Pseudoexponentials and covers

Model theory meets some arithmetic geometry

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Our steps today

Very classical covers: exp and j

Model theory - classifying theories (and beyond)

exp and j as covers

Focus: quasiminimality and categoricity

The model theory of covers

Galois theory in model theory

1

Very classical covers: exp and j

The general quest

Some interactions between Model Theory and Arithmetic Geometry:

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	Classical $j : \mathbb{H} \to \mathbb{C}$		

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Schanuel Conjecture	Pseudo-exp $\exp : \mathbb{C} \to \mathbb{C}$	Zilber	(Covers) Categoricity
Mumford-Tate	Shimura curves, modular curves	Daw, Harris	Categoricity
	Mirror Maps uniformization	Baldwin, Cruz, V.	Categoricity

Classical j invariant

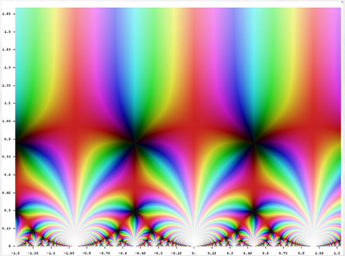
Klein defines the function (we call) "classical j"

$$j : \mathbb{H} \to \mathbb{C}$$

(where $\mathbb{H}$ is the complex upper half-plane)
through the explicit rational formula

$$j(\tau) = 12^3 \cdot \frac{g_2(\tau)^3}{g_2(\tau)^3 - 27g_3(\tau)^3}$$

with g_2 and g_3 certain functions ("Eisenstein" series).



j-invariant on $\mathbb{C}$ (Wikipedia article on j-invariant)

Basic facts about classical j

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- $$j(\tau) = j\left(\frac{a\tau + b}{c\tau + d}\right)$$

  if $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$.

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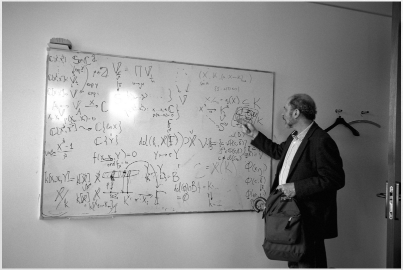
- (Hilbert's 12th...)

Zilber: transcendent functions - arithmetic geometry



1. Zilber started around 2001 to use axiomatizations in $L_{\omega_1,\omega}$ of **covers** (the most famous, **complex exponentiation**, to try to capture categorical versions of exponentiation, and thereby study classical conjectures,
2. with A. Harris, he extended this work to j invariants; they linked model theoretic categoricity to a positive answer to a Serre Conjecture on Tate modules linked to elliptic curves (and later extended to Shimura varieties by Daw et al.)

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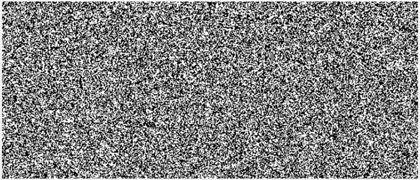
Model theory - classifying theories (and beyond)

Some mathematical structures seem less random, more “canonical” than others.
Why?

The context for classification



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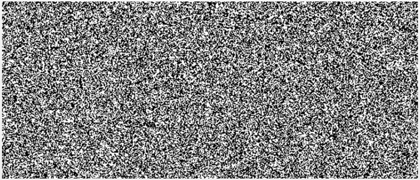

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Is it a homogeneous world? What kind of classification is there?

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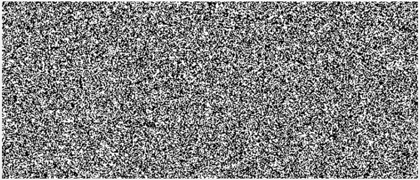

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Perhaps not quite! Groups, fields, algebraic varieties, Sobolev spaces, inner models of set theory, function spaces, you name them...

Is it a homogeneous world? What kind of classification is there? Model theory has a strong classification of all FIRST ORDER structures.

Model Theory - a theory of invariants?

$\langle \mathbb{N}, +, <, \cdot, 0, 1 \rangle$ - arithmetics

$\langle \mathbb{C}, +, \cdot, 0, 1 \rangle$ - algebraic geometry

$\langle \mathbb{R}, +, <, \cdot, 0, 1 \rangle$ - real alg. geom.

vector spaces (modules, etc.)

elliptic curves

some combinatorial graphs

Hilbert spaces, ℓ_2, etc.

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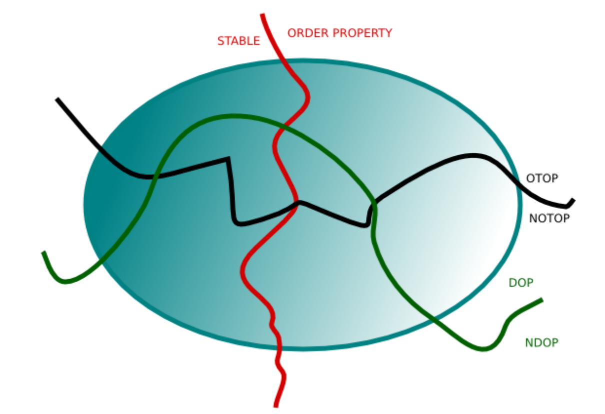
Words like "dimension", "rank", "degree", "density character" - seem to appear attached to those structures, and control them and allow us to capture them

Model Theory: perspective and fine-grain

1. Arbitrary **structures**.
2. Hierarchy of types of structures (or their theories): Stability Theory.
3. In the "best part" of the hierarchy: generalized Zariski topology - Zariski Geometries due to Hrushovski and Zilber: "arbitrary" structures whose place in the hierarchy ends up automatically giving them strong similarity to elliptic curves.
4. Uni-dimensional objects in Zariski structures are exactly finite covers of algebraic curves - these correspond to structures built to capture non-commutative phenomena in Physics!
5. More recently, Model Theory has dealt with "limit structures" of various kinds: limiting processes of constructions. Mathematical approximation and perturbation end up being model-theoretical.

Taxonomy

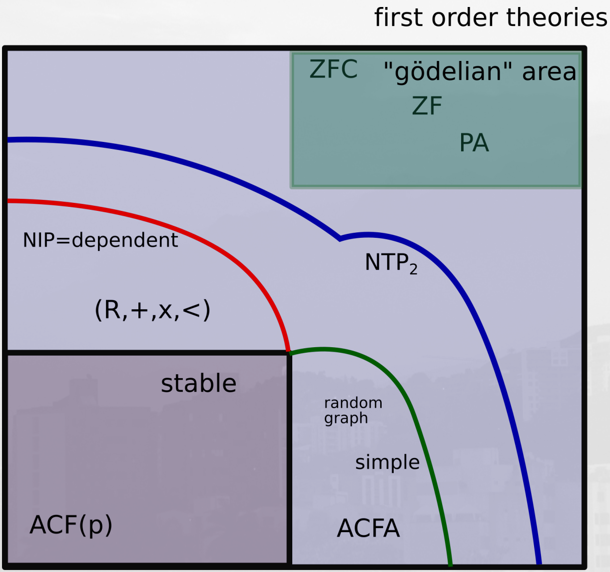


A "taxonomy" of classes of structures.

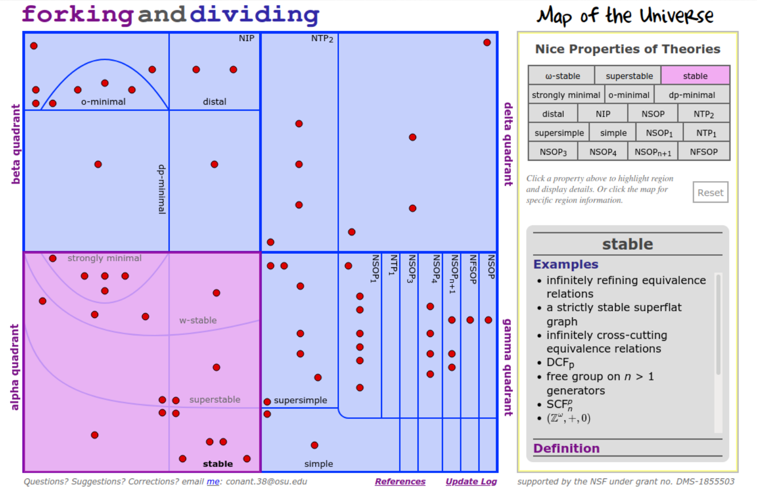
Dividing Lines: stable/unstable, etc.

stable	unstable	order property
NDOP	DOP	dimensional order property
NOTOP	OTOP	(omitting types) order property
superstable	unsuperstable	local control of $\mathop{\smile\hspace{-0.9em}\vert\hspace{0.5em}}$
depend. (NIP)	IP	codifying $a \in b \subset \omega$
etc. (NTP_2)	TP_2	tree properties...

A “map” of first order theories



High resolution (forkinganddividing.com / G. Conant)

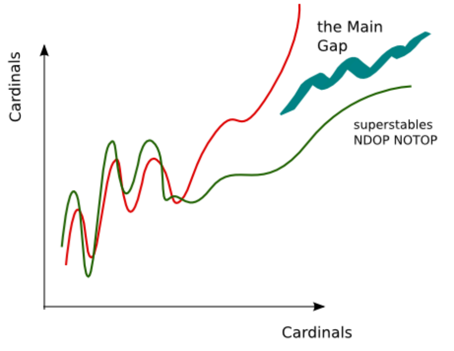


A big aim in Model Theory, triggered by Stability

Given a countable theory T, the spectrum function $I(T, \cdot)$ either always achieves the maximum values, else it has a bound:

$$I(T, \aleph_\alpha) < \beth_{\omega_1}(|\omega + \alpha|)$$

Notice that the result reveals asymptotic behaviour!



Later spin-off from Stability Theory:

- Hrushovski's proof of the Mordell-Lang Conjecture (ca. 1990)
- Model-Theoretic Analysis of the André-Oort Conjecture
- Proof (Casale-Freitag-Nagloo) of a Conjecture by Painlevé from ca. 1895, using the model theory of Differentiably Closed Fields
- Model Theoretic Analysis of analytic functions and Grothendieck's Standard Conjectures (Zilber, since ca. 2000) - this part not only in First Order Model Theory
- ...

The Main Dividing Line: Stability

The original dividing line, from whose name the whole subject inherited its name, is the notion of stability.

Classical Model Theory: definability. Formulas.

Initially, Model Theory allows two basic things:

Capturing classes of models

Isolating definible sets within each class

Examples:

Classes: (models of) Peano axioms, group axioms, field axioms, algebraically closed fields, etc. (Hilbert spaces, ...).

Definable Sets: two levels: through a formula or through infinite sets of formulas that still have "solution sets" (loci)

Definable Sets

In Model Theory, a set D is definable in a structure $\mathfrak{A}$ if there exists a formula $\varphi(x)$ such that $D = \varphi(\mathfrak{A}) = \{a \in A | \mathfrak{A} \models \varphi[a]\}$.
D is then the **locus** of a formula φ, in $\mathfrak{A}$.
Classical examples of definable sets in a field include affine varieties: the elliptic curve given by

$$y^2 = x^3 + ax + bc + c$$

can be understood as the definable set D_C over $\langle \mathbb{C}, +, \cdot, a, b, c \rangle$,

$$D_C = \varphi_C(\mathbb{C}, a, b, c) = \{(x, y) \in \mathbb{C}^2 | \varphi_C(x, y, a, b, c)\},$$

where $\varphi_C(x, y, a, b, c)$ is the formula $y^2 = x^3 + ax + bc + c$.

Type-definible sets (I)

Given a and C, the type of a over C in M is the set

$$\mathrm{tp}(a/C, M) := \{\varphi(x, \bar{c}) | \bar{c} \in C, M \models \varphi[a, \bar{c}]\}.$$

1. $\mathrm{tp}(2/\mathbb{Q}, \mathbb{C}) = \{x = 1 + 1, \cdots\}$
2. $\mathrm{tp}(\sqrt{2}/\mathbb{Q}, \mathbb{C}) = \{x \cdot x = 1 + 1, \cdots\}$
3. $\mathrm{tp}(\pi/\mathbb{Q}, \mathbb{C}) = \{\neg(P(x) = 0) | P(x) \in \mathbb{Q}[x]\}$

1 (realized type): one formula, one solution.
2 is also determined by one formula (it is a principal type) but it has several solutions (finitely many): it is algebraic

3 is a non algebraic type: not determined by a single formula, infinite solutions

Type-definable sets (II)

Turning things around, we may regard a set of formulas

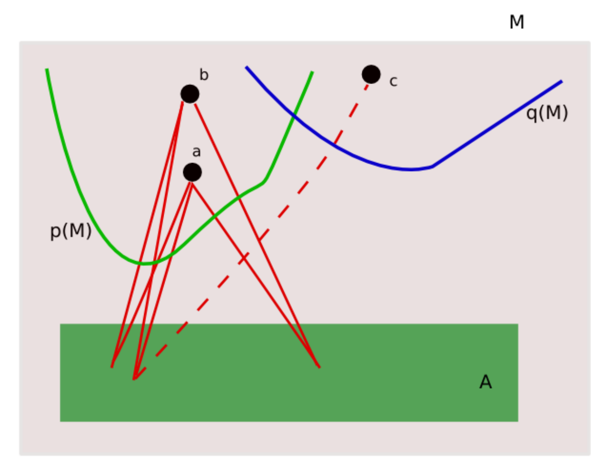
$$p = \{\varphi_i(x, \bar{b}) | i \in I\}$$

as corresponding to a definable (type-definable): look for the solutions (**locus**) in some model M yields the type-definible set

$$p(M) = \{a \in M | a \models p\}$$

Naturally, the same process can be applied to find type-definable subsets of M^2, M^3, etc. even $M^{\mathbb{N}}$...

$\mathrm{tp}(a/A, M)$, etc.



Rôle of definable and type-definable sets

Akin to the rôle of ideals in algebraic geometry: by Hilbert's Nullstellensatz, the crucial information on varieties is captured by radical ideals.

In $(\mathbb{C}, +, \cdot, 0, 1)$, "*a* and *b* have the same type over *V*" is captured by ideals (this is called "quantifier elimination" in logical lingo).

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More in general, types are basic blocks of information, generating, counting structures and controlling embeddings between them.

Variants

Types in model theory can be seen as

1. Sets of formulas
2. Zariski-closed sets
3. Orbits under automorphisms of monster models (strongly homogeneous, saturated)
4. More recently, measures/states or distributions.

Advantages of First Order Logic (Elementary Classes)

The classical relation

$$T \overset{\sim}{\Longleftrightarrow} Mod(T)$$

has been extremely productive in the case of First Order theories for decades (the content of a usual first course in Model Theory):

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has been extremely productive in the case of First Order theories for decades (the content of a usual first course in Model Theory):

- Theories, elementary maps, types
- Ultraproducts, compactness
- Typespaces (types topologized by definable sets)
- Unions of chains, Löwenheim-Skolem, saturated and homogeneous models.
- Quantifier elimination.
- Omitting types. Categoricity transfer.
- Stability, independence, simplicity, dependence.

General descriptions of Model theory

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More recently (2009) Hrushovski has described Model Theory as the geography of tame mathematics.

Very classical: Decidability of the complex numbers.

- Steinitz Theorem: all uncountable algebraically closed fields of characteristic zero, of same cardinality, must be isomorphic.
- The theory ACF_0 is included in $Th(\mathbb{C}, +, \cdot, 0, 1)$ — this one is obviously complete.
- The theory ACF_0 is axiomatizable (field axioms, solutions to all nonconstant polynomials, charact. zero) in a recursive way.
- The categoricity (in uncountable cardinals, Steinitz) here implies that ACF_0 is complete and therefore equal to $Th(\mathbb{C}, +, \cdot, 0, 1)$.
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Up and down phenomenon, strengthened in extreme ways later.

Classical: Ax-Kochen

For each positive integer d there is a finite set Y_d of prime numbers, such that if p is a prime not in Y_d then every homogeneous polynomial of degree d over the p-adic numbers in at least $d^2 + 1$ variables has a nontrivial zero.

Hrushovski & co. - the strongest classical results

Theorem (Hrushovski)
A solution to Mordell-Lang's conjecture over fields of arbitrary characteristic.

This uses much more sophisticated ideas: analysis of "modular sets" in (generalized Zariski) contexts, analysis of unidimensional sets, ... "geometric model theory" at its deepest.

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Other developments along this line include results on differentially closed fields, Galois theory for differential-difference equations, tec. - (Pillay, Macintyre, van den Dries, . . . and the André-Oort Conjecture [Scanlon]).

(Algebraically-minded) model theory...

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- (Abstract Elementary Classes [AECs] - in Baldwin's quick and rough (but very much to the point) description in a recent email:

  "Anatoly: For what's worth AECs are a style of model theory that approaches mathematics is a more familiar fashion. Instead of syntax and semantic, one investigates class of structures that satisfy certain properties (like closure under limits) - indeed there is a purely category theory definition.)"

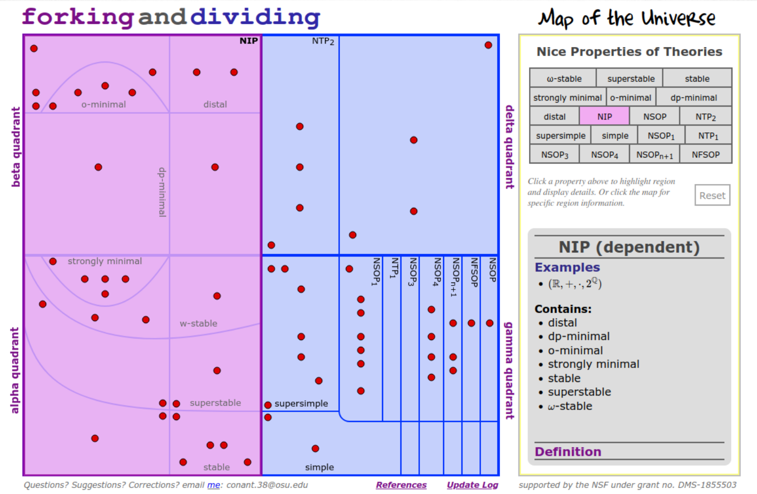
Abstract Elementary Classes are smoothly forward closed, generative and cumulative/coherent model classes

Abstract Elementary Classes

Abstract Elementary Classes are smoothly forward closed, generative and cumulative/coherent model classes:

- In addition to the model class $\mathcal{K}$, there is a notion of strong embedding $\prec_{\mathcal{K}}$, ordering $\mathcal{K}$ and refining algebraic extension,
- Closed under direct limits
- Endowed with an effective scheme of generating submodels
- With a cumulative/coherent behavior with respect to the existence of abstract "solutions"

The "outer layer of a well-known map" ?!?



The Conant map (again)

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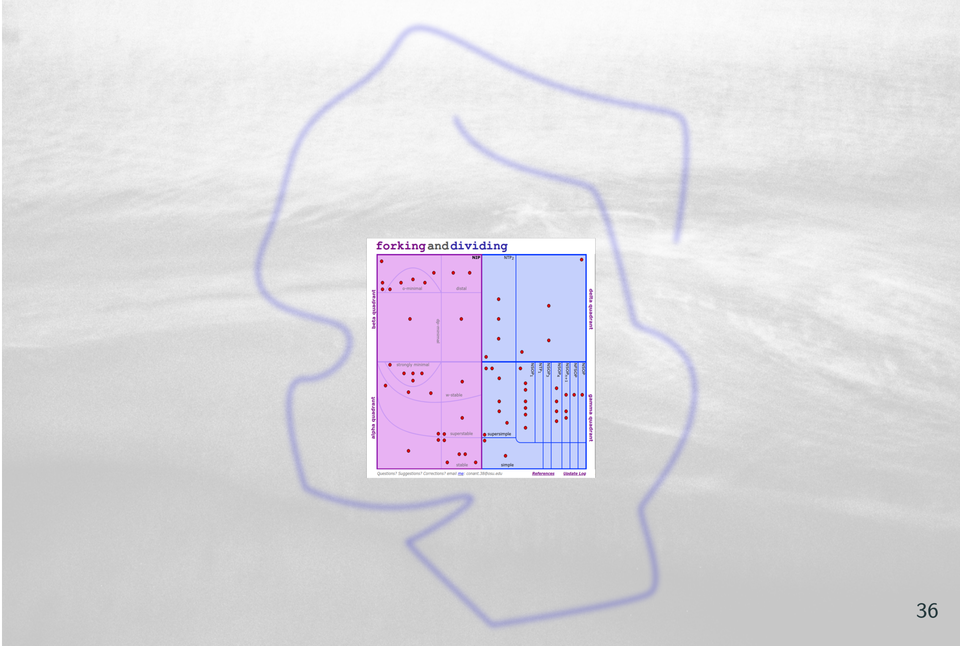

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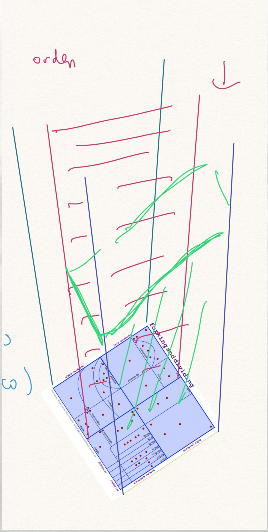


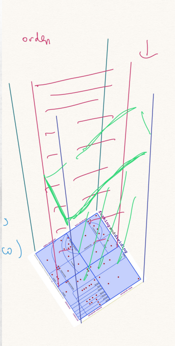
The New Century: picking up speed

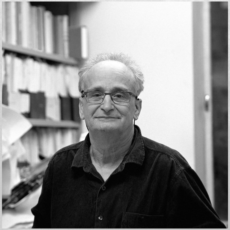
The first quarter of the 21st Century has seen enormous progress towards filling up the new map:

- Categoricity (Shelah-V. 1999, Boney-Grossberg-Vasey 2016),
- Superstability ("noetherianity" of model classes; Grossberg-VanDieren-V. 2010, Vasey, Boney, Mazari-Armida 2016-2022),
- Other structural dividing lines extended, revealing "faults" along many classes of structures.

Shelah-V.: internal logic of an AEC

1. Getting axiomatization of AECs in a logic was obtained early by Shelah, at the price of expanding the language (Presentation Theorem),
2. We improved this result significantly (joint work, 2021) - independent related results by Leung: a sentence $\psi_{\mathcal{K}}$ axiomatizing an AEC $\mathcal{K}$, in infinitary logic, in the original vocabulary,
3. This offers new avenues to study definability of types, extending the stability map by better syntactic control...





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$\exp$ and j as covers

j-covers and a path to categoricity

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- Generalization of this analysis to higher dimensions (Shimura varieties).

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- An axiomatization in $L_{\omega_1,\omega}$ of j
- A convoluted proof of categoricity of this version of j
- Generalization of this analysis to higher dimensions (Shimura varieties).
- Analogies to pseudoexponentiation ("Zilber field") are strong, but the structure of j seems to have a much higher degree of complexity even than *exp*.

j is also a cover...

An $L_{\omega_1,\omega}$ axiomatization of j:

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An $L_{\omega_1,\omega}$ axiomatization of j:

Let L be a language for two-sorted structures of the form

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Really, j is a **cover** from the action structure into the field $\mathbb{C}$.

The $L_{\omega_1,\omega}$-axiom - Crucial point: Standard fibers of the cover j

Let then

$$Th_{\omega_1,\omega}(j) := Th(\mathbb{C}_j) \cup \forall x \forall y (j(x) = j(y) \rightarrow \bigvee_{i<\omega} x = \gamma_i(y))$$

for $\mathbb{C}_j$ the "standard model" $(\mathbb{H}, \langle \mathbb{C}, +, \cdot, 0, 1 \rangle, j : \mathbb{H} \rightarrow \mathbb{C})$.

This captures all the first order theory of j (not the analyticity!) plus the fact that fibers are "standard" ("fibers are orbits")

Categoricity of classical j

Theorem (Harris, assuming Mumford-Tate Conj.)

The theory $Th_{\omega_1,\omega}(j) + \mathrm{trdeg}(F) \geq \aleph_0$ is categorical in all infinite cardinalities. I.e.,

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($\mathcal{H}_i = (H_i, \{g_j^i\}_{j\in\mathbb{N}})$ and $\mathcal{F}_i = (F_i, +_i, \cdot_i, 0, 1)$)
there are isomorphisms φ_H, φ_F such that

$$\begin{array}{ccc} \mathcal{H}_1 & \xrightarrow[\varphi_H]{} & \mathcal{H}_2 \\ \Big\downarrow j_1 & & \Big\downarrow j_2 \\ \mathcal{F}_1 & \xrightarrow[\varphi_F]{} & \mathcal{F}_2 \end{array}$$

commutes.

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- Quasiminimal abstract elementary classes. These must be categorical (the model theory of these harks back to results of Shelah from the late 1980s - excellent classes, then combined with quasiminimal classes and much more dramatically simplified - in some cases - by Bays, Hart, Hyttinen, Kesälä and Kirby). [Linguistic closure, homogeneity, uniqueness of generic, CC, alg. control.]

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- On the way to the previous, reduction of types of elliptic curves to the torsion information, readable by limits of N-covers on the field structure. A quite strong form of QE.
- A theorem by Keisler on the number of types of categorical sentences of $\mathcal{L}_{\omega_1,\omega}$... (this will give a surprising twist).

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Arithmetic Geometry:

An instance of the adelic Mumford-Tate conjecture for products of elliptic curves to show this. The strategy to build an isomorphism between two models M and M' consists (as expected) in

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- realizing the field type of a finite subset of a Hecke orbit over any parameter set (algebraicity of modular curves),...
- then show that the information in the type is contained in a finite subset ("Mumford-Tate" open image theorem used here) ... every point $\tau \in \mathbb{H}$ corresponds to an elliptic curve E — the type of τ is determined by algebraic relations between torsion points of E.

The types

If $\langle H, F\rangle \models Th(j)$, want a description of $\mathrm{tp}_H(\tau)$ and $\mathrm{tp}_F(z)$.

The two-sorted type of a non-special tuple $\tau \subset H$ may be studied in the field sort.

More specifically...

Since the algebraic curves $Z_g = \Gamma_g \setminus \mathbb{H}$ encode info on geometric interactions in $\langle H, F \rangle$, we may reduce to the field: (τ) is determined by

$$\bigcup_{g \subset G} {}_F(p_g(\tau)/\mathrm{dcl}(\emptyset) \cap F).$$

The key point is

$$\exists g \in G(g\tau_i = \tau_j) \qquad \underline{\text{iff}} \qquad \exists g \in G(j(\tau_i), j(\tau_j)) \in Z_g.$$

Later: the key info on types is contained in the Galois representations.

Building the model

Consider the "pro-étale cover"

$$\hat{\mathbb{C}} = \varprojlim_{g \subset G} Z_g .$$

(Morphisms: definable maps already discussed.) $\hat{\mathbb{C}}$ is pro-definable in $\langle \mathbb{C}, +, \cdot, \mathbb{Q}(j(S)) \rangle$; use $\hat{j} : \hat{\mathbb{C}} \to \mathbb{C}$.

The Galois action on $\hat{\mathbb{C}}$ is our tool.

$$\pi_1' := \varprojlim_{N} \mathrm{Aut}_{\mathrm{Fin}}(Z_N / Z_1).$$

The approach to categoricity

Types of independent tuples over the model-theoretic étale cover

$$\hat{\mathcal{U}} := \langle \hat{U}, F \rangle \xrightarrow{\hat{j}} \langle \mathbb{C}, +, \cdot, \mathbb{Q}(j(S)) \rangle$$

are the same as those in the standard model.

Categoricity will then be described in terms of Galois representation in the geometric étale fundamental group!

New Mathematical Connections

The new century opened with new, surprising, connections between AEC model and

- arithmetic/algebraic geometry

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- arithmetic/algebraic geometry
- modules
- (possibly) o-minimal and specific NIP AECs

The New Century: new connections

- 2001: the Zilber school: arithmetic geometry studied under the lens of Abstract Elementary Classes (really, axiomatized in $L_{\omega_1\omega}$ or under "quasiminimality hypotheses"): (pseudo)-exponential covers, modular forms and functions (the most famous of these analyses for the so-called j-mapping

$$j : \mathbb{H} \to \mathbb{C},$$

transferring the <u>group action structure</u> $\mathbb{H} \setminus \Gamma$ (of Fuchsian groups, e.g. $\Gamma = SL_2(\mathbb{Q})$) on (a structure H elementarily equivalent to) $\mathbb{H}$, to the <u>field structure</u> of a model of ACF_0 such as $(\mathbb{C}, +, \cdot, 0, 1)$),
- 2012-2023: The model theory of covers. . .

Starting in 2001: model theory of covers

This work happened (at least initially) with the context of

quasiminimal (excellent) pregeometries.

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An abstract closure notion (generalizing acl from model theory) $\mathrm{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$ satisfying monotonocity, exchange, idempotence and generally providing notions of dimension, independence, generated objects.

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NO a priori syntax!

Building from the exponential and the j-map (see [BV23])

	topic	paper	method/context	section
1	Complex exponentiation	[8]	quasiminimality	§1
2	cov mult group	[9]	quasiminimality	§1
3		[3]	quasiminimality	
4	j-function	[7]	background	§4.1
5	Modular/Shimura Curves	[5]	quasiminimality	§4
6	Modular/Shimura Curves	[4]	quasiminimality	
7	finite Morley rank groups	[1]	fmr & notop	§5.1
8	Abelian Varieties	[2]	fmr & notop / qm	§5.3
9	Shimura varieties	[6]	notop	§6
10	Smooth varieties	[10]	o-quasiminimality	§8

4

Focus: quasiminimality and categoricity

Main "take home"

Our two main themes have been:

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Categoricity in λ: there is a perfect collection of invariants

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Categoricity in λ: there is a perfect collection of invariants

Quasiminimality: there is only one way of being transcendental

Categoricity

A theory T (or, much more generally, an AEC $(\mathcal{K}, \prec_{\mathcal{K}})$) is categorical in λ iff for each $M, N \models T$ $(M, N \in \mathcal{K})$ of cardinality λ,

$$M \approx N.$$

Categoricity of the infinitary theory of j, or of pseudoexponentiation, is a very strong indicator of plausibility of Schanuel conjecture, of other arithmetic properties.

Quasiminimality

An abstract closure notion $cl : \mathcal{P}(X) \to \mathcal{P}(X)$ satisfying

- Monotonicity: $A \subset B \subset X \to cl(A) \subset cl(B)$,
- Idempotence: $cl(cl(A)) = cl(A)$,
- Exchange: $a \in cl(A \cup \{b\}) \setminus cl(A) \to b \in cl\,(A \cup \{a\})$.

is called a preogeometry (= matroid).

In pregeometries, one may define **independence**, **generated**, **bases** and therefore **dimension.**

A pregeometry (X, cl) is quasiminimal if

- There is only ONE KIND of "non-algebraic element" over every closed set: is $a, a' \notin cl(B)$ then there is an automorphism of X mapping a to a' while fixing $cl(B)$,
- The structure (X, cl) has strong homogeneity properties.

5

The model theory of covers

The cover structure

Remember the main example started like this:

$$\mathbb{C}_{\mathsf{exp}} = (\mathbb{C}, +, \cdot, exp)$$

the infinitary language $\mathbb{L}_{\omega_1 \omega}(Q)$ allows coding transcendental number theory (e.g. period conjecture).

The cover structure

Remember the main example started like this:

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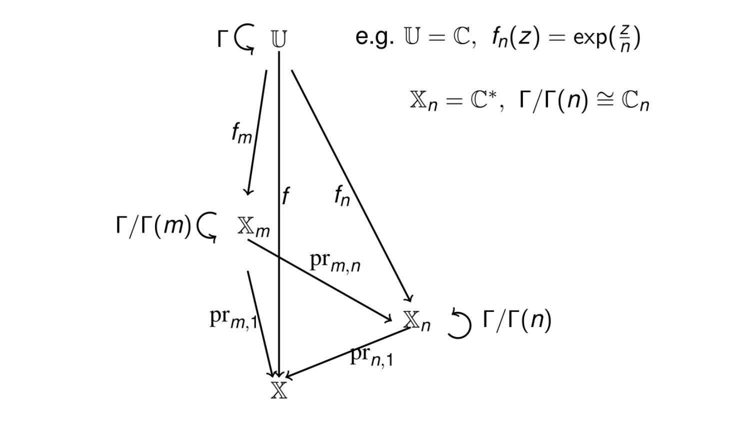
the infinitary language $\mathbb{L}_{\omega_1\omega}(Q)$ allows coding transcendental number theory (e.g. period conjecture).

However, in order to study **complex algebraic geometry** of a variety $\mathbb{X}$, one must use covers (more basic, more general) of algebraic subvarieties of $\mathbb{X}^n$ (for all n) that are definable

$$\mathbb{U} \to \mathbb{X}.$$

The cover structure (Zilber)



(Image from Zilber's lecture in our Conexión de GALoiS Seminar in Bogotá, 2024)

The cover structure (Zilber)

- This structure has enormous information on the **metric topology** of $\mathbb{X}(\mathbb{C})$ and is also **stable** (that is, essentially, it behaves as algebraic geometry),
- Without the cover $\mathbb{U}$ we only have the étale topology of $\mathbb{X}$; that is, we only know $\mathbb{X}$ up to abstract automorphisms of $\mathbb{C}$.

In this sense, postulating the existence of non-trivial covers is something quite strong (about $\mathbb{X}$)!

Logic and geometry, again

Zilber's school theorems (this century): the theory in infinitary logic $\mathbb{L}_{\omega_1\omega}$ is many cases (exp, j, $\mathfrak{p}$) is uncountably categorical.

Proving this mixes the theory of AECs with arithmetic theorems: (proved) versions of Mumford-Tate's Conjecture, extensions of Galois and Kummer theories.

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Categoricity ends being equivalent to deep facts in arithmetic geometry!

Complete topological invariants $\Sigma(\mathbb{X}_a)$ for complex (Shimura, abelian) varieties $\mathbb{X}_a$

New possibilities: the “Mirror Map program”

- The **Conexión de GALoiS** Seminar (Universidad Nacional de Colombia, University of Illinois at Chicago)
- Alexander Cruz: observing strong analogies with another model-theoretic work for functions similar to j (Freitag-Scanlon, Casale-Freitag-Nagloo and work with Blázquez-Sanz (UNAL-Med)) using (Schwarzian) **differential equations** and the regularity of the group action associated to $j \dots$

In his paper Does model theory have something to say about mirror symmetry?, Cruz proposes a program for mirror symmetry in mathematical physics.

Mirror maps (originating in superconformal field theories in physics) link symplectic geometry with complex geometry, the **A Side** with the **B Side** in geometry.

Mirror symmetry predicts that Calabi-Yau varieties are always given by pairs $\left(V, \hat{V}\right)$ linked by a **mirror map** reflecting Hodge type symmetries of V and $\hat{V}$.

More on the MM+MT-program

Observations (Cruz):

- Classical j is a mirror map (for elliptic curves), satisfying the classic Schwarzian equation that was crucial to model-theoretic analysis using differential Galois theory in model theoretic style (DCF).
- Hypersuperfaces in $\mathbb{CP}^3$ given by

$$X_s = x_0^3 + x_1^3 + x_2^3 + x_3^3 + s x_0 x_1 x_2 x_3$$

correspond to mirror symmetry for K3 surfaces. A differential operator (annihilating periods) given by

$$\Theta^3 - 8z(1+2\Theta)(1+4\Theta)(3-4\Theta), \text{ con } \Theta = z\frac{d}{dz}$$

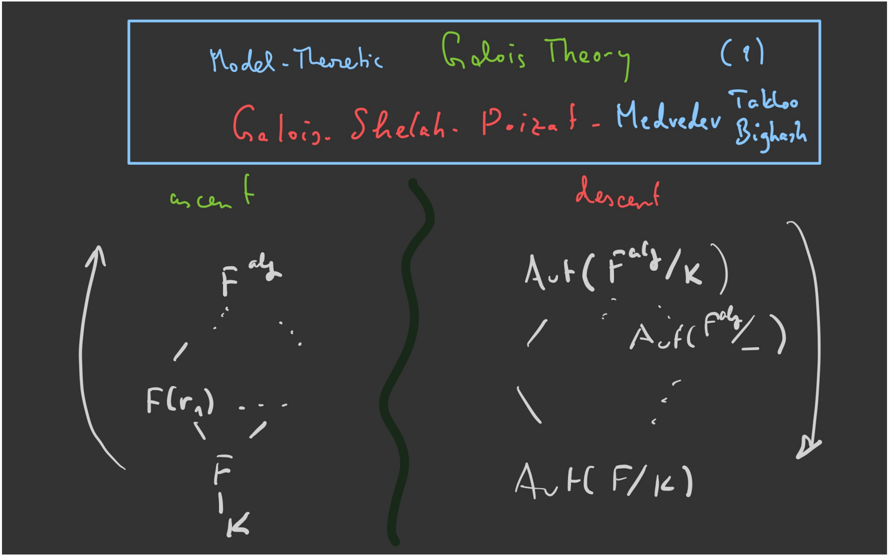
generates a mirror map $z(q)$ with modularity properties.

- Cruz observed that the mirror map $z(q)$ for K3 surfaces has infinitary theory $Th_{\omega_1,\omega}(z) + \text{trdeg}(F) \geq \aleph_0$ that is categorical in each uncountable cardinal.
- This starts suggesting new frameworks for other mirror maps.

6

Galois theory in model theory

Model Theory: a natural Galois-theoretic framework



In first order, the key role of imaginaries

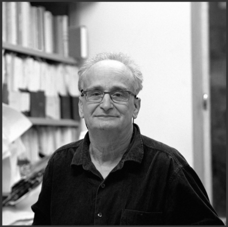
~ 1980: Shelah defines imaginaries

for ARBITRARY first order theories T:

equivalence classes of definable functions

$$a/_E = \{ b \in M \mid a E^M b \}$$

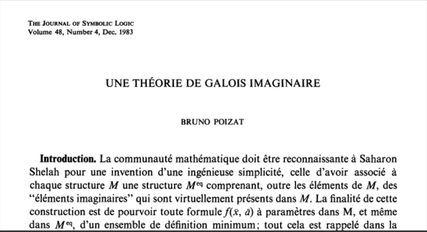
$$\varphi(a,b) \iff aEb$$



S. Shelah – 2016

1983 : Poizat notes that Shelah's imaginaries are exactly what's needed for a general Galois theory (for models of T)

Bruno Poizat (1970s)



THE JOURNAL OF SYMBOLIC LOGIC
Volume 48, Number 4, Dec. 1983

UNE THÉORIE DE GALOIS IMAGINAIRE

BRUNO POIZAT

Introduction. La communauté mathématique doit être reconnaissante à Saharon Shelah pour une invention d'une ingénieuse simplicité, celle d'avoir associé à chaque structure M une structure M^{eq} comprenant, outre les éléments de M, des "éléments imaginaires" qui sont virtuellement présents dans M. La finalité de cette construction est de pourvoir toute formule $f(\bar{x}, \bar{a})$ à paramètres dans M, et même dans M^{eq}, d'un ensemble de définition minimum; tout cela est rappelé dans la

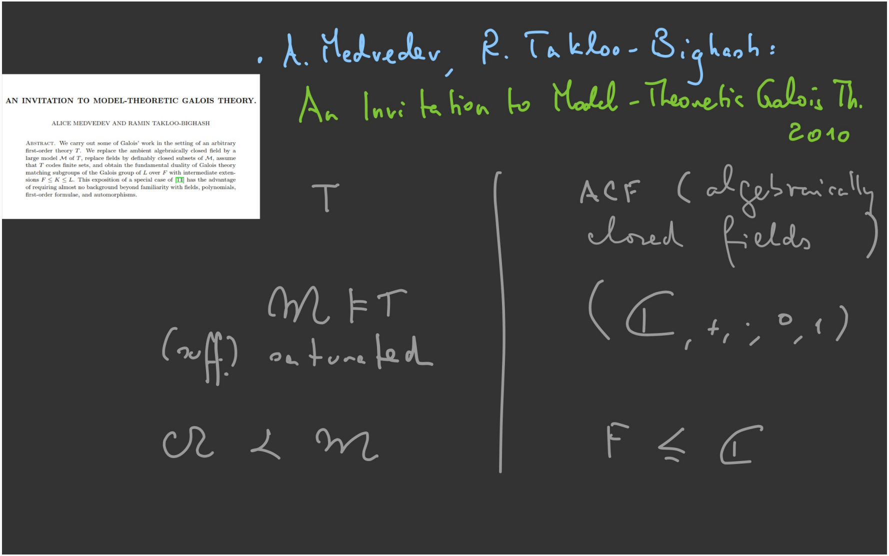
In the stable context, a MUCH CLOSER proof to the original

> Il n'est pas de grande gloire à démontrer un résultat universellement connu depuis 150 ans (qui est exprimé ici de manière anachronique : au temps de Galois, il n'est question ni de corps, ni a fortiori de clôture algébrique) ; cependant cette démonstration a le mérite d'être beaucoup plus proche des preuves, ou des indications de preuve, qu'on trouve dans les manuscrits de Galois, dont elle est en définitive une traduction en langage contemporain ; elle est plus directe que celle qu'on enseigne habituellement dans les cours d'algèbre, qui repose sur l'étude des extensions de corps de degré fini, et qui n'a été mise au point qu'à la fin du siècle dernier ; ceci tout en satisfaisant aux exigences de la sacro-sainte rigueur des mathématiciens d'aujourd'hui (à l'exception de ceux qui ne voient pas en la théorie des modèles une activité mathématique).

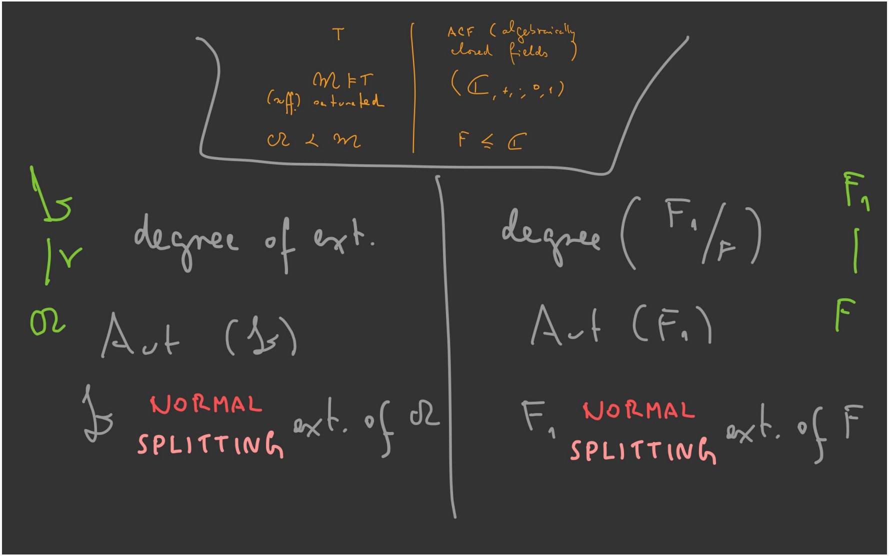
Poizat: the model-theoretic proof of Galois' Correspondence is MUCH CLOSER to the (indications of) proof in Galois' manuscripts!

- In what sense?
- With which obstructions?

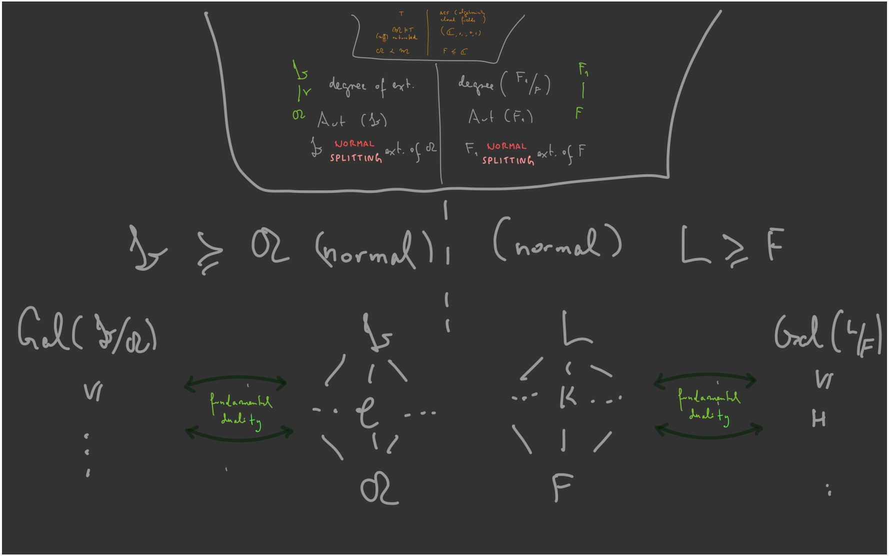
Some translations (following Medvedev/Takloo-Bighash)



Normal and Splitting Extensions (def. later)



Galois duality I



Some differences (lost in translation)

What is lost?

- L might not have function symbols!
- Polynomials $\rightsquigarrow$ Formulas (QE)
- $L \supset K$ might be far from linear/K
- the degree of an extension might not be given by vector space DIM
- NO norms, traces, determinants!

A couple of notions for the translation

- $\mathcal{M} \models T$ suff. saturated
- $\varphi(x,y)$ L-fmla; $b \in \mathcal{M}$ is a solution of $\varphi(a,y)$ if $\mathcal{M} \models \varphi(a,b)$.
- we have acl, dcl notions:

  $b \in \mathrm{acl}(A)$ if $\mathrm{orb}(b/A)$ is finite

- ... degree of b/A = $|\mathrm{orb}(b/A)|$

Normal extensions

$\mathfrak{M}$
$|$
$\mathfrak{L}$
$|$
$\mathfrak{A}$

* $\mathfrak{L}$ is a finite extension of $\mathfrak{A}$ iff

  $\exists\, b$ tuple of $\mathfrak{L} \cap \mathrm{acl}(\mathfrak{A})$ s.t. $\mathfrak{L} \subseteq \mathrm{dcl}(\mathfrak{A} \cup b)$

* $\mathfrak{L}$ is NORMAL $/\mathfrak{A}$ iff

  $\mathrm{orb}(c/\mathfrak{A}) \subseteq \mathfrak{L}, \quad \forall c \in \mathfrak{L}$

Splitting extensions

- $\mathfrak{F}$ is **SPLITTING** over $\operatorname{irr}(b/A)$, over $\mathfrak{A}$

  if $\begin{cases} \operatorname{orb}(b/\mathfrak{A}) \subseteq \mathfrak{B} \\ \mathfrak{B} \subseteq \operatorname{dcl}(\mathfrak{A} \cup \operatorname{orb}(b/\mathfrak{A})) \end{cases}$

- $\mathfrak{F}$ def. closed $\Rightarrow$ $\mathfrak{F}$ is normal $/\mathfrak{A}$, splits $/\mathfrak{A}$
- $\mathfrak{A} \hookrightarrow \mathfrak{F} \hookrightarrow \mathfrak{E}$ fin.

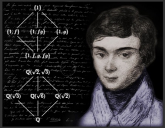
  $\Rightarrow$ $\deg(\mathfrak{E}/\mathfrak{A}) = \deg(\mathfrak{E}/\mathfrak{F}) \cdot \deg(\mathfrak{F}/\mathfrak{A})$

... If T codifies finite sets

$\mathcal{C} = \mathrm{dcl}(\mathcal{C})$ is a normal ext. of $\mathfrak{a}$ $\left(= \mathrm{dcl}(\mathfrak{a}) \right)$

$G = \mathrm{Aut}(\mathcal{C}/\mathfrak{a})$

$\Rightarrow$ Subgr. of G



Definably closed intermediate extensions

$G \geq H$ ⟵ Gal ⟶ $\mathrm{Fix}(H) = \{ c \in \mathcal{C} \mid \forall h \in H \; h(c) = c \} \geq \mathfrak{a}$

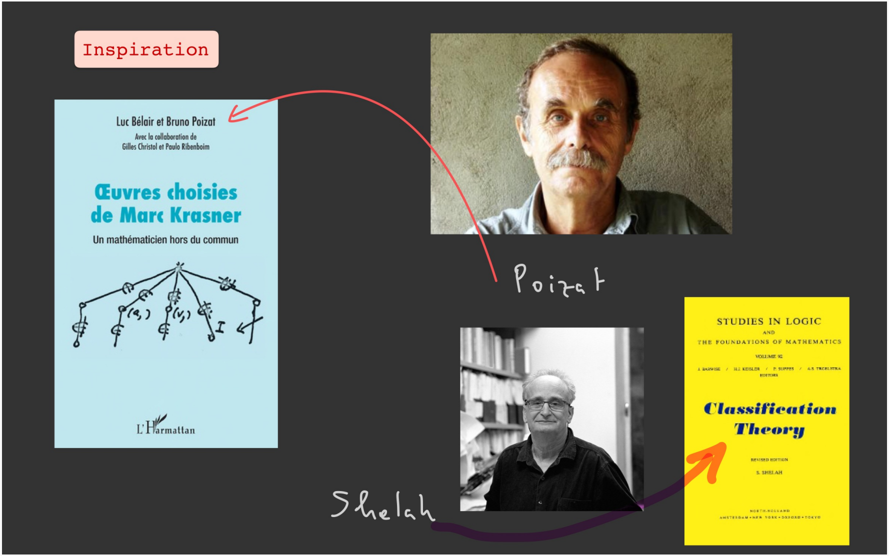
T codifies finite SETS of tuples i.e.

alike SYMMETRIC polynomials

$\forall n \in \mathbb{N}$

$\forall F \subset M^n$ finite $\quad \exists b \left(\forall \sigma \in Aut(M) \quad \sigma(b) = b \Updownarrow \sigma(F) = F \right)$

Summary of the first rapprochement: the two sources

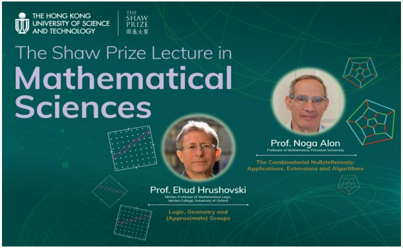


A crucial early hypothesis: stability

The key hypothesis to the early possibility of defining a good Galois group of a theory was stability: roughly, a solid theory of definability of orbits (Galois-types) of the action of the automorphism group of a large structure $M \models T$. We now take a détour.

Hrushovski: Galois theory/definability

1. Hrushovski isolated patterns of definability and the core of a first order theory T,
2. A way of "balancing the Galois theory" of first order theories - one of the strongest versions of Galois theory available, linked to structural Ramsey theory - the "correct Galois object" for correspondence,
3. This has been extended a little beyond First Order (positive theories)



But we stop for now. . .

Thank you!
¡Mil gracias!

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