Some recent (and some not so recent) interactions between Set Theory and Model Theory

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Universidad Nacional de Colombia / Bogotá

Contents

At the boundary between Logic and Geometry...

> *If the Greeks were so attached to geometry, wasn't it that they thought by tracing lines, with no words? However (or maybe just because of that?) [they produced] a perfect axiomatic! Euclid's Postulates, construction. Limiting what one is allowed to trace.*

Simone Weil, Cahier III

A story of compactness (and what remains in its absence)

Today, two topics:

- The quest for the natural logic for Abstract Elementary Classes (AECs): removing compactness, while keeping a very weak remnant! (The apparent paradox of a rich model theory with "very" pale compactness)
- Another logic similar to First Order, but much stronger: $\mathbb{L}^1_\kappa$. And some connections to strong and weak compactness properties...

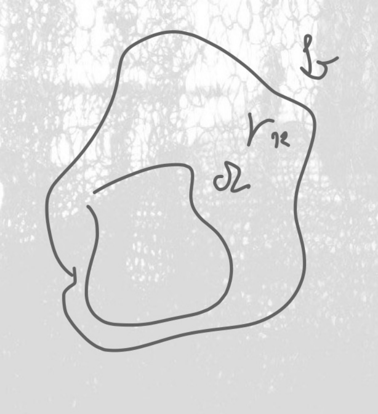
Plan

Axiomatizing AECs: logics with a tiny remnant of compactness?

Connections with large cardinals and forcing

AECs: why so much stability theory?

And our first question was the title of this slide!



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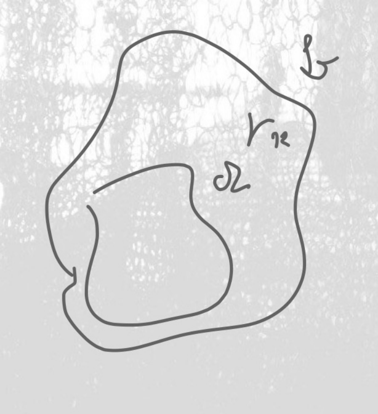


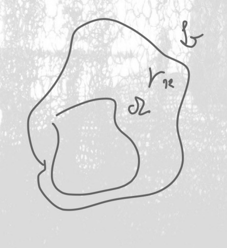
In AECs, we replace from the start the usual extreme emphasis on φ, T, compactness by more **semantical** notions:
$\prec_{\mathcal{K}}$, f a morphism,
f $\in$ Aut($\mathbb{C}$), etc.

φ

T

$T_0 \subseteq^{\text{fin}} T$

$\vdots$



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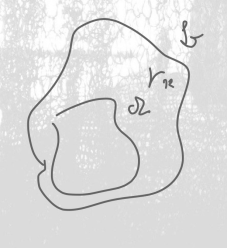
emphasis shift

towards 1980



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: Instead of extracting $\prec$, f, etc. from T, φ, we turn $\prec$, f a strong embedding into the primitive notions!

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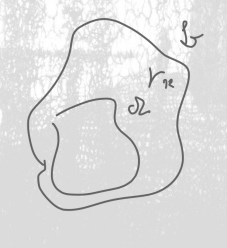

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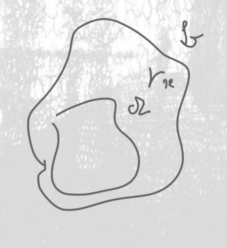
strong substructure



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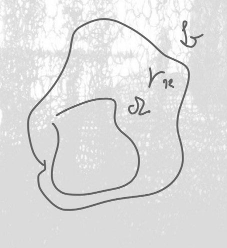

$\mathfrak{A} \prec_{\mathcal{K}} \mathfrak{B}$

"perfect" extension, algebraically closed, etc.

AEC - the axioms, briefly

Fix $\mathcal{K}$ be a class of τ-structures, $\prec_{\mathcal{K}}$ a binary relation on $\mathcal{K}$.

Definition

$(\mathcal{K}, \prec_{\mathcal{K}})$ is an abstract elementary class iff

- $\mathcal{K}$, $\prec_{\mathcal{K}}$ are **closed under isomorphism**,
- $M, N \in \mathcal{K}$, $M \prec_{\mathcal{K}} N \Rightarrow M \subset N$,
- $\prec_{\mathcal{K}}$ is a partial order,
- (**TV**) $M \subset N \prec_{\mathcal{K}} \bar{N}, M \prec_{\mathcal{K}} \bar{N} \Rightarrow M \prec_{\mathcal{K}} N$,
- ($\searrow$**LS**) There is some $\kappa = LS(\mathcal{K}) \geq \aleph_0$ such that for every $M \in \mathcal{K}$, for every $A \subset |M|$, there is $N \prec_{\mathcal{K}} M$ with $A \subset |N|$ and $\|N\| \leq |A| + LS(\mathcal{K})$,
- (**Unions of $\prec_{\mathcal{K}}$-chains**) A union of an arbitrary $\prec_{\mathcal{K}}$-chain in $\mathcal{K}$ belongs to $\mathcal{K}$, is a $\prec_{\mathcal{K}}$-extension of all models in the chain and is the sup of the chain.

AECs, as described by a model theorist to a geometer

> ``Anatoly: For what's worth, AECs are a style of model theory that approaches mathematics in a more familiar fashion. Instead of syntax and semantic, one investigates class of structures that satisfy certain properties (like closure under limits) - indeed there is a purely category theory definition.''

John Baldwin, in an email to Anatoly Libgober (2023)

Abstract Elementary Classes, in a nutshell

Abstract Elementary Classes are smoothly forward closed, generative and cumulative/coherent model classes

And really, a lot of examples (and model theory)

Many natural constructions in Mathematics are examples of AECs (or metric AECs)

1. Complete first order theories
2. Excellent, quasiminimal classes
3. Various classes axiomatizable in $L_{\omega_1,\omega}$ or $L_{\kappa\omega}$
4. Covers of Abelian algebraic groups

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7. Model Theory of Modules ($\mathrm{Hom}, \mathrm{Ext}_1, \ldots$ Mazari-Armida)
8. AECs of C^*-algebras (Argoty, Berenstein, V.)
9. Zilber analytic classes (pseudoexponentiation, j-map, Shimura)
10. Classes of Valued Fields...

Elusive, “embedded” definability?

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- and of course, many variants of “forking independence”, very specially versions of “splitting” (weak definability of types)!

So, the question of finding **right notions of definability** responsible for all these inner workings is important...

Axiomatizing an AEC: attempts (old and new)

- Shelah, V. 2020,
- Leung 2021.

Fix $(\mathcal{K}, \prec_{\mathcal{K}})$ an AEC with $\mathrm{LS}(\mathcal{K}) = \kappa$. We also assume wlog that all models in $\mathcal{K}$ are of cardinality $\geq \kappa$.

Earlier results:

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- **Shelah-Vasey:** If $\mathsf{LS}(\mathcal{K}) = \aleph_0$, $\mathcal{K}$ is $\aleph_0$-stable and has the $\aleph_0$-AP, and $\mathsf{I}(\aleph_0, \mathcal{K}) \leq \aleph_0$ <u>then</u> $\mathcal{K}$ is $\mathsf{PC}_{\aleph_0}$.

More

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2020

Shelah-V.

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(The $\omega \cdot \omega$ refers to quantification of an EF game of length $\omega \cdot \omega$)

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In 2022, a better bound: in $\mathbb{L}_{(2^\kappa)^+,\kappa^+}$

a better bound, with the high price of using

$\forall x_0 \exists y_0 \ldots \forall x_i \exists x_i \ldots, \ (i < \omega \cdot \omega)$

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INFINITARY LOGICS AND ABSTRACT ELEMENTARY CLASSES

SAHARON SHELAH AND ANDRÉS VILLAVECES

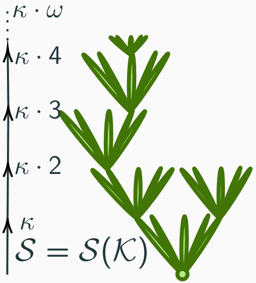
(Communicated by Heike Mildenberger)

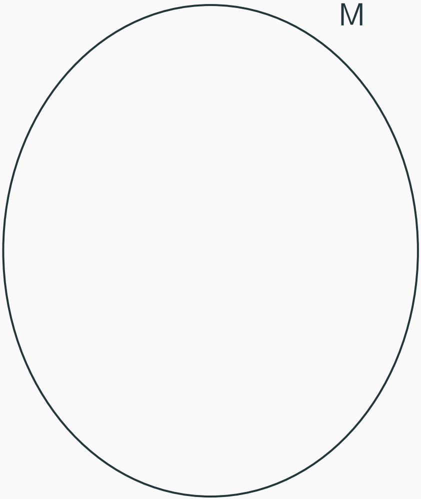
Abstract. We prove that every abstract elementary class (a.e.c.) Löwenheim–Skolem–Tarski (LST) number κ and vocabulary τ of cardin $\leq \kappa$ can be axiomatized in the logic $\mathbb{L}_{\beth_2(\kappa)^{+++},\kappa^+}(\tau)$. An a.e.c. $\mathcal{K}$ in voc lary τ is therefore an EC class in this logic, rather than merely a PC class. constitutes a major improvement on the level of definability previously by the Presentation Theorem. As part of our proof, we define the *cano tree* $\mathcal{S} = \mathcal{S}_{\mathcal{K}}$ of an a.e.c. $\mathcal{K}$. This turns out to be an interesting combina object of the class, beyond the aim of our theorem. Furthermore, we s a connection between the sentences defining an a.e.c. and the relatively infinitary logic L^1_λ.

Introduction

Given an abstract elementary class (a.e.c.) $\mathcal{K}$, in vocabulary τ of $LST(\mathcal{K})$, we prove the two following results:

Testing an arbitrary L-structure M against $\mathcal{S}_{\mathcal{K}}$





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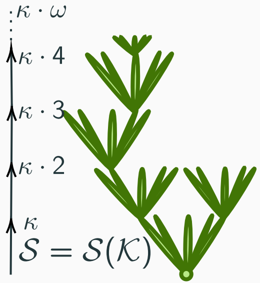

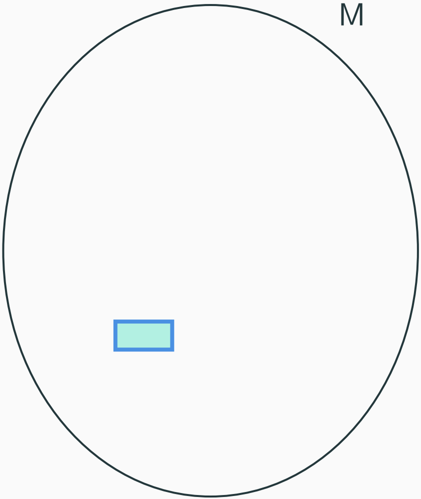

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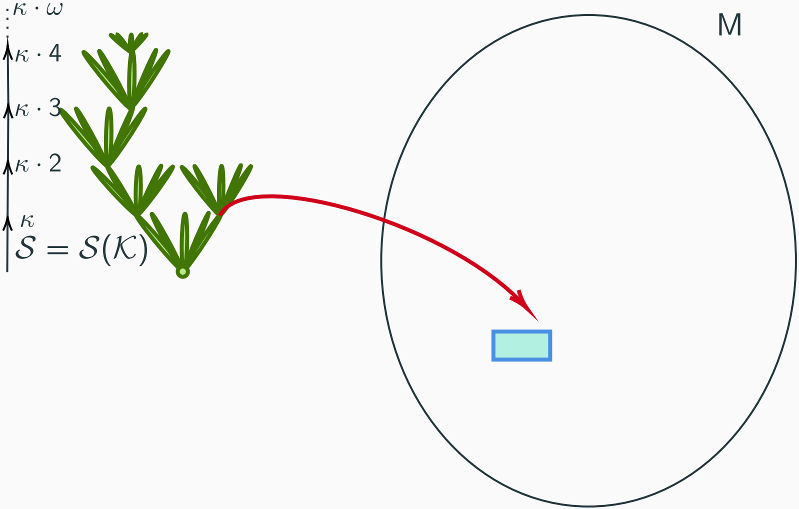

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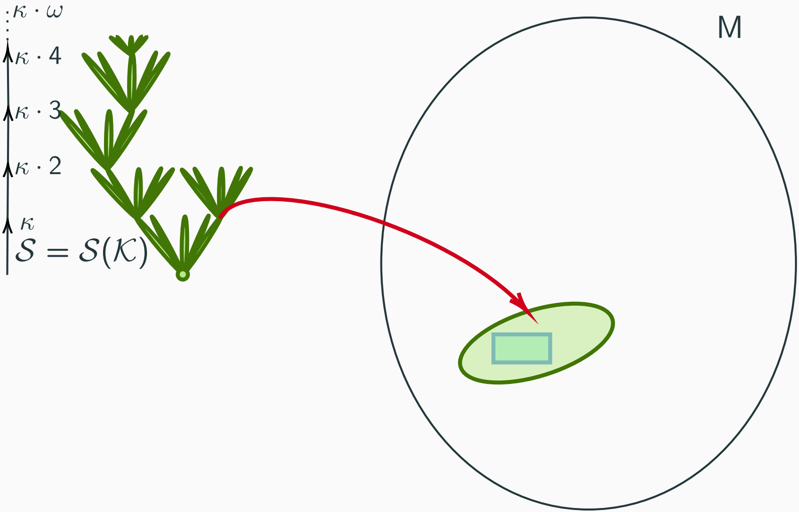

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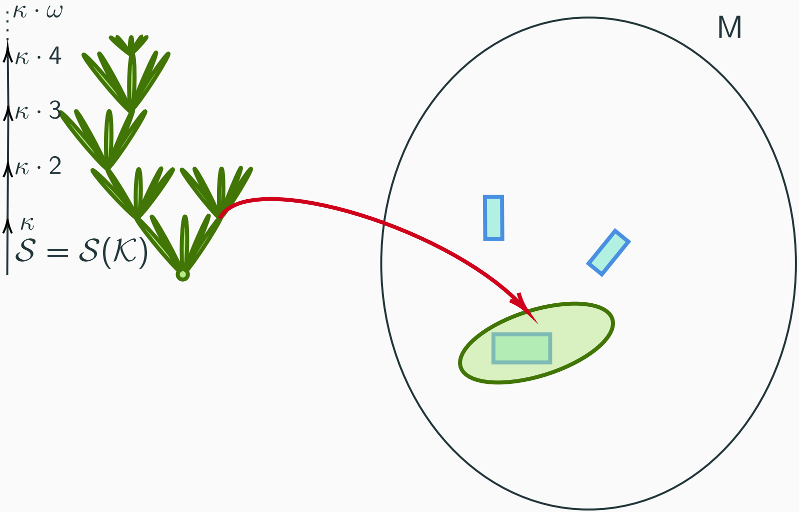

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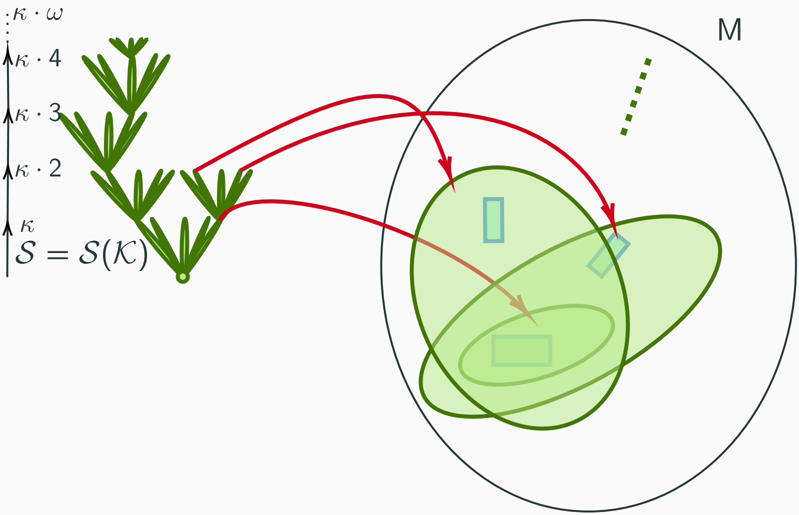

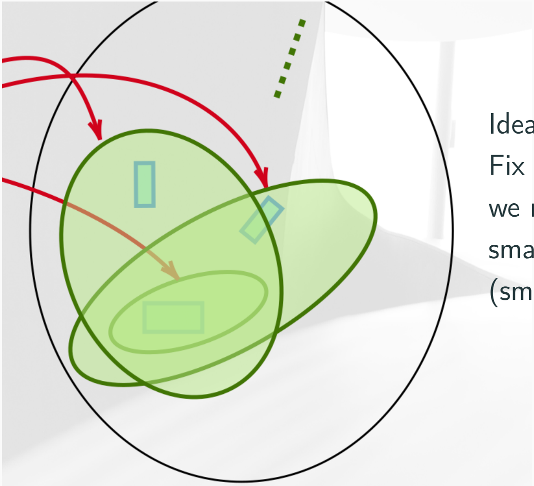

Idea of our axiomatization:
Fix an L-structure M. How can we realize M as a direct limit of small models N $\in \mathcal{K}$?
(small = size $\kappa = LS(\mathcal{K})$)

Realizing an arbitrary model as a limit

$$M = \varinjlim \{N \subseteq M | N \in \mathcal{K}\} \qquad ???$$

(Of course, we need a lot of constraints!)

Towards this goal

We use the canonical tree of $\mathcal{K}$: models of size $\kappa = \mathsf{LS}(\mathcal{K})$, with universes

$$\kappa, \kappa + \kappa, \kappa + \kappa + \kappa, \ldots$$

and a whole "system of $\prec_{\mathcal{K}}$-elementary embeddings" between those models:

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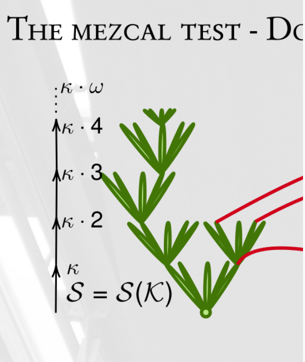
$\mathcal{S}_{\mathcal{K}}$: the canonical tree of $\mathcal{K}$.

In $\mathcal{S}_{\mathcal{K}}$, $N_1 \lhd N_2$ iff $N_1 \prec_{\mathcal{K}} N_2$.



We now use syntax to. . .

. . . to "test" the model M **- the test membership in** $\mathcal{K}$

M **must "pass"** $\beth_2(\kappa)^{++} + 2$ **tests**

(a newer proof reduces this number to $(2^\kappa)^+$**)**

$I_2(\kappa)^{++}+2$ (2020) / $\alpha < (2^\kappa)^+$ (2021)

sentences, "approximating" $\mathcal{K}$:

$\varphi_{0,0} = \top$

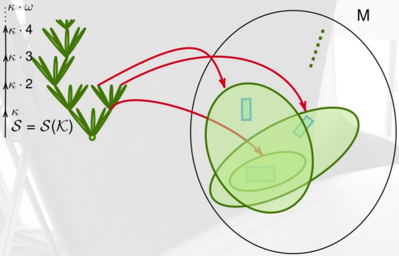
$\varphi_{1,0}$ iterate the "test" against the tree $\mathcal{S}_\mathcal{K}$

$\vdots$

$\varphi_{\alpha,0}$

$\vdots$

The mezcal test - Does $M \in \mathcal{K}$?



Formulas $\varphi_{M,\gamma,n}(\bar{x}_n)$

For M in the canonical tree $\mathcal{S}$ at level n, a formula with $\kappa \cdot n$ free variables, defined by induction on γ.

- $\gamma = 0$: $\varphi_{0,0} = \top$ ("truth"). If $n > 0$,

$$\varphi_{M,0,n} := \bigwedge \mathrm{Diag}^n_\kappa(M),$$

  the atomic diagram of M in $\kappa \cdot n$ variables.
- γ limit: Then

$$\varphi_{M,\gamma,n}(\bar{x}_n) := \bigwedge_{\beta<\gamma} \varphi_{M,\beta,n}(\bar{x}_n).$$

- $\gamma = \beta + 1$: Then $\varphi_{M,\gamma,n}(\bar{x}_n)$ is the $L_{\lambda^+,\kappa^+}(\tau)$ formula

$$\forall \bar{z}_{[\kappa]} \bigvee_{\substack{N \succ_{\mathcal{K}} M \\ N \in \mathcal{S}_{n+1}}} \exists \bar{x}_{=n} \left[\varphi_{N,\beta,n+1}(\bar{x}_{n+1}) \wedge \bigwedge_{\alpha<\alpha_n[N]} \bigvee_{\delta \in S[N]} z_\alpha = x_\delta \right]$$

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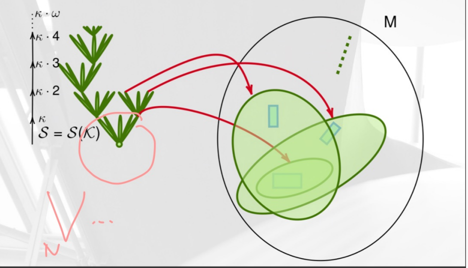

$$\varphi_{1,0} : \forall z \bigvee_{N \in \mathcal{S}_1} \exists \bar{x}_1 \left[\varphi_{N,0,1}(\bar{x}_1) \wedge \bigwedge_{\alpha<\kappa} \bigvee_{\delta<\kappa} z_\alpha = x_\delta \right]$$

size κ

a copy of N

the copy covers z

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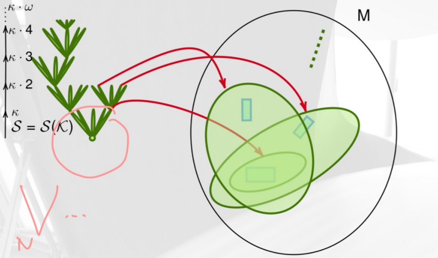

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size κ

"copy of N"

covers z

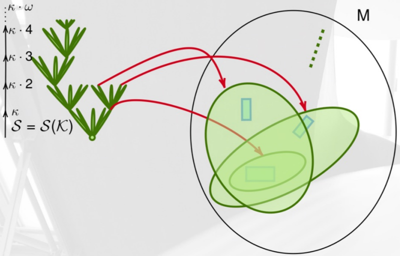
$M \models \varphi_{1,0}$ if it may "be covered" by levels $N \in \mathcal{K}$, of size κ "

$$\varphi_{1,0}: \forall z \bigvee_{N \in S_1} \exists \bar{x}_1 \left[\varphi_{N,0,1}(\bar{x}_1) \wedge \bigwedge_{\alpha<\kappa} \bigvee_{\delta<\kappa} z_\alpha = x_\delta \right]$$

size κ — version of N — covers z

"S_1 covers M"

The mezcal test - Does $M \in \mathcal{K}$?

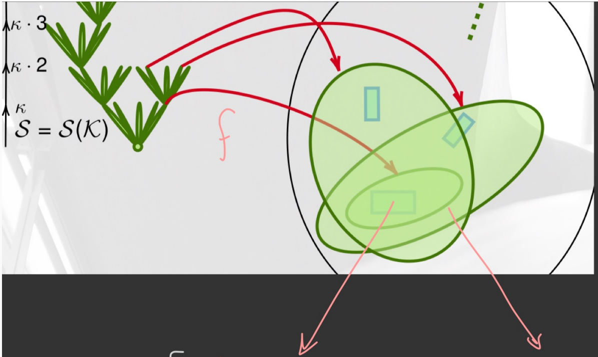


$$M \models \varphi_{2,0} = \forall z \bigvee_{N \in S_1} \exists \bar{x}_2 \left[\varphi_{N,1,2}(\bar{x}_2) \wedge \text{"} z \subseteq \bar{x}_2 \text{"} \right]$$

κ ?

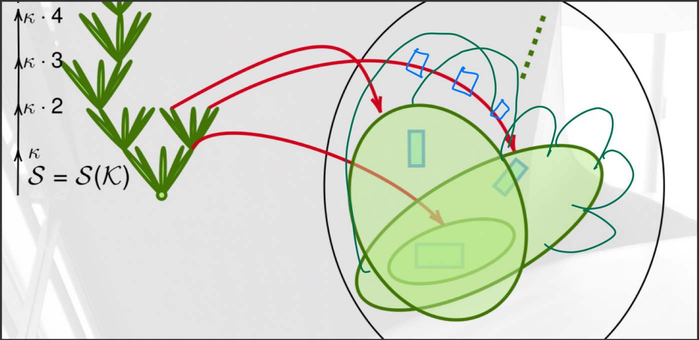
Covers and then Covers

$$\forall z \bigvee_{N \in S_1} \exists \bar{x}_2 \left[\forall z' \bigvee_{\substack{N' > N \\ N' \in S_2}} \exists \bar{x}_1 \left((\bar{x}_2{}^\frown \bar{x}_k) \wedge z' \subset \bar{x}_2{}^\frown \bar{x}_k \right) \wedge z \subset \bar{x}_2 \right]$$



Covering notions, refining...

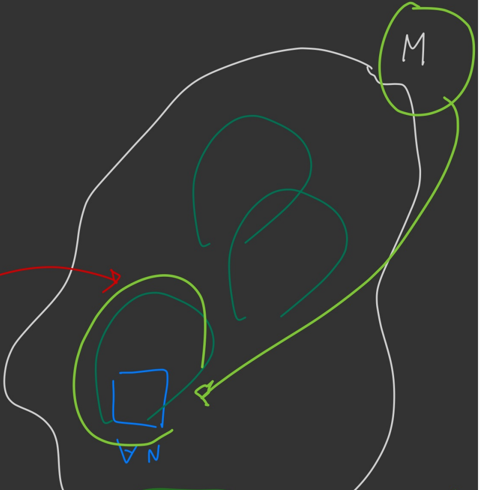
$$\varphi_{2,0}\left[\begin{array}{l}\forall z \text{ some } N \in \delta_1 \text{ covers } z \text{ (via } f\text{)} \\ \forall z' \text{ some } N' \in \delta_2\ \ N' \geq_{\kappa} N \text{ covers } z'\end{array}\right.$$



$\varphi_{3,0}$: better cover yet...

Problem: M is big!

As this way of covering may be insufficient, we iterate transfinitely:



$M \models \varphi_{\omega+1,0}$:

$\forall N \bigvee$ covers N BUT $M \models \varphi_{\omega,N,1}(\dots) \rightarrow$ info of depth ω

$N \in \ell_1$

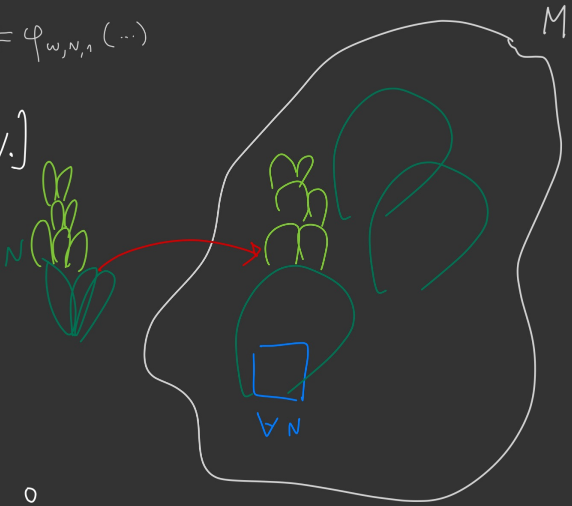
$\forall N \bigvee_{N \in \ell_1} \text{Cov. } N$ But $M \models \varphi_{\omega, N, 1}(\ldots)$

Theorem [Shelah, V.]

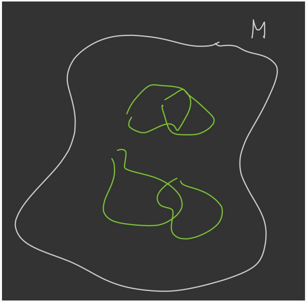
$M \in \mathcal{K}$

$\Updownarrow$

$M \models \varphi_{\beth_2(\kappa)^{++} + 2, 0}$



Key Idea



Inside M (because of the sentences $\varphi_{\alpha,0}$ it satisfies), there are "densely" many models of size κ, from the class $\mathcal{K}$.

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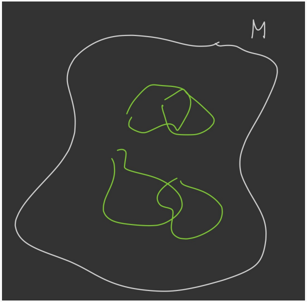

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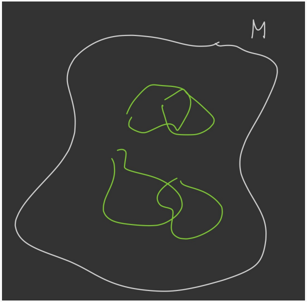

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Now, this per se would be too weak to guarantee that $M \in \mathcal{K}$.
However. . . , since $M \models \varphi_{\beth_2(\kappa)^+ + 2, 0} \cdots$
and the $\subseteq$-directed system will also turn out to be a $\prec_{\mathcal{K}}$-directed system!

Why $\prec_{\mathcal{K}}$-directed? ("Model-completeness" inside M)

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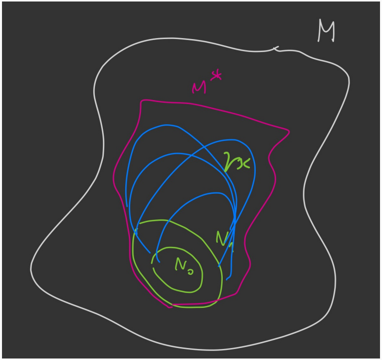

Two combinatorial arguments:

- In 2020, using a **partition relation** for well-founded trees due to Komjáth and Shelah.
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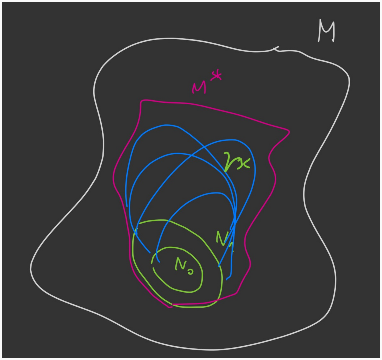

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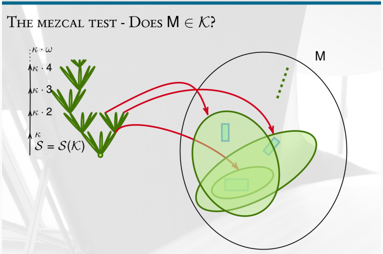
Assuming $N_0 \not\prec_{\mathcal{K}} N_1$, using the tree $\mathcal{S}_{\mathcal{K}}$ and the fact that $M \models \varphi_{\alpha,0}$, we build a **tree of models** converging to the same model - by the axioms of AEC's we may conclude that $N_0 \prec_{\mathcal{K}} N_1$!

Steps:



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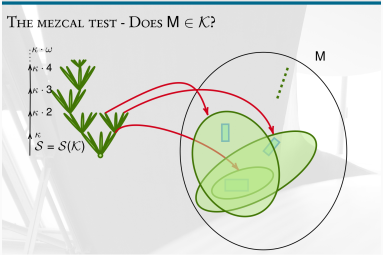

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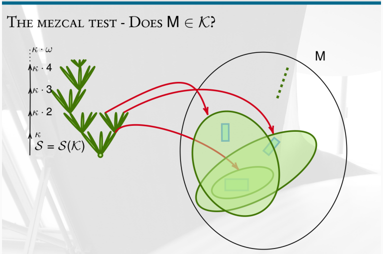

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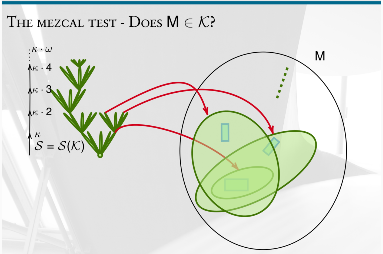

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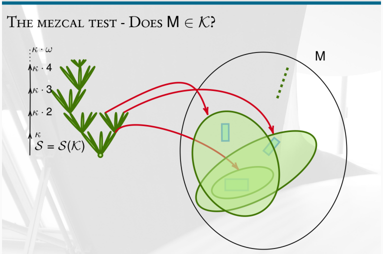
- M $\models \varphi_{\alpha,0}$ for α "high enough" implies (by quite non-trivial combinatorics) that... M is a $\prec_{\mathcal{K}}$-**direct limit of small models** from the class $\mathcal{K}$ (and therefore in $\mathcal{K}$)!

Leung's strategy:



Leung's strategy has similarities, but he replaces the combinatorics by the game quantifier

$$\forall x_0 \exists y_0 \forall x_1 \exists y_1 \ldots \forall x_i \exists y_i \ldots$$

of length $\omega \cdot \omega$ (this has unclear semantics. . .).

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- Bi-interpretability in AECs (Galois theory),
- $\mathcal{K}$'s behaviour in forcing extensions,
- $\mathcal{K}$'s behaviour under large cardinal embeddings

$$j : V \to_{\lambda} M \ldots$$

On improving the bounds

The sentence axiomatizing an AEC $\mathcal{K}$ with Löwenheim-Skolem-Tarski number κ has been reduced to the logic $\mathsf{L}_{(2^\kappa)^+,\kappa^+}$.

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The same proof yields more, as we shall see:

A syntactic “Tarski-Vaught test” for AECs,

Definability of types (with Nájar)

On a variant of the "Tarski-Vaught Test" for AECs

Theorem
Let $\mathcal{K}$ be an a.e.c., $\tau = \tau(\mathcal{K}) \leq \kappa = \mathrm{LST}(\mathcal{K})$, $\lambda = \beth_2(\kappa)^{++}$. Then, given τ-models $\mathrm{M}_1 \subseteq \mathrm{M}_2$, the following are equivalent:

1. $\mathrm{M}_1 \prec_{\mathcal{K}} \mathrm{M}_2$
2. *if* $\bar{\mathrm{a}} \in {}^{\kappa \geq}(\mathrm{M}_1)$ *then there are* $\bar{\mathrm{b}}$, N *and* f *such that*
   - 2.1 $\bar{\mathrm{b}} \in {}^{\kappa \geq}(\mathrm{M}_1)$ *and* $\mathrm{N} \in \mathcal{S}_1$
   - 2.2 $\mathrm{Rang}(\bar{\mathrm{a}}) \subseteq \mathrm{Rang}(\bar{\mathrm{b}})$
   - 2.3 f *is an isomorphism from* N *onto* $\mathrm{M}_1 \restriction \mathrm{Rang}(\bar{\mathrm{b}})$
   - 2.4 $\mathrm{M}_2 \models \varphi_{\mathrm{N},\lambda+1,1}[\langle \mathrm{f}(\mathrm{a}^*_\alpha) \mid \alpha < \kappa \rangle]$.

The Internal Logic of an AEC

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However, the fact they support so many constructions from stability theory (towers of models, structural control by [Galois] types, type omission, minimal pairs, stability spectrum, canonical forking notions for stable AECs, group configuration, etc.) raises the question of finding the natural internal logic of the AEC.

We have now embarked (with Shelah and Nájar) on this large scale project.

Two Internal Logics of an AEC

$$\mathbb{L}_{\mathcal{K}}^{1,\mathsf{aec}} < \mathbb{L}_{\mathcal{K}}^{1,\mathsf{II},\mathsf{aec}}$$

The two logics

$$\mathbb{L}^{1,\mathrm{aec}}_{\mathcal{K}} < \mathbb{L}^{2,\mathrm{aec}}_{\mathcal{K}}$$

$\psi_{\mathcal{K}} \in \mathbb{L}^{1,\mathrm{aec}}_{\mathcal{K}}$,
fragment of $\mathbb{L}_{(2^\kappa)^+,\kappa^+}$ containing $\psi_{\mathcal{K}}$ (Shelah-V. 2021)

$\psi_{\mathcal{K}} \in \mathbb{L}^{1,\mathrm{II},\mathrm{aec}}_{\mathcal{K}}$,
second order interpretability of $\mathcal{K}$ (Shelah-V. in progress)

We close $\mathbb{L}_{(2^\kappa)^+,\omega}$ under $\forall x$, $\exists x$, $\bigwedge_{i<2^\kappa} \psi_i$, $\neg$ and $\psi_{\mathcal{K}}$.

This can very easily define well-ordering!

("Non-well orders" form an AEC, of very low "Scott rank", in a natural way!)

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Other possibilities:

- Removing $\neg$ from $\mathbb{L}^{1,\mathrm{aec}}_{\mathcal{K}}$?
- Comparing/adapting $\mathbb{L}^{1}_{\kappa}$?
- Developing stability theory for $\mathbb{L}^{1}_{\kappa}$?
- Transfer stability theory to $\mathbb{L}^{1,\mathrm{aec}}_{\mathcal{K}}$?
- Omitting Types for these logics ?

Plan

Axiomatizing AECs: logics with a tiny remnant of compactness?

Connections with large cardinals and forcing

Virtually Large Cardinals

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  can be "virtualized" by requiring the embedding to exist in a set-forcing extension of V.
- Virtual(ized) large cardinals are still large cardinals, but are now in the neighborhood of an ω-Erdős cardinal; they are consistent with L.

Virtually Large Cardinals

- A cardinal κ is **virtually supercompact** (remarkable) if for every $\lambda > \kappa$, there is $\alpha > \lambda$ and a transitive M with ${}^{\lambda}\mathsf{M} \subseteq \mathsf{M}$ such that there is a virtual elementary embedding $\mathsf{j} : \mathsf{V}_\alpha \to_\kappa \mathsf{M}$ with $\mathsf{j}(\kappa) > \lambda$.

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Back to logic: the strong compactness cardinal of a logic

In 1971, Magidor proved that extendible cardinals are **strong compactness cardinals** for second-order infinitary logic $\mathbb{L}^{\text{II}}_{\kappa,\kappa}$. This means that every $< \kappa$-satisfiable theory **in this logic** is satisfiable.

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In their preprint Model Theoretic Characterizations of Large Cardinals, Boney, Dimopoulos, Gitman and Magidor [BDGM] generalize Magidor's early result to virtually extendible cardinals.

Theorem (BDGM)
*κ is virtually extendible iff every $< \kappa$-satisfiable $\mathbb{L}^{\text{II}}_{\kappa,\kappa}$-theory has a... **pseudo-model**.*

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They introduce the filtering of "being a model" (compactness) to "being a pseudo-model" (pseudo-compactness) and get the equivalence with virtuality.

Pseudo-models and forth-systems

So... what are these “filtered” models?

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Definition
Let T be a τ-theory in some logic $\mathcal{L}$, let M be a τ^*-structure.
A **forth system** $\mathcal{F}$ from τ to τ^* is a collection of renamings $f : \sigma \to \sigma^*$, $f \in \mathcal{F}$ with σ, σ^* finite subsets of τ, τ^* respectively, such that

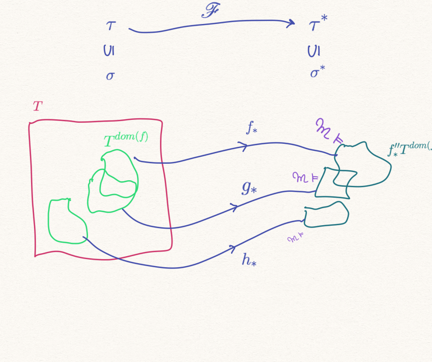
1. $\emptyset \in \mathcal{F}$,
2. If $f \in \mathcal{F}$ and $\tau_0 \subseteq^{\mathsf{fin}} \tau$ then there is $g \in \mathcal{F}$ with $f \subseteq g$ and $\tau_0 \subseteq \mathrm{dom}(g)$

M is a **pseudomodel** of T if there is a forth system $\mathcal{F}$ from τ to τ^* such that for every $f \in \mathcal{F}$, $M \models f''_* T^{\mathrm{dom}(f)}$.

Pseudo-models: a picture

The notion of **pseudomodel** deals with

- localizing in coherent ways (sheaf-like construction) the notion of being a model. . .



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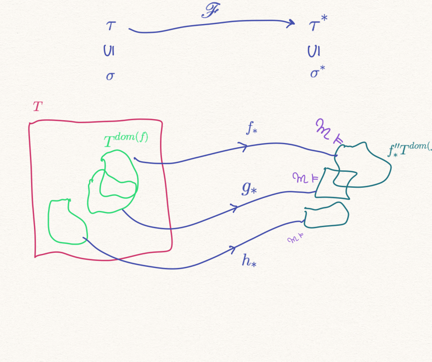

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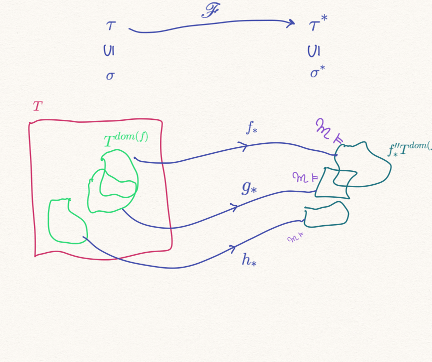

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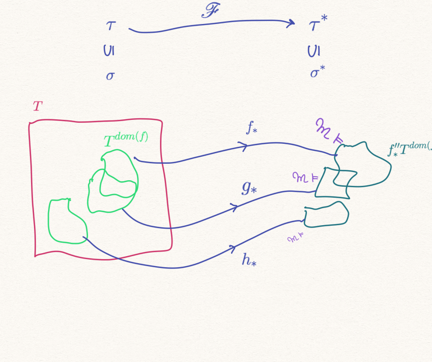

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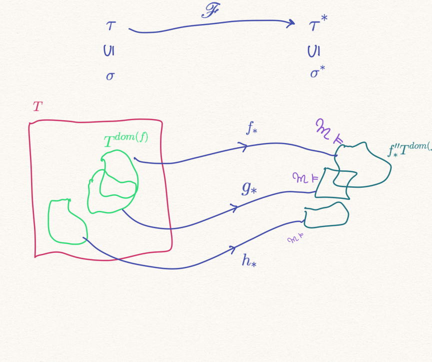
- The other direction uses the virtual embedding to obtain the forth system.



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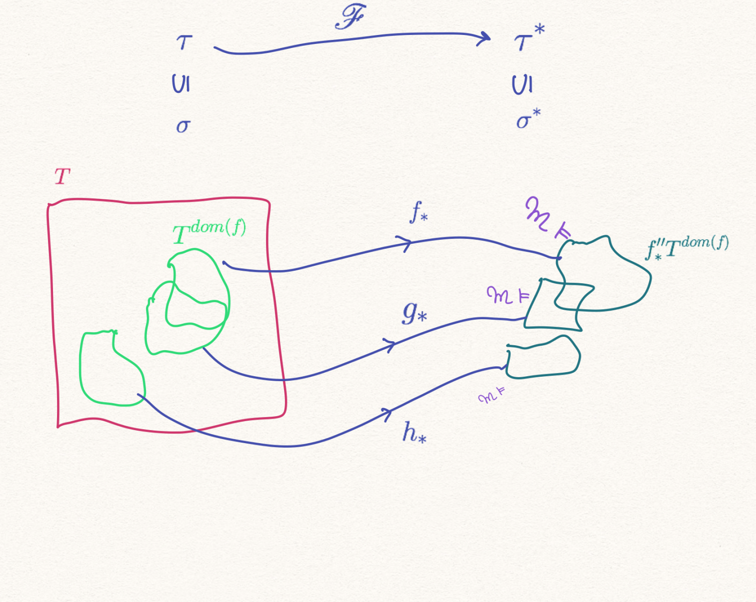
Motto

Forth-systems between vocabularies $\equiv$ forcing notions for virtuality

Pseudomodels



Virtualization of a Logic

A related notion: the virtualization of a logic. Now using forth-systems **for models** (and not for vocabularies, as before).

An $\mathcal{L}$-**forth system** $\mathcal{P}$ from M to N (both τ – structures) is a collection of $\mathcal{L}$-elementary embeddings with the "forth property":

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This is equivalent to playing the classical Ehrenfeucht-Fraïssé game but with $\forall$ picking only challenges "from the left" (from M).

Virtualization of a logic (II)

[BDGM] use those $\mathcal{L}$-forth systems to get Löwenheim-Skolem-Tarski style characterizations of virtual cardinals: **the existence of a virtual elementary embedding** $f : M \to_\kappa N$ is equivalent to the existence of a forth system from M to N or that N satisfies the **virtualized logic** theory of M (or player $\exists$ having a winning strategy in the half (virtual) game)...

A direction worth looking at: $\mathbb{L}^1_\theta$ for θ strongly compact

Shelah has been able to extract interesting model theory from the blend of the definition of his logic $\mathbb{L}^1_\theta$ **under the additional assumption that** θ is a strongly compact cardinal:

- A "Keisler-Shelah"-like theorem ($\mathbb{L}^1_\theta$-elementarily equivalent models have isomorphic iterated ultrapowers)
- Special models (unions of ω-chains of iterated ultrapowers) are unique... giving easier proofs of Craig (essentially, showing Robinson and using compactness).
- Connections to stability theory.

The methods are connected with Malliaris-Shelah's constructions and also with careful use of saturation, not unlike the use of forth models in [BDGM].

So, what is $\mathbb{L}^1_\kappa$?

- $\mathbb{L}_{\kappa\omega} \leq \mathbb{L}^1_\kappa \leq \mathbb{L}_{\kappa\kappa}$,
- $\mathbb{L}^1_\kappa$ has **interpolation**
- $\mathbb{L}^1_\kappa$ is maximal among extensions of $\mathbb{L}_{\kappa\omega}$ with interpolation and a form of undefinability of well-order (Lindström-type characterization by Shelah, in 2012)
- $\mathbb{L}^1_\kappa$ satisfies a weak version of unions of **countable** chains: if

$$\mathsf{M}_0 \prec_{\mathbb{L}_{\kappa\kappa}} \mathsf{M}_1 \prec_{\mathbb{L}_{\kappa\kappa}} \dots \mathsf{M}_\mathsf{n} \dots$$

then

$$\mathsf{M}_\mathsf{i} \prec_{\mathbb{L}^1_\kappa} \mathsf{M}_\omega \quad \forall \mathsf{i} < \omega,$$

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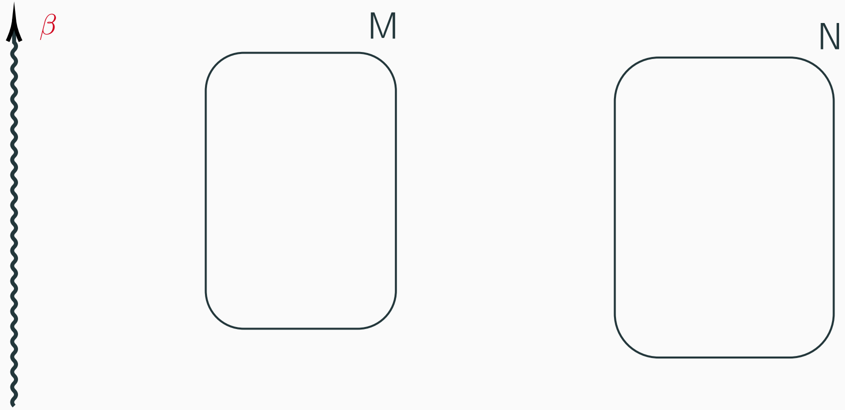
- $\mathbb{L}^1_\kappa$ does not have a **syntax** in the usual sense (no recognizable formulas or sentences), but rather
- $\mathbb{L}^1_\kappa$ has a game-theoretic "fake syntax" given by a "delayed game" (the Shelah game $\mathfrak{D}^\beta_\theta(\mathsf{M}, \mathsf{N})$) ...

Shelah's game $\mathfrak{D}^{\beta}_{\theta}(M, N)$.



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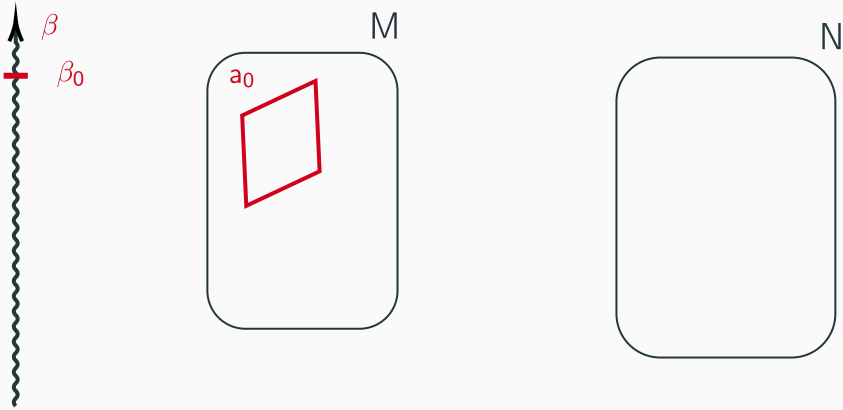
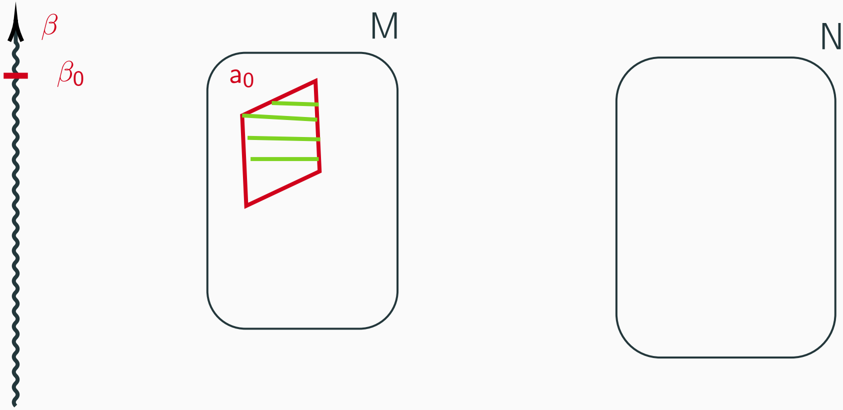


Shelah's game $\Game_{\theta}^{\beta}(M, N)$.



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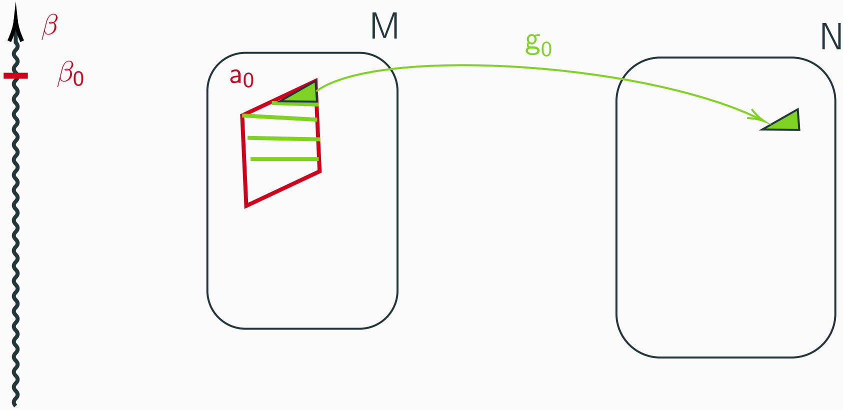

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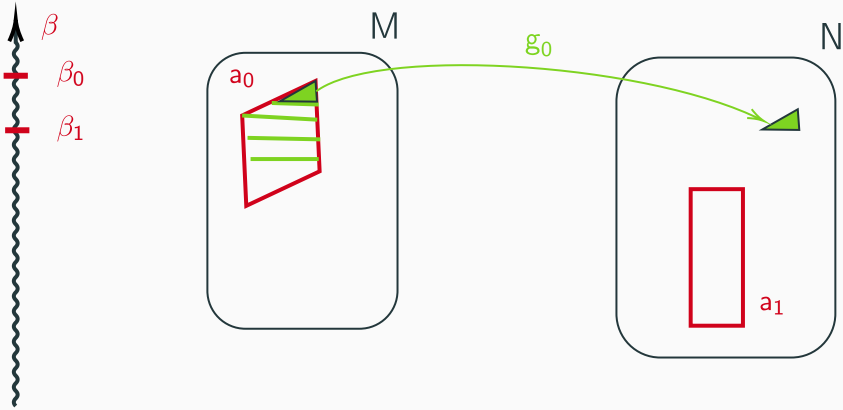
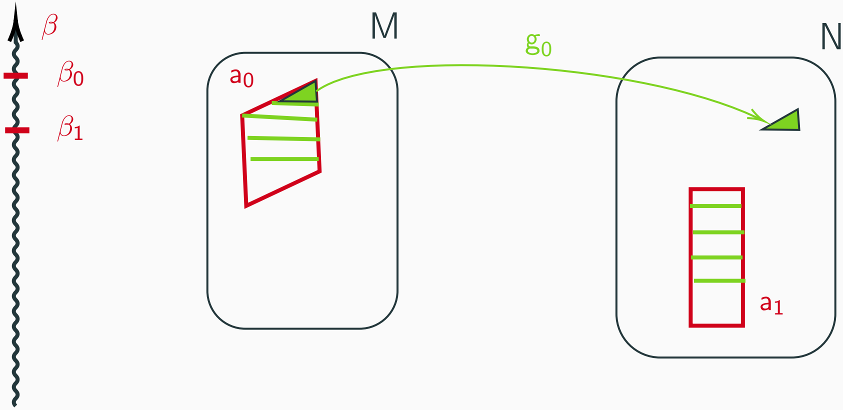


Shelah's game $\mathfrak{D}^{\beta}_{\theta}(M, N)$.



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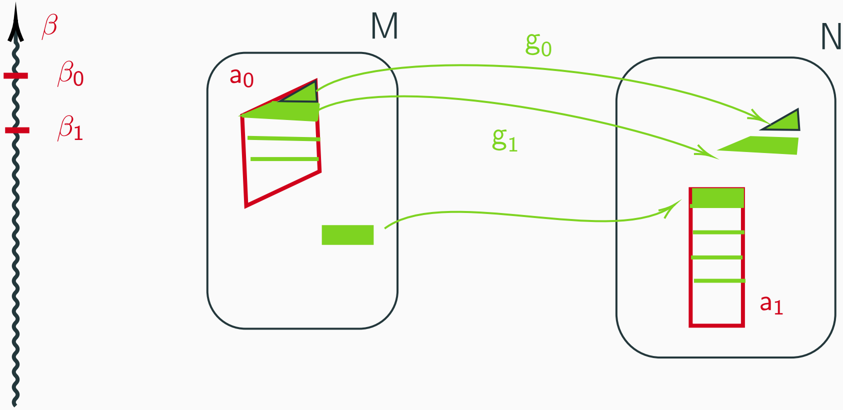

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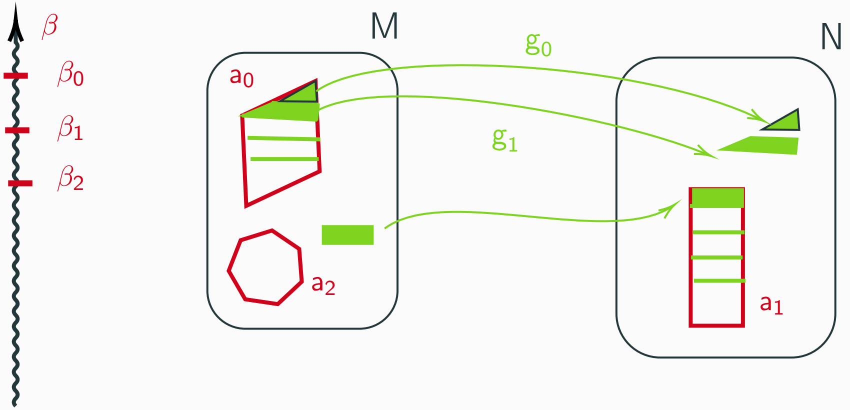

Shelah's game $\mathfrak{D}^{\beta}_{\theta}(M, N)$ of ordinal clock β.

$\forall$	$\exists$
$\beta_0 < \beta$, $\vec{a^0}$	
	$f_0 : \vec{a^0} \to \omega$, $g_0 : M \to N$ a p.i.
$\beta_1 < \beta_0$, $\vec{b^1}$	$f_1 : \vec{a^1} \to \omega$, $g_1 : M \to N$ a p.i., $g_1 \supseteq g_0$
$\vdots$	$\vdots$

Constraints:

- $\text{len}(\vec{a^n}) \leq \theta$
- $f^{-1}_{2n}(m) \subseteq \text{dom}(g_{2n})$ for $m \leq n$.
- $f^{-1}_{2n+1}(m) \subseteq \text{ran}(g_{2n})$ for $m \leq n$.

$\exists$ wins if she can play all her moves, otherwise $\forall$ wins.

Shelah's game equivalence (not [nec.] transitive!)

- M $\sim_\theta^\beta$ N iff $\exists$ has a winning strategy in the game.
- M $\equiv_\theta^\beta$ N is defined as the transitive closure of M $\sim_\theta^\beta$ N.
- A union of $\leq \beth_{\beta+1}(\theta)$ equivalence classes of $\equiv_\theta^\beta$ for some $\theta < \kappa$ and $\beta < \theta^+$ is called a sentence of $\mathbb{L}_\kappa^1$.

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Notice the weirdness!

(With Kiivimäki and Väänänen, we constructed a logic **with generative syntax** (usual sentences) whose Δ-closure is $\mathbb{L}_\kappa^1$: **Cartagena logic** $\mathbb{L}_\kappa^{1,c}$.)

Virtualizing $\mathbb{L}^1_\kappa$, $\mathbb{L}^{1,c}_\kappa$, ...

There are at least two competing virtualizations of these logics:

- Use the definition from [BDGM]... but with the difficulty of not having a good grasp (in the case of Shelah's logic) of elementary embeddings...

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There are at least two competing virtualizations of these logics:

- Use the definition from [BDGM]... but with the difficulty of not having a good grasp (in the case of Shelah's logic) of elementary embeddings...
- Use a "virtualized" version of the Shelah (or the Cartagena) game $\mathfrak{D}^\beta_\theta$, $\mathfrak{D}^{\beta,c}_\theta$ (Kiivimäki, Väänänen, V.)

Both virtualized versions are essentially existential closures of the logics. They would give rise to two competing notions of virtual embeddings (or different notions of genericity!). So... which one?

Delayable, virtually delayable...

Definition
A cardinal κ is a delayable cardinal if it is a compactness cardinal for the second-order version of Shelah's logic $\mathsf{L}^{1,\mathsf{II}}_{\kappa}$. It is a virtually delayable cardinal if it is a pseudo-compactness cardinal for $\mathsf{L}^{1,\mathsf{II}}_{\kappa}$. If we replace $\mathsf{L}^{1,\mathsf{II}}_{\kappa}$ by $\mathsf{L}^{1,\mathsf{II},\mathsf{c}}_{\kappa}$ we get the corresponding two notions of Cart-delayable cardinal and virtually Cart-delayable cardinal.

1. Where are these cardinals located? What kind of reflection properties do they capture?
2. The deeper issue is: what kind of virtuality do they actually correspond to? What version of forth-systems?

Thank you for your attention!

We go wrong when we believe in facts; only signs exist. We go wrong when we believe in truth; only interpretations exist. The sign is an always equivocal, implicit and implied meaning. "Throughout my existence, I had followed a path inverse to those of those peoples who only after having considered characters as mere successions of symbols start using them as a phonetic writing." [La prisonnière, Proust]

G. Deleuze - Proust and signs