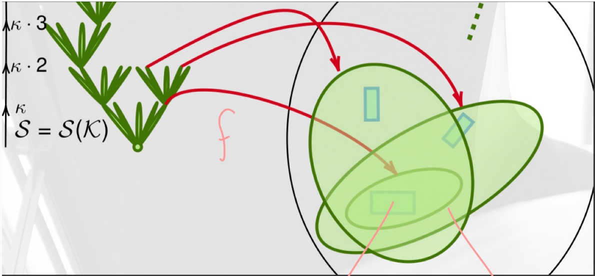


$$M \models \varphi_{2,0} = \forall z \bigvee_{N \in \mathcal{S}_1} \exists \bar{x}_2 \left[\varphi_{N,1,2}(\bar{x}_2) \wedge \text{“} z \subseteq \bar{x}_2 \text{”} \right]$$

κ ?

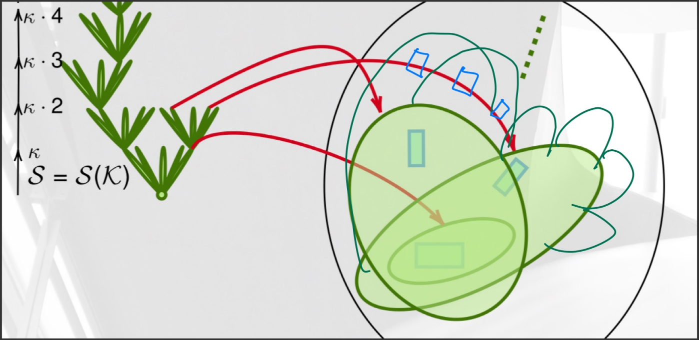
Covers and then covers

$$\forall z \bigvee_{N \in \mathcal{S}_1} \exists \bar{x}_2 \left[\forall z' \bigvee_{\substack{N' > N \\ N' \in \mathcal{S}_2}} \exists \bar{x}_1 \left((\bar{x}_2 {}^\frown \bar{x}_\kappa) \wedge z' \subset \bar{x}_2 {}^\frown \bar{x}_\kappa \wedge z \subset \bar{x}_2 \right) \right]$$



Covering notions, refining ...

$$\varphi_{2,0} \begin{cases} \forall z \text{ some } N \in \mathcal{S}_1 \text{ covers } z \text{ (via } f) \\ \forall z' \text{ some } N' \in \mathcal{S}_2 \;\; N' \geq_\kappa N \text{ covers } z' \end{cases}$$



$\varphi_{3,0}$: better cover yet ...

Problem: M is big !

As this way of covering may be insufficient, we iterate transfinitely:

$M \models \varphi_{\omega+1,0}$:

$\forall N \bigvee$ covers N

$N \in \aleph_1$

BUT $M \models \varphi_{\omega,N,1}(\dots)$ → info of depth ω

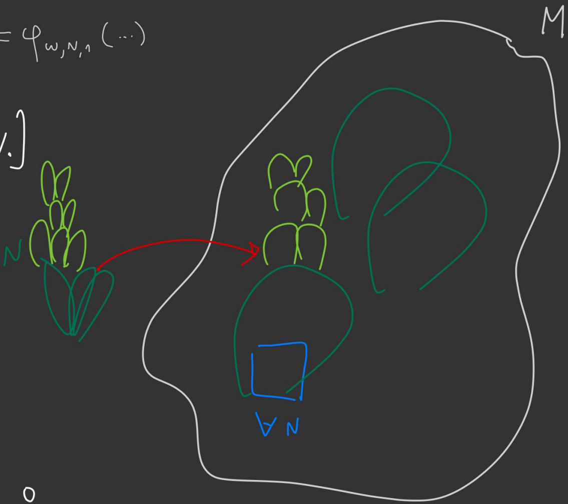
$\forall N \bigvee_{N \in \mathfrak{I}_1}$ cov. N But $M \models \varphi_{\omega,N,1}(\dots)$

Theorem [Shelah, V.]

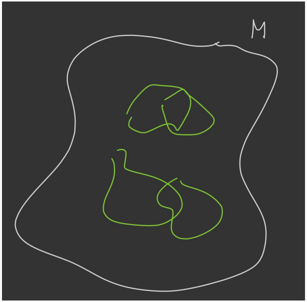
$M \in \mathcal{K}$

$\Updownarrow$

$M \models \varphi_{\beth_2(\kappa)^{++}+2,\,0}$



Key Idea



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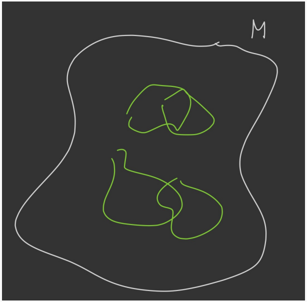

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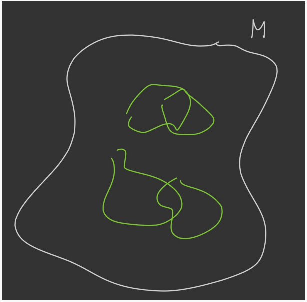

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However..., since $M \models \varphi_{\beth_2(\kappa)^+ + 2,0} \cdots$ **and the $\subseteq$-directed system will also turn out to be a $\prec_{\mathcal{K}}$-directed system!**

Why $\prec_{\mathcal{K}}$-directed? ("Model-completeness" inside M)

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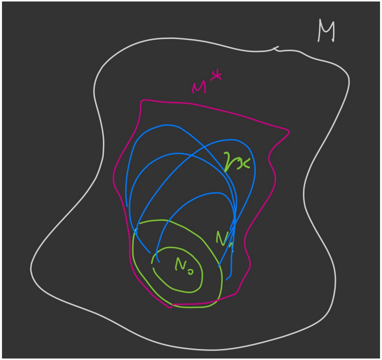

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- In 2020, using a **partition relation** for well-founded trees due to Komjáth and Shelah.
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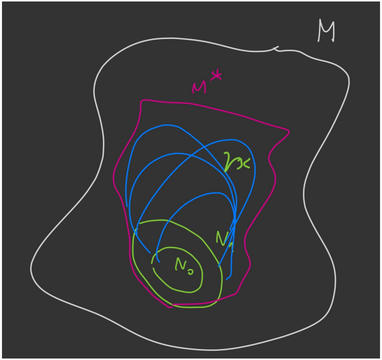

Two combinatorial arguments:

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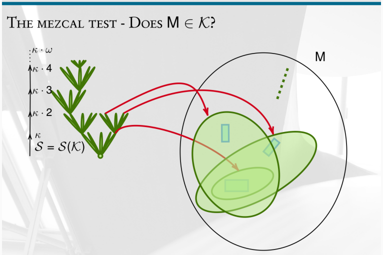
- In 2021, we reduced complexity:

Assuming $N_0 \not\prec_{\mathcal{K}} N_1$, using the tree $\mathcal{S}_{\mathcal{K}}$ and the fact that $M \models \varphi_{\alpha,0}$, we build a **tree of models** converging to the same model - by the axioms of AEC's we may conclude that $N_0 \prec_{\mathcal{K}} N_1$!

Steps:



- Build the tree $\mathcal{S}_{\mathcal{K}}$ (ω levels $\kappa \cdot$ n, n $< \omega$)

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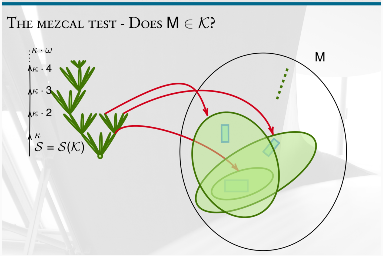

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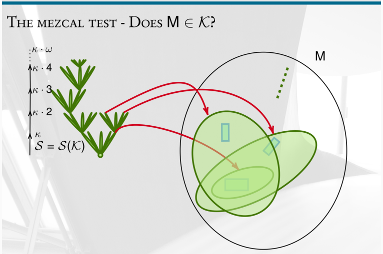

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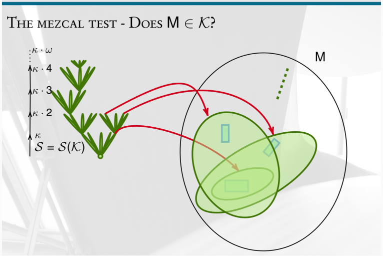

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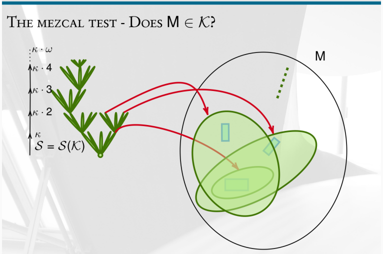
- M $\models \varphi_{\alpha,0}$ for α "high enough" implies (by quite non-trivial combinatorics) that... M is a $\prec_{\mathcal{K}}$-**direct limit of small models** from the class $\mathcal{K}$ (and therefore in $\mathcal{K}$)!

Leung's strategy:



Leung's strategy has similarities, but he replaces the combinatorics by the game quantifier

$$\forall x_0 \exists y_0 \forall x_1 \exists y_1 \ldots \forall x_i \exists y_i \ldots$$

of length $\omega \cdot \omega$ (this has unclear semantics. . .).

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- Bi-interpretability in AECs (Galois theory),
- $\mathcal{K}$'s behaviour in forcing extensions,
- $\mathcal{K}$'s behaviour under large cardinal embeddings

$$j : V \to_{\lambda} M \ldots$$

Plan

Axiomatizing AECs: logics with a tiny remnant of compactness?

Connections with large cardinals and forcing

Tameness and strongly compact cardinals

More Model Theory, More Set Theory

Virtually Large Cardinals

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  can be "virtualized" by requiring the embedding to exist in a set-forcing extension of V.
- Virtual(ized) large cardinals are still large cardinals, but are now in the neighborhood of an ω-Erdős cardinal; they are consistent with L.

- A cardinal κ is **virtually supercompact** (remarkable) if for every $\lambda > \kappa$, there is $\alpha > \lambda$ and a transitive M with ${}^{\lambda}\mathsf{M} \subseteq \mathsf{M}$ such that there is a virtual elementary embedding $\mathsf{j} : \mathsf{V}_\alpha \to_\kappa \mathsf{M}$ with $\mathsf{j}(\kappa) > \lambda$.

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In their preprint Model Theoretic Characterizations of Large Cardinals, Boney, Dimopoulos, Gitman and Magidor [BDGM] generalize Magidor's early result to virtually extendible cardinals.

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Theorem (BDGM)
κ is virtually extendible iff every $< \kappa$-satisfiable $\mathbb{L}^{\mathrm{II}}_{\kappa,\kappa}$-theory has a... ***pseudo-model****.*

They introduce the filtering of "being a model" (compactness) to "being a pseudo-model" (pseudo-compactness) and get the equivalence with virtuality.

Pseudo-models and forth-systems

So... what are these "filtered" models?

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Definition
Let T be a τ-theory in some logic $\mathcal{L}$, let M be a τ^*-structure.
A **forth system** $\mathcal{F}$ from τ to τ^* is a collection of renamings $\mathsf{f} : \sigma \to \sigma^*$, $\mathsf{f} \in \mathcal{F}$ with σ, σ^* finite subsets of τ, τ^* respectively, such that

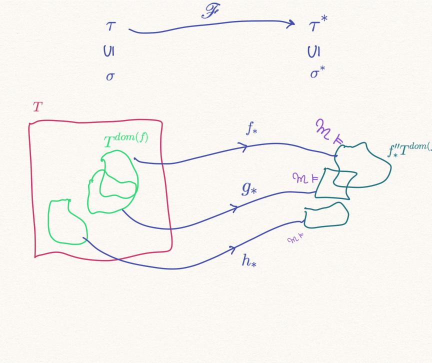
1. $\emptyset \in \mathcal{F}$,
2. If $\mathsf{f} \in \mathcal{F}$ and $\tau_0 \subseteq^{\mathsf{fin}} \tau$ then there is $\mathsf{g} \in \mathcal{F}$ with $\mathsf{f} \subseteq \mathsf{g}$ and $\tau_0 \subseteq \mathrm{dom}(\mathsf{g})$

M is a **pseudomodel** of T if there is a forth system $\mathcal{F}$ from τ to τ^* such that for every $\mathsf{f} \in \mathcal{F}$, $\mathsf{M} \models \mathsf{f}_*'' \mathsf{T}^{\mathrm{dom}(\mathsf{f})}$.

Pseudo-models: a picture

The notion of **pseudomodel** deals with

- localizing in coherent ways (sheaf-like construction) the notion of being a model. . .



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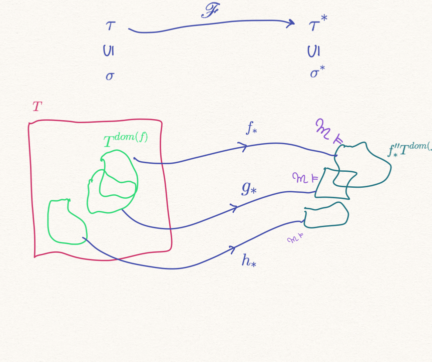

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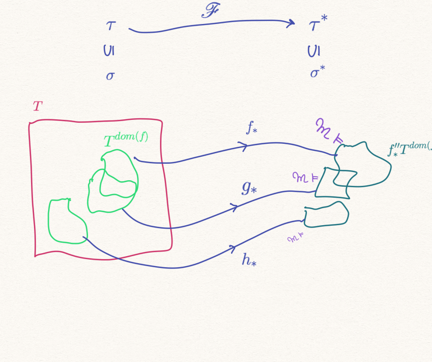

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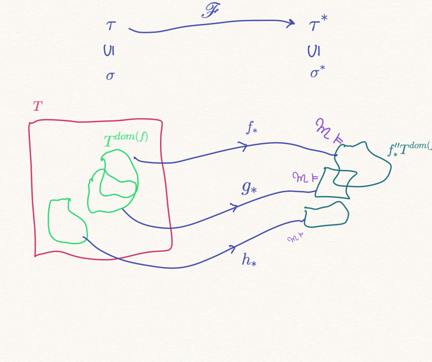

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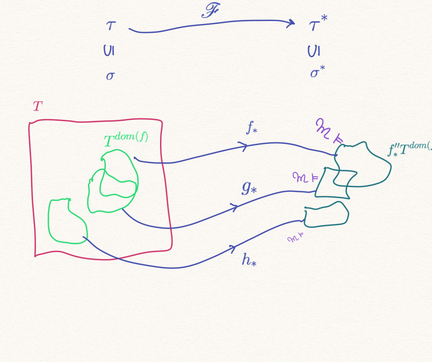
- The other direction uses the virtual embedding to obtain the forth system.



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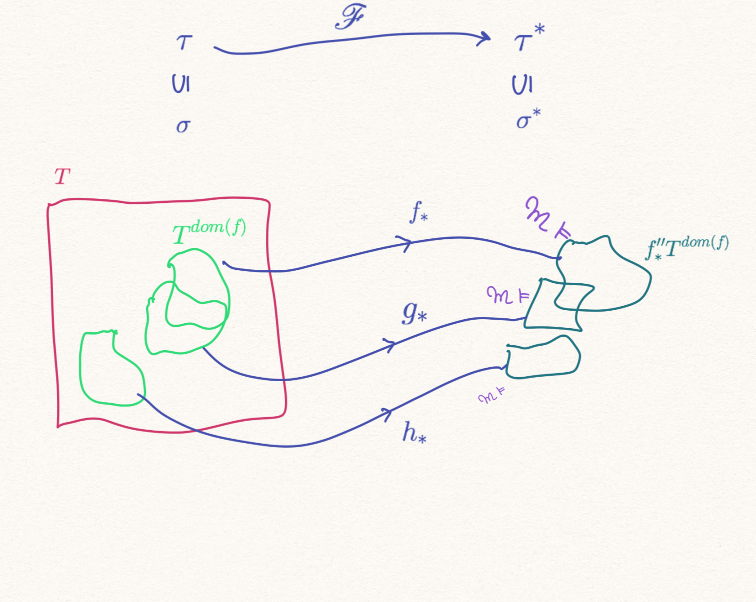
Motto

Forth-systems between vocabularies $\equiv$ forcing notions for virtuality

Pseudomodels



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The Shelah Conjecture (early version)

A key test problem in model theory in the past two or three decades: finding versions of the Morley Theorem and Shelah's Categoricity Transfer theorems, for wider contexts: abstract elementary classes (semantically-centered extensions of the model theory of $L_{\lambda^+,\omega}(Q)$).

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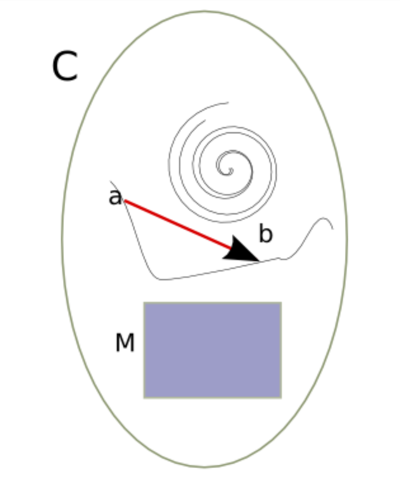
Conjecture (Shelah)
Given any cardinal λ, there exists μ_λ such that if ψ is an $\mathsf{L}_{\omega_1,\omega}$-sentence that satisfies a "Löwenheim-Skolem" theorem down to λ and is categorical is some cardinality $\geq \mu_\lambda$, then it is categorical in all cardinalities above μ_λ.

Galois (orbital) types

The correct notion of type in an AEC (with the amalgamation and joint embedding properties (AP, JEP), and no maximal models (NMM):



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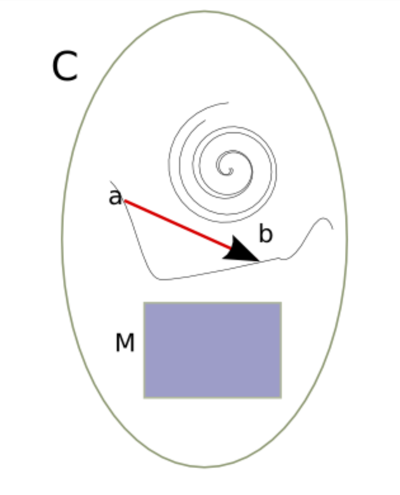

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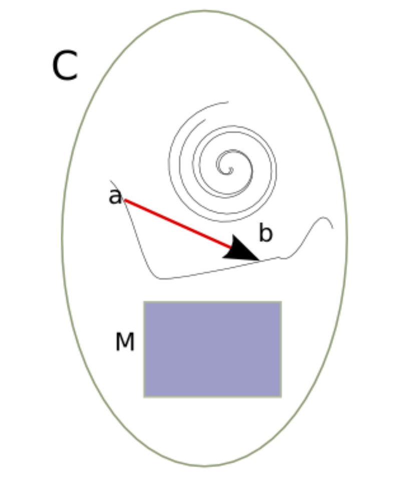

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3. Then (under AP, JEP, NMM) Galois types over M are orbits under the action of the group $Aut_M(\mathbb{C})$, the automorphisms of the monster that fix M pointwise.

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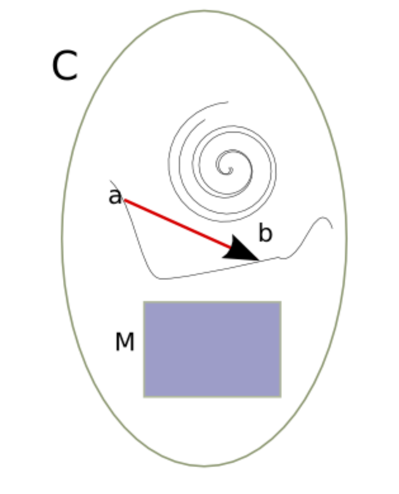

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4. (This generalizes the classical (syntactic) notion of a type.)

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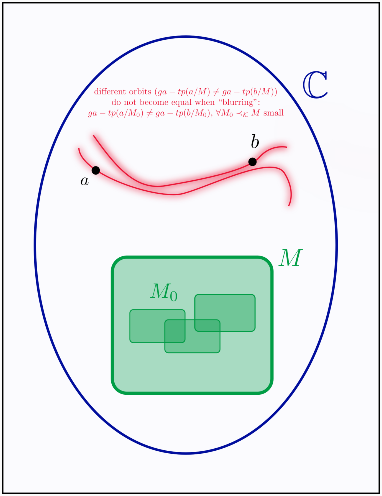
Their proof built on a previous proof of the "downward" transfer by Shelah but has a crucial element: isolating the notion of tameness ("buried" in Shelah's proof of the downward part - fleshing out the notion allows Grossberg/VanDieren to prove the upward categoricity).

Localizing difference

Idea: "localizing" the condition of... extending a map f that fixes a model M in an aec $\mathcal{K}$ to a $\mathcal{K}$-embedding:



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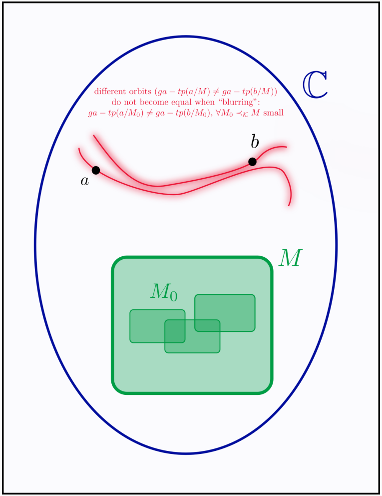

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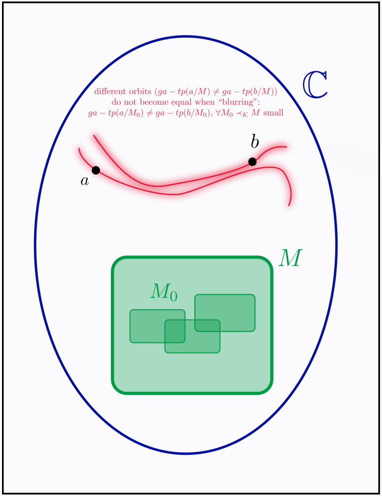
- <u>we want</u>: to localize this to checking that there is some $M_0 \in \mathcal{P}^*_\kappa(M)$ and $X_0 \in \mathcal{P}_\kappa(N_0)$ such that

$$\mathrm{gatp}(X_0/M_0) \neq \mathrm{gatp}(f(X_0)/M_0)$$



Getting Tameness from Large Cardinals

In 2013, Boney changed a bit the direction of the approach: why not look directly at the impact of large cardinals on tameness and similar notions?

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Theorem (Boney)
If κ is strongly compact and $\mathcal{K}$ is essentially below κ (i.e. $\mathsf{LS}(\mathcal{K}) < \kappa$ or $\mathcal{K} = \mathsf{Mod}(\psi)$ for some $\mathsf{L}_{\kappa,\omega}$-sentence ψ) then $\mathcal{K}$ is $(< (\kappa + \mathsf{LS}(\mathsf{K})^+, \lambda$-tame for all λ.

Boney and Unger proved (2015) that under strong inaccessibility of κ, the $(< \kappa, \kappa)$-tameness of all aecs implies κ's strong compactness.

Reframing slightly Boney's proof

A cardinal κ is strongly compact iff for every $\lambda > \kappa$ there exists an elementary embedding $j : V \to M$ with critical point κ, and there exists some $Y \in M$ such that $j''\lambda \subset Y$ and $|Y|^M < j(\kappa)$.

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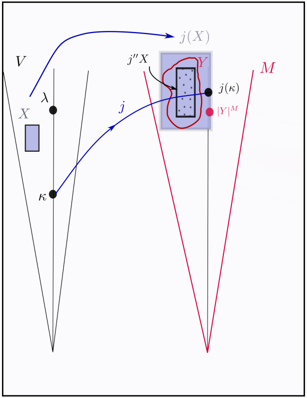

Definition
Let $j : V \to M$ be an elementary embedding. j has the (κ, λ)-cover property if for every X with $|X| \leq \lambda$ there exists $Y \in M$ such that $j''X \subset Y \subset j(X)$ and $|Y|^M < j(\kappa)$.

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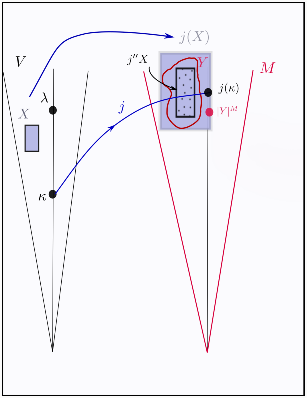

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κ measurable	j has the (κ, κ)-cp
κ λ-strongly compact	j has the (κ, λ)-cp

The "image" of an AEC under $j : V \to M$

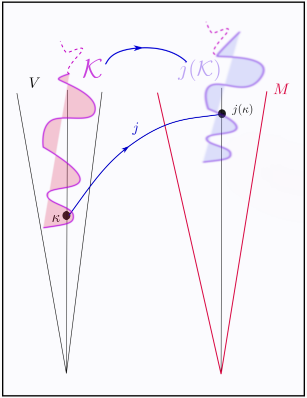
Let $(\mathcal{K}, \prec_{\mathcal{K}})$ be an AEC in τ.
Shelah's Presentation Theorem gives

- $\tau' \supset \tau$,
- T' a τ'-theory and
- Γ' a set of T'-types

such that

$$\mathcal{K} = PC(\tau, T', \Gamma') = \{M' \restriction \tau \mid M' \models T' \text{ and } M' \text{ omits } \Gamma'\},$$

We define $j(\mathcal{K})$ as the class $PC^M(j(\tau), j(T'), j(\Gamma'))$.
By elementarity, $M \models j(\mathcal{K})$ is an AEC with LS number equal to $j(LS(\mathcal{K}))$.



Attempt at getting $\mathrm{j}(\mathcal{K}) \subset \mathcal{K}$ and $\prec_{\mathrm{j}(\mathcal{K})} \subset \prec_{\mathcal{K}}$.

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- for every $\mathcal{M} \in \mathcal{K}$, $\mathrm{j}''\mathcal{M} \prec_{\mathcal{K}} \mathrm{j}(\mathcal{M}) \restriction \tau$.

Examples

1. Let first $j : V \to M$ be a nontrivial elementary embedding with critical point κ and let $\mathcal{K}$ be an AEC with $LS(\mathcal{K}) < \kappa$. Then $\mathcal{K} = PC(\tau', T', \Gamma')$, with $|\tau'| + |T'| + |\Gamma'| < \kappa$; wlog we can assume $\tau', T', \Gamma' \in V_\kappa$ and therefore

$$j(\mathcal{K}) = PC^M(\tau, T', \Gamma') = (\mathcal{K} \cap M, \prec_{\mathcal{K}} \cap M),$$

(we have to use correctness of $\models$). Clearly, j respects $\mathcal{K}$.

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2. $\mathcal{K}$ is given as $Mod(\varphi)$ for φ in $L_{\kappa,\omega}$, with $\prec_{\mathcal{K}} = \subset^{TV}_{\mathcal{F}}$, $\mathcal{F}$ some fragment of $L_{\kappa,\omega}$. Then j respects $\mathcal{K}$.

Getting Tameness

We prove then that whenever $\mathcal{K}$ is an AEC with LS($\mathcal{K}$) $< \kappa < \lambda$, and j : V $\to$ M has the (κ, λ)-cover property and respects $\mathcal{K}$ then $\mathcal{K}$ is $(< \kappa, \lambda)$-tame.

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Let $\mathcal{M} \in \mathcal{K}_\lambda$ and $p_1 = \mathrm{gatp}(\vec{a}/\mathcal{M}, \mathcal{N}_1)$, $p_2 = \mathrm{gatp}(\vec{b}/\mathcal{M}, \mathcal{N}_2)$ be two types such that for every $\mathcal{N} \prec_{\mathcal{K}} \mathcal{M}$ of size $< \kappa$ we have

$$p_1 \upharpoonright \mathcal{N} = p_2 \upharpoonright \mathcal{N}.$$

(Here, $\vec{a} = (a_i)_{i \in I}, \vec{b} = (b_i)_{i \in I}$.)

Getting Tameness

Let now $Y \in M$ by such that $j''|\mathcal{M}| \subset Y \subset j(|\mathcal{M}|)$ and $|Y|^M < j(\kappa)$.

But in M, $LS(j(\mathcal{K})) = j(LS(\mathcal{K})) < j(\kappa)$ so there is $\mathcal{M}' \in j(\mathcal{K})$ such that $Y \subset |\mathcal{M}'|$, $\|\mathcal{M}'\| < j(\kappa)$ and $\mathcal{M}' \prec_{j(\mathcal{K})} j(\mathcal{M})$; by transitivity, $\mathcal{M}' \prec_{j(\mathcal{K})} j(\mathcal{M})$.

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$$
\begin{aligned}
p_1' &= \text{gatp}(j(\vec{a})/\mathcal{M}' \restriction \tau, j(\mathcal{N}_1) \restriction \tau) \\
&= \text{gatp}(j(\vec{b})/\mathcal{M}' \restriction \tau, j(\mathcal{N}_2) \restriction \tau) = p_2'
\end{aligned}
$$

in $\mathcal{K}$ (again by our hypothesis on j).

Getting Tameness

Since $j''\mathcal{M} \prec_{\mathcal{K}} j(\mathcal{M})$ we get that $j''\mathcal{M} \prec_{\mathcal{K}} \mathcal{M}' \upharpoonright \tau$ (coherence axiom), so restricting we have

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Restricting "above" we get

$$\mathrm{gatp}(j(\vec{a})/j''\mathcal{M}, j''\mathcal{N}_1) = \mathrm{gatp}(j(\vec{b}))/j''\mathcal{M}, j''\mathcal{N}_2),$$

and therefore

$$p = q. \qquad \square$$

Back to the Reflection Property

So, we use the λ-strong compactness of κ to show first that the embedding $j : V \to M$ has the (κ, λ)-property and respects $\mathcal{K}$ and then apply the previous. One may also show that the (κ, λ)-cover of $j : V \to M$ for $\kappa > LS(\mathcal{K})$ implies

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So, we are in a good position to use the Grossberg-VanDieren theorem to conclude the consistency of the Shelah Categoricity Conjecture.

Plan

Axiomatizing AECs: logics with a tiny remnant of compactness?

Connections with large cardinals and forcing

Tameness and strongly compact cardinals

More Model Theory, More Set Theory

Other interactions
Mod Th / Set Th

More on the two sides

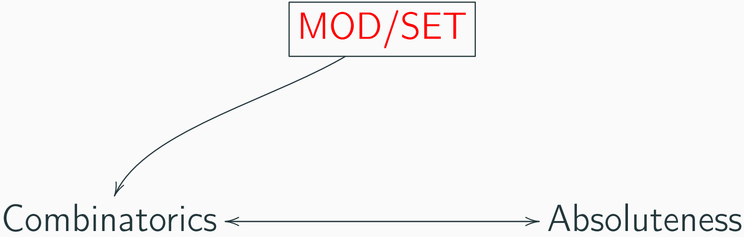
Oh... I had a very strange referee report on the (proper forcing) paper. I think Moschovakis was the editor. So he thought "Saharon is a model theorist" well, he knew me - I was even a year in UCLA before, so he sent it to a model theorist. And the problem was in model theory, [of the form] "the consistency of...", and the referee report said "well, there is very little model theory". . .

Saharon Shelah, in (forthcoming) interview, 2017.

MOD/SET

MOD/SET

Combinatorics


MOD/SET
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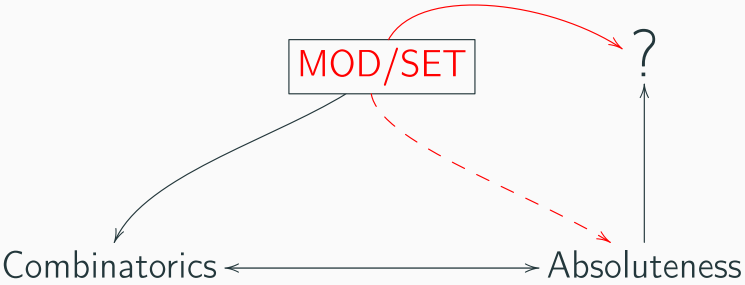
MOD/SET
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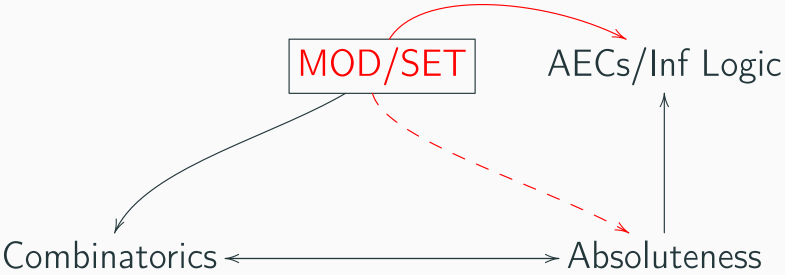
MOD/SET
AECs/Inf Logic
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Virtualization of a Logic

A related notion: the virtualization of a logic. Now using forth-systems **for models** (and not for vocabularies, as before).

An $\mathcal{L}$-**forth system** $\mathcal{P}$ from M to N (both τ – structures) is a collection of $\mathcal{L}$-elementary embeddings with the "forth property":

1. $\emptyset \in \mathcal{P}$,
2. if f $\in \mathcal{P}$, a $\in$ M then there is g $\supseteq$ f in $\mathcal{P}$ such that a $\in$ dom(g).

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This is equivalent to playing the classical Ehrenfeucht-Fraïssé game but with $\forall$ picking only challenges "from the left" (from M).

Virtualization of a logic (II)

[BDGM] use those $\mathcal{L}$-forth systems to get Löwenheim-Skolem-Tarski style characterizations of virtual cardinals: **the existence of a virtual elementary embedding** f : M $\rightarrow_\kappa$ N is equivalent to the existence of a forth system from M to N or that N satisfies the **virtualized logic** theory of M (or player $\exists$ having a winning strategy in the half (virtual) game)...

A direction worth looking at: $\mathbb{L}^1_\theta$ for θ strongly compact

Shelah has been able to extract interesting model theory from the blend of the definition of his logic $\mathbb{L}^1_\theta$ **under the additional assumption that** θ is a strongly compact cardinal:

- A "Keisler-Shelah"-like theorem ($\mathbb{L}^1_\theta$-elementarily equivalent models have isomorphic iterated ultrapowers)
- Special models (unions of ω-chains of iterated ultrapowers) are unique. . . giving easier proofs of Craig (essentially, showing Robinson and using compactness).
- Connections to stability theory.

The methods are connected with Malliaris-Shelah's constructions and also with careful use of saturation, not unlike the use of forth models in [BDGM].

So, what is $\mathbb{L}^1_\kappa$?

- $\mathbb{L}_{\kappa\omega} \leq \mathbb{L}^1_\kappa \leq \mathbb{L}_{\kappa\kappa}$,
- $\mathbb{L}^1_\kappa$ has **interpolation**
- $\mathbb{L}^1_\kappa$ is maximal among extensions of $\mathbb{L}_{\kappa\omega}$ with interpolation and a form of undefinability of well-order (Lindström-type characterization by Shelah, in 2012)
- $\mathbb{L}^1_\kappa$ satisfies a weak version of unions of **countable** chains: if

$$M_0 \prec_{\mathbb{L}_{\kappa\kappa}} M_1 \prec_{\mathbb{L}_{\kappa\kappa}} \ldots M_n \ldots$$

then

$$M_i \prec_{\mathbb{L}^1_\kappa} M_\omega \quad \forall i < \omega,$$

where $M_\omega = \bigcup_{n<\omega} M_n$ ... BUT

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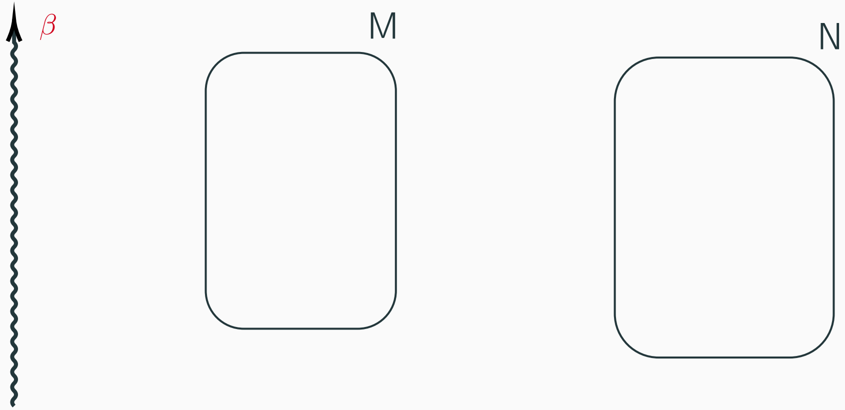
  where $M_\omega = \bigcup_{n<\omega} M_n$ ... BUT
- $\mathbb{L}^1_\kappa$ does not have a **syntax** in the usual sense (no recognizable formulas or sentences), but rather
- $\mathbb{L}^1_\kappa$ has a game-theoretic "fake syntax" given by a "delayed game" (the Shelah game $\mathfrak{D}^\beta_\theta(M, N)$) ...

Shelah's game $\mathfrak{D}_\theta^\beta(M, N)$.



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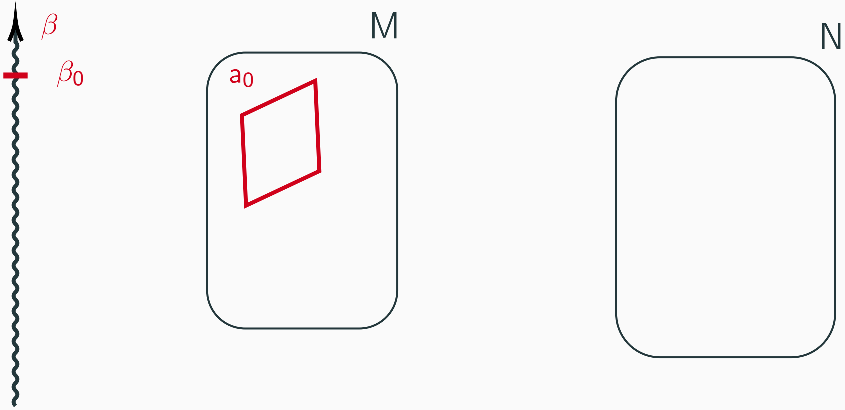
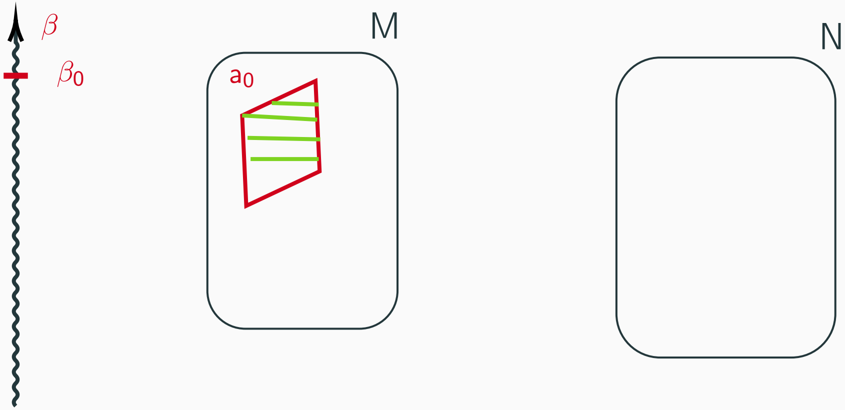


Shelah's game $\Game_\theta^\beta(M, N)$.



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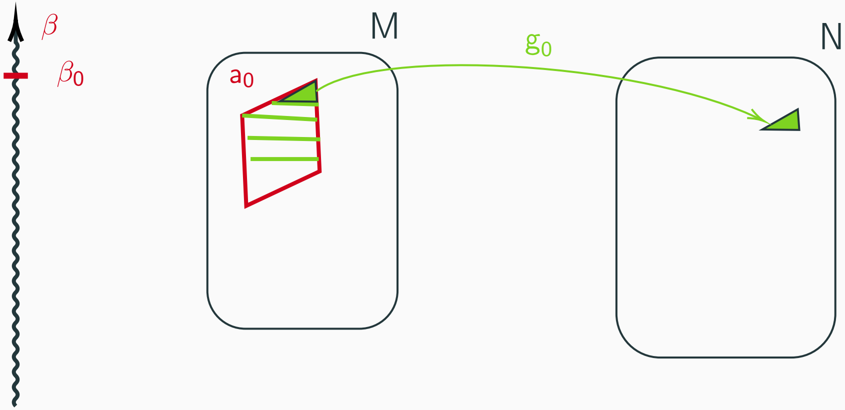

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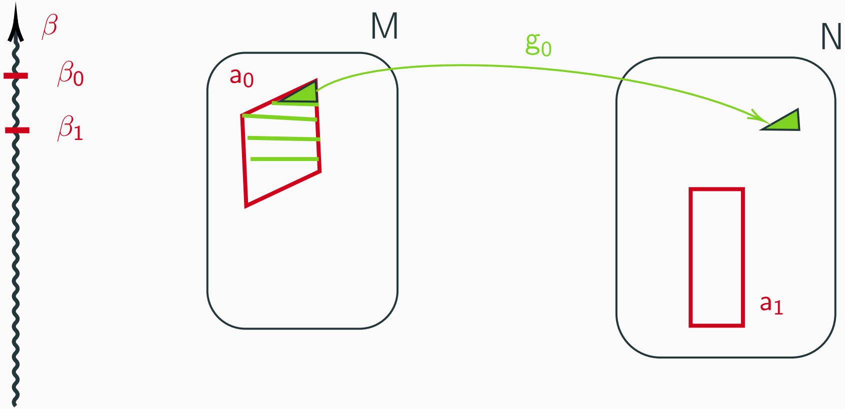

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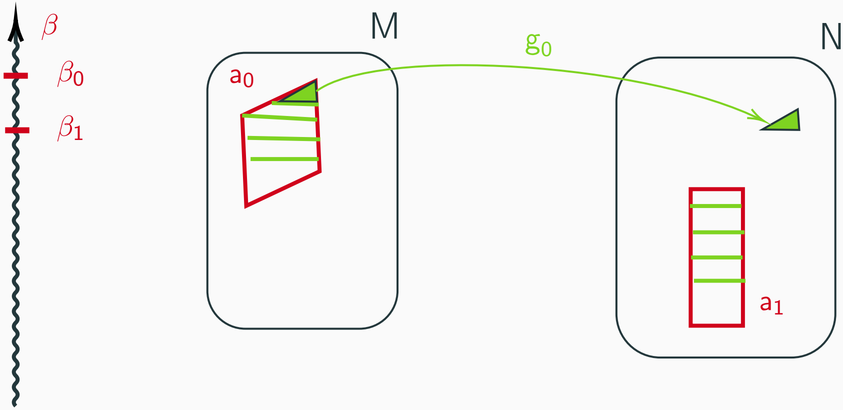

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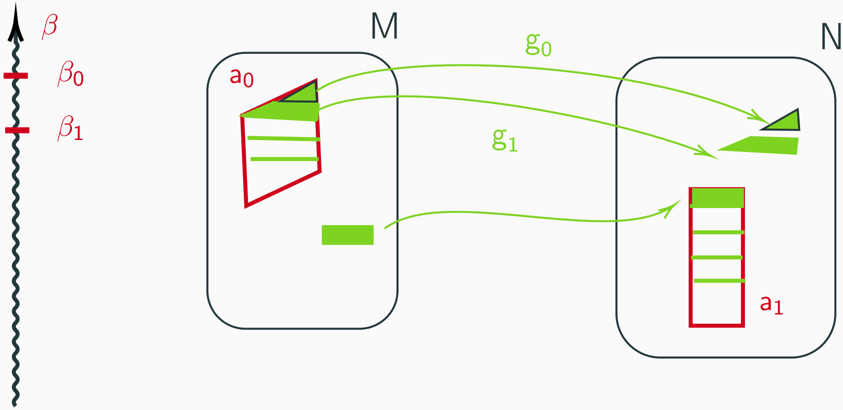

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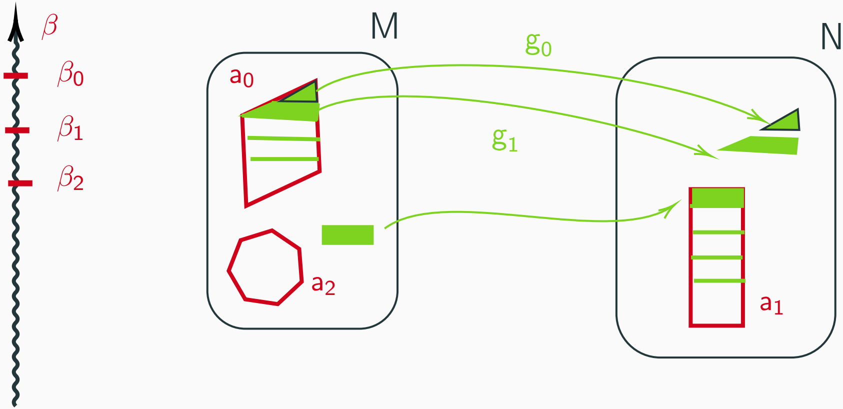

Shelah's game $\mathfrak{D}_\theta^\beta(M, N)$ of ordinal clock β.

$\forall$	$\exists$
$\beta_0 < \beta$, $\vec{a^0}$	
	$f_0 : \vec{a^0} \to \omega$, $g_0 : M \to N$ a p.i.
$\beta_1 < \beta_0$, $\vec{b^1}$	$f_1 : \vec{a^1} \to \omega$, $g_1 : M \to N$ a p.i., $g_1 \supseteq g_0$
$\vdots$	$\vdots$

Constraints:

- $\text{len}(\vec{a^n}) \leq \theta$
- $f_{2n}^{-1}(m) \subseteq \text{dom}(g_{2n})$ for $m \leq n$.
- $f_{2n+1}^{-1}(m) \subseteq \text{ran}(g_{2n})$ for $m \leq n$.

$\exists$ wins if she can play all her moves, otherwise $\forall$ wins.

Shelah's game equivalence (not [nec.] transitive!)

- M $\sim_\theta^\beta$ N iff $\exists$ has a winning strategy in the game.
- M $\equiv_\theta^\beta$ N is defined as the transitive closure of M $\sim_\theta^\beta$ N.
- A union of $\leq \beth_{\beta+1}(\theta)$ equivalence classes of $\equiv_\theta^\beta$ for some $\theta < \kappa$ and $\beta < \theta^+$ is called a sentence of $\mathbb{L}_\kappa^1$.

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- M $\sim_\theta^\beta$ N iff $\exists$ has a winning strategy in the game.
- M $\equiv_\theta^\beta$ N is defined as the transitive closure of M $\sim_\theta^\beta$ N.
- A union of $\leq \beth_{\beta+1}(\theta)$ equivalence classes of $\equiv_\theta^\beta$ for some $\theta < \kappa$ and $\beta < \theta^+$ is called a sentence of $\mathbb{L}_\kappa^1$.

Notice the weirdness!

(With Kiivimäki and Väänänen, we constructed a logic **with generative syntax** (usual sentences) whose Δ-closure is $\mathbb{L}_\kappa^1$: **Cartagena logic** $\mathbb{L}_\kappa^{1,c}$.)

Virtualizing $\mathbb{L}^1_\kappa$, $\mathbb{L}^{1,c}_\kappa$, ...

There are at least two competing virtualizations of these logics:

- Use the definition from [BDGM]... but with the difficulty of not having a good grasp (in the case of Shelah's logic) of elementary embeddings...

Virtualizing $\mathbb{L}^1_\kappa$, $\mathbb{L}^{1,c}_\kappa$, ...

There are at least two competing virtualizations of these logics:

- Use the definition from [BDGM]. . . but with the difficulty of not having a good grasp (in the case of Shelah's logic) of elementary embeddings. . .
- Use a "virtualized" version of the Shelah (or the Cartagena) game $\mathfrak{D}^\beta_\theta$, $\mathfrak{D}^{\beta,c}_\theta$ (Kiivimäki, Väänänen, V.)

Both virtualized versions are essentially existential closures of the logics. They would give rise to two competing notions of virtual embeddings (or different notions of genericity!). So. . . which one?

Definition
A cardinal κ is a delayable cardinal if it is a compactness cardinal for the second-order version of Shelah's logic $\mathsf{L}^{1,\mathsf{II}}_{\kappa}$. It is a virtually delayable cardinal if it is a pseudo-compactness cardinal for $\mathsf{L}^{1,\mathsf{II}}_{\kappa}$. If we replace $\mathsf{L}^{1,\mathsf{II}}_{\kappa}$ by $\mathsf{L}^{1,\mathsf{II},\mathsf{c}}_{\kappa}$ we get the corresponding two notions of Cart-delayable cardinal and virtually Cart-delayable cardinal.

1. Where are these cardinals located? What kind of reflection properties do they capture?
2. The deeper issue is: what kind of virtuality do they actually correspond to? What version of forth-systems?

Thank you for your attention!